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Two nontrivial solutions for a nonhomogeneous fractional Schrödinger–Poisson equation in $\mathbb{R}^3$

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Abstract

In this paper, we consider the following nonhomogeneous fractional Schrödinger–Poisson equations:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u) + g(x) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0,1]$, $2t + 4s > 3$, $(-\Delta)^s$ denotes the fractional Laplacian. By assuming more relaxed conditions on the nonlinear term f, using some new proof techniques on the verification of the boundedness of Palais–Smale sequence, existence and multiplicity of solutions are obtained.

MSC: 35B38; 35A15; 35J60

Keywords: Fractional Schrödinger–Poisson; Variational methods; Critical point theorem

1 Introduction

In this paper, we are concerned with nontrivial solutions for the following nonhomogeneous fractional Schrödinger–Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u) + g(x) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{E}$$

where $s, t \in (0,1]$, $2t + 4s > 3$, $(-\Delta)^s$ denotes the fractional Laplacian.

The nonlinear fractional Schrödinger–Poisson system (E) comes from the following fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = m(x,u), \quad x \in \mathbb{R}^3,$$

used to study the standing wave solutions $\psi(x,t) = u(x)e^{-iwt}$ for the equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hbar(-\Delta)^{\alpha}\psi + W(x)\psi - m(x,\psi), \quad x \in \mathbb{R}^3,$$

where $\hbar$ is Planck's constant, $W : \mathbb{R}^3 \to \mathbb{R}$ is an external potential, and m is a suitable nonlinearity. In the research of fractional quantum mechanics, this equation is an important model; therefore, it has been extensively studied, for example, see [4, 5, 13, 20, 22, 25, 26, 29, 32, 33] and their references.

When $s = t = 1$, system (E) reduces to the following Schrödinger–Poisson type equations:

$$\begin{cases} -\Delta u + V(x)u + \phi u = m(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{E$'$}$$

This problem was first introduced by Benci and Fortunato in [2]. They took it as a physical model describing solitary waves for nonlinear Schrödinger type equations interacting with a known electrostatic field. The first equation of (E′) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field. We refer the readers to [1, 16, 17] and the references for the details about the physical background. Moreover, equation (E′) has been dealt with by several papers, for example, [3, 7, 14, 15, 21, 24, 27, 30, 34, 35]. The authors in [27] researched the case when m was asymptotically linear at infinity. In [34], when m is superlinear at infinity, the authors proved the existence of ground state solutions. In [21], the researcher considered the existence of infinitely many solutions via the fountain theorem.

However, the fractional Schrödinger–Poisson system was first introduced by Giammetta in [12] and the diffusion is fractional only in the Poisson equation. In the last decades, there have been many papers devoted to the existence and multiplicity of solutions for the system like

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{E$''$}$$

Under different assumptions on the potential $V(x)$ and the nonlinearity f, the tools are variational methods and critical point theory, see [5, 10, 11, 31] and the references therein. In [31], with the following super-quadratic conditions, the researcher proved the existence of infinitely many solutions for the fractional Schrödinger–Poisson equation:

(A_1) $V \in C(\mathbb{R}^3,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a positive constant. Moreover, for each $b > 0$, $|\{x \in \mathbb{R}^3 | V(x) \leq b\}| < +\infty$, where $|\cdot|$ is the Lebesgue measure;

(A_2) $f \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$ for every $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\bigl|f(x,u)\bigr| \leq C\bigl(1 + |u|^{p-1}\bigr)$$

for some $p \in (2, 2_s^*)$, where $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent. Moreover, $f(x,t) = o(|u|)$, $|u| \to 0$, uniformly on $\mathbb{R}^3$;

(A_3) $\frac{F(x,u)}{|u|^4} \to +\infty$ as $|u| \to \infty$ uniformly on $\mathbb{R}^3$;
(A_4) There exists a constant $\theta \geq 1$ such that

$$\theta \mathfrak{F}(x,u) \geq \mathfrak{F}(x,\tau u), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tau \in [0,1],$$

where $\mathfrak{F}(x,u) = \frac{1}{4}uf(x,u) - F(x,u)$;
(A_5) There exists $r_1 > 0$ such that

$$4F(x,u) \leq uf(x,u), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, |u| \geq r_1;$$

(A_6) $f(x,-u) = -f(x,u), \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}$.
Under conditions (A_1)–(A_4), (A_6) or (A_1)–(A_3), (A_5), and (A_6), the author obtained infinitely many solutions for (E″).
In [6], in order to prove the existence of high energy solutions, the following Ambrosetti and Rabinowitz condition was assumed:
(A_0) (Also known as (AR) condition) There exist $\mu > 4$ and $r_1 > 0$ such that

$$0 < \mu F(x,u) \leq uf(x,u), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, |u| \geq r_1.$$

It is known that the (AR) condition is important to verify the boundedness of a $(PS)_c$ ($c \in \mathbb{R}$) sequence of the corresponding functional. Without the (AR) condition, the problem becomes more complicated. In order to overcome the difficulty, we will assume that $f(x,u)$ in problem (E) satisfies some weaker conditions. We also use the conditions on the potential $V(x)$ to get the boundedness; to the best of our knowledge, about this, little information could be found in the existing references. In this paper, we assume that the functions V and f satisfy the following conditions:
(V_0) $V \in C(\mathbb{R}^3,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where V_0 is a positive constant;
(V_1) For each $b > 0$, $|\{x \in \mathbb{R}^3 | V(x) \leq b\}| < +\infty$, where $|\cdot|$ is the Lebesgue measure;
(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$ for every $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\bigl|f(x,t)\bigr| \leq C\bigl(1 + |t|^{p-1}\bigr)$$

for some $p \in (2, 2_s^*)$, where $2_s^* = \frac{6}{3-2s}$ is the fractional critical Sobolev exponent;
(f_2) $f(x,t) = o(|t|)$, $|t| \to 0$, uniformly on $\mathbb{R}^3$;
(f_3) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to \infty$ uniformly on $\mathbb{R}^3$;
(f_4) There exist $L > 0$ and $d \in [0, \frac{V_0}{2}]$ such that

$$4F(x,t) - f(x,t)t \leq d|t|^2$$

for a.e. $x \in \mathbb{R}^3$ and $\forall |t| \geq L$. Now we state our results as follows.

Theorem 1.1 *Assume that* $s,t \in (0,1]$, $2t + 4s > 3$, $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, (V_0), (V_1), *and* (f_1)–(f_4) *hold. Then there exists a constant* g_0 *such that problem* (E) *has at least two different solutions whenever* $|g|_2 < g_0$, *one is a negative energy solution, and the other is a positive energy solution.*

Remark 1.2 Note that condition (f_3) is more general than the condition

$$0 < \mu F(x,t) \leq tf(x,t), \quad \mu > 4, t \neq 0.$$

(Similar to the proof of Lemma 2.2 of [9].)

Theorem 1.3 *Assume that* $t \in (0,1]$, $s \in (3/4,1)$, $2t + 4s > 3$, $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, (V_0), (V_1), *and* (f_2) *and*

(f_1') $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ *for every* $x \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$, *there exists a constant* $C > 0$ *such that*

$$\left|f(x,t)\right| \leq C\left(1 + |t|^{p-1}\right)$$

for some $p \in (4, 2_s^*)$, *where* $2_s^* = \frac{6}{3-2s}$ *is the fractional critical Sobolev exponent*;

(f_3') $\inf_{x\in\mathbb{R}^3,|t|=1} F(x,t) > 0$, *where* $F(x,t) = \int_0^t f(x,s)\,ds$;

(f_4') *There exist* $\mu > 4$ *and* $r > 0$ *such that*

$$\mu F(x,t) - f(x,t)t \leq 0, \quad \forall x \in \mathbb{R}^3, |t| \geq r$$

hold. Then there exists a constant m_0 *such that problem* (E) *has at least two different solutions whenever* $|g|_2 < m_0$, *one is a negative energy solution, and the other is a positive energy solution.*

Remark 1.4 Conditions (f_3') and (f_4') imply that the range of p in (f_1') should be $(4, 2_s^*)$ not $(2, 2_s^*)$. If $p \leq 4$, by (f_1'), one has

$$\left|F(x,t)\right| \leq \int_0^1 \left|f(x,st)t\right|\,ds \leq C\int_0^1 \left(1 + |st|^{p-1}\right)|t|\,ds \leq C\left(|t| + |t|^p\right)$$

for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, then we conclude that

$$\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^4} \leq C \quad \text{uniformly in } x \in \mathbb{R}^3. \tag{1.1}$$

For any $x \in \mathbb{R}^3$, $r \in \mathbb{R}$, define

$$m(t) := F\left(x, t^{-1}r\right)t^{\mu}, \quad t \geq 1.$$

Then, by (f_4'), one has

$$m'(t) = t^{\mu-1}\left[\mu F\left(x, t^{-1}r\right) - t^{-1}rf\left(x, t^{-1}r\right)\right] \leq 0$$

for $|r| \geq 1$, $t \in [1, |r|]$. That is, $m(1) \geq m(|r|)$. Therefore, by (f_3'), we deduce

$$F(x,r) \geq F\left(x, \frac{r}{|r|}\right)|r|^{\mu} \geq \inf_{x\in\mathbb{R}^3,|t|=1} F(x,t)|r|^{\mu}$$

for $x \in \mathbb{R}^3$ and $|r| \geq 1$, which contradicts (1.1). That is, (f_3') and (f_4') imply that the range of p in (f_1') should be $(4, 2_s^*)$.

Theorem 1.5 *Assume that $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, (V_0), (V_1), (f_1), (f_2) and*

(f_3'') *there exists $L' > 0$ such that*

$$c' = \inf_{x \in \mathbb{R}^3, |t| = L'} F(x,t) > 0;$$

(f_4'') *there exist $\mu > 4$ and $d' \in [0, \frac{c'(\mu-2)}{L'^2})$ such that*

$$\mu F(x,t) - f(x,t)t \le d'|t|^2 \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and } \forall |t| \ge L'.$$

Then there exists a constant g_0 such that problem (E) has at least two different solutions whenever $|g|_2 < g_0$, one is a negative energy solution, and the other is a positive energy solution.

2 Preliminaries

In this paper, we make use of the following notations: the L^r-norm $(1 \le r \le +\infty)$ by $|\cdot|_r$. C denotes various positive constants, which may vary from line to line.

We define the Gagliardo seminorm by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^p}{|x-y|^{3+sp}} \, dx \, dy \right)^{1/p},$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ is a measurable function. Then we define a fractional Sobolev space by

$$W^{s,p}(\mathbb{R}^3) = \left\{ u \in L^p(\mathbb{R}^3) : u \text{ is measurable and } [u]_{s,p} < \infty \right\}$$

endowed with the norm

$$\|u\|_{s,p} = \left([u]_{s,p}^p + |u|_p^p \right)^{1/p},$$

where $|u|_p = (\int_{\mathbb{R}^3} |u(x)|^p \, dx)^{1/p}$.

For $p = 2$, the space $W^{s,2}(\mathbb{R}^3)$ is an equivalent definition of the fractional Sobolev spaces, which is based on the Fourier analysis, that is,

$$H^s(\mathbb{R}^3) = W^{s,2}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\tilde{u}|^2 \, d\xi < \infty \right\}$$

endowed with the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\tilde{u}|^2 \, d\xi + \int_{\mathbb{R}^3} |\tilde{u}|^2 \, d\xi \right)^{1/2},$$

where $\tilde{u}$ denotes the usual Fourier transform of u.

Furthermore, we know that $\|\cdot\|_{H^s}$ is equivalent to the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} \left| (-\triangle)^{s/2} u \right|^2 dx + \int_{\mathbb{R}^3} u^2 \, dx \right)^{1/2}.$$

In view of the potential $V(x)$, we consider the subspace

$$X = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < \infty \right\}.$$

Thus, X is a Hilbert space with the inner product

$$(u,v) = \int_{\mathbb{R}^3} \bigl(|\xi|^{2s}\tilde{u}(\xi)\tilde{v}(\xi) + \tilde{u}(\xi)\tilde{v}(\xi)\bigr)\,d\xi + \int_{\mathbb{R}^3} V(x)u(x)v(x)\,dx$$

and the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^3} \bigl(|\xi|^{2s}\bigl|\tilde{u}(\xi)\bigr|^2 + \bigl|\tilde{u}(\xi)\bigr|^2\bigr)\,d\xi + \int_{\mathbb{R}^3} V(x)u^2(x)\,dx \right)^{1/2}.$$

Moreover, $\|\cdot\|_X$ is equivalent to the norm

$$\|u\| := \|u\|_X = \left(\int_{\mathbb{R}^3} \bigl|(-\triangle)^{s/2}u\bigr|^2\,dx + \int_{\mathbb{R}^3} V(x)u^2\,dx \right)^{1/2},$$

where the corresponding inner product is

$$(u,v)_X = \int_{\mathbb{R}^3} \left((-\triangle)^{s/2}u(-\triangle)^{s/2}v + \int_{\mathbb{R}^3} V(x)uv \right) dx.$$

The homogeneous Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined by

$$D^{s,2}(\mathbb{R}^3) = \bigl\{ u \in L^{2_s^*}(\mathbb{R}^3) : |\xi|^s\tilde{u}(\xi) \in L^2(\mathbb{R}^3) \bigr\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{s,2}} = \left(\int_{\mathbb{R}^3} \bigl|(-\triangle)^{s/2}u\bigr|^2\,dx \right)^{1/2} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}\bigl|\tilde{u}(\xi)\bigr|^2\,d\xi \right)^{1/2}$$

endowed with the inner product

$$(u,v)_{D^{s,2}} = \int_{\mathbb{R}^3} (-\triangle)^{s/2}u(-\triangle)^{s/2}v\,dx.$$

Then $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$, that is, there exists a constant $C_0 > 0$ such that

$$|u|_{2_s^*} \le C_0\|u\|_{D^{s,2}}.$$

Lemma 2.1 ([5]) *Space X is continuously embedded in $L^p(\mathbb{R}^3)$ for $2 \le p \le 2_s^*$ and compactly embedded in $L^p(\mathbb{R}^3)$ for all $p \in [2, 2_s^*)$.*

By Lemma 2.1, *one obtains that there exists a constant $\nu_p > 0$ such that*

$$|u|_p \le \nu_p\|u\|, \tag{2.1}$$

where $|u|_p$ denotes the usual norm in $L^p(\mathbb{R}^3)$ for $2 \le p \le 2_s^*$.

Lemma 2.2 (Theorem 2.1, [8]) *For any $s \in (0, \frac{3}{2})$, $D^{s,2}(\mathbb{R}^3)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^3)$, that is, there exists $c_s > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2_s^*}\, dx\right)^{2/2_s^*} \le c_s \int_{\mathbb{R}^3} \left|(-\triangle)^{\frac{s}{2}} u\right|^2 dx, \quad u \in D^{s,2}\bigl(\mathbb{R}^3\bigr).$$

If $2t + 4s > 3$, then $X \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. For $u \in X$, the linear operator $T_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$T_u(v) = \int_{\mathbb{R}^3} u^2 v\, dx.$$

By Hölder's inequality and Lemma 2.2,

$$\left|T_u(v)\right| \le \|u\|^2_{12/(3+2t)} \|v\|_{2_t^*} \le C\|u\|^2 \|v\|_{D^{t,2}}. \tag{2.2}$$

Set

$$\eta(u,v) = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} u \cdot (-\Delta)^{\frac{t}{2}} v\, dx, \quad u, v \in D^{t,2}\bigl(\mathbb{R}^3\bigr).$$

It is clear that $\eta(u,v)$ is bilinear, bounded, and coercive. The Lax–Milgram theorem implies that, for every $u \in X$, there exists unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that $T_u(v) = \eta(\phi_u, v)$ for any $v \in D^{t,2}(\mathbb{R}^3)$, that is,

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t \cdot (-\Delta)^{\frac{t}{2}} v\, dx = \int_{\mathbb{R}^3} u^2 v\, dx. \tag{2.3}$$

Therefore, $(-\Delta)^t \phi_u^t = u^2$ in a weak sense. Moreover,

$$\left\|\phi_u^t\right\|_{D^{t,2}} \le C\|u\|^2. \tag{2.4}$$

Since $t \in (0,1]$ and $2t + 4s > 3$, then $\frac{12}{3+2t} \in (2, 2_s^*)$. From Lemma 2.2, (2.2), and (2.3), it follows that

$$\left\|\phi_u^t\right\|_{D^{t,2}} = \int_{\mathbb{R}^3} \left|(-\triangle)^{\frac{t}{2}} \phi_u^t\right|^2 dx = \int_{\mathbb{R}^3} u^2 \phi_u^t\, dx \le C|u|^2_{\frac{12}{3+2t}} \left\|\phi_u^t\right\|_{D^{t,2}}. \tag{2.5}$$

Then

$$\left\|\phi_u^t\right\|_{D^{t,2}} \le C|u|^2_{\frac{12}{3+2t}}. \tag{2.6}$$

For $x \in \mathbb{R}^3$, we have

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}}\, dy,$$

which is the Riesz potential [23], where

$$c_t = \frac{\Gamma(\frac{3-2t}{2})}{\pi^{3/2} 2^{2t} \Gamma(t)}.$$

Substituting ϕ_u^t in (E), we have the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u + \phi_u^t u = f(x,u) + g(x), \quad x \in \mathbb{R}^3. \tag{2.7}$$

The energy functional $I: X \to \mathbb{R}$ corresponding to problem (2.7) is defined by

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[\left|(-\Delta)^s u\right|^2 + V(x)u^2\right]dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx, \quad u \in X. \end{aligned} \tag{2.8}$$

We say that $(u,\phi) \in X \times D^{t,2}(\mathbb{R}^3)$ is a weak solution of problem (E) if u is a weak solution of (2.7). By a similar argument as [28], we also can get the following lemma, which is important for our research.

Lemma 2.3 *Assume that $g \in L^2(\mathbb{R}^3)$, (V_0), (V_1) and (f_1), (f_2) hold. Then I is well defined in X and $I \in C^1(\mathbb{R}^3,\mathbb{R})$, and*

$$\begin{aligned} \langle I'(u), v\rangle &= \int_{\mathbb{R}^3}\left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + V(x)uv\right]dx + \int_{\mathbb{R}^3}\phi_u^t uv\,dx \\ &\quad - \int_{\mathbb{R}^3} f(x,u)v\,dx - \int_{\mathbb{R}^3} g(x)v\,dx, \quad u,v \in X. \end{aligned} \tag{2.9}$$

Moreover, let $\Psi(u) = \int_{\mathbb{R}^3} F(x,u)\,dx + \int_{\mathbb{R}^3} g(x)u\,dx$, then $\Psi' : X \to X^*$ is compact.

In order to get a negative energy solution, our tool is Ekeland's variational principle. For readers' convenience, we give it in the following.

Lemma 2.4 (Theorem 4.1, [18]) *Let M be a complete metric space with metric d, and let $I: M \to (-\infty,+\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $u \in M$ be such that*

$$I(u) \le \inf_M I + \varepsilon.$$

Then there exists $v \in M$ such that

$$I(v) \le I(u), \qquad d(u,v) \le 1,$$

and for each $w \in M$, one has

$$I(v) \le I(w) + \varepsilon d(v,w).$$

At the end of this section, we recall the mountain pass theorem, which is necessary to obtain the main results. This theorem allows us to find a Palais–Smale type sequence. Recall that a sequence $\{u_n\} \subset X$ is said to be a Palais–Smale sequence at the level $c \in \mathbb{R}$ ($(PS)_c$-sequence for short) if $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$. I_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$-sequence has a convergent subsequence.

Lemma 2.5 (Theorem 2.2, [19]) *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfying (PS) conditions. Suppose $I(0) = 0$ and*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$,*
(ii) *there is $u_1 \in X \setminus \bar{B}_\rho$ such that $I(u_1) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\}$.

3 Proof of Theorem 1.1

Lemma 3.1 *Assume that $g \in L^2(\mathbb{R}^3)$ and (f_1), (f_2) hold. Then there exist some constants ρ, α, and $\beta > 0$ such that $I(u) \geq \alpha$ whenever $\|u\|_X = \rho$ and $|g|_2 < \beta$.*

Proof For any $\varepsilon > 0$, by (f_1), (f_2), there exists $C(\varepsilon) > 0$ such that

$$\left|f(x,u)\right| \leq \varepsilon|u| + C(\varepsilon)|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{3.1}$$

$$\left|F(x,u)\right| \leq \int_0^1 \left|f(x,st)t\right| ds \leq \varepsilon|u|^2 + C(\varepsilon)|u|^p, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.2}$$

By (2.1), (2.8), (3.2), and Hölder's inequality,

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \left(\varepsilon|u|_2^2 + C(\varepsilon)|u|_p^p\right) - |g|_2|u|_2 \\
&\geq \frac{1}{2}\|u\|^2 - \left(\nu_2^2\varepsilon\|u\|^2 + \nu_p^p C(\varepsilon)\|u\|^p\right) - \nu_2|g|_2\|u\| \\
&= \|u\|\left[\left(\frac{1}{2} - \nu_2^2\varepsilon\right)\|u\| - \nu_p^p C(\varepsilon)\|u\|^{p-1} - \nu_2|g|_2\right].
\end{aligned}$$

Choose $\varepsilon = \frac{1}{4\nu_2^2} > 0$, and take

$$m(t) = \frac{1}{4}t - \nu_p^p C(\varepsilon)t^{p-1}, \quad \forall t \geq 0.$$

Note that $2 < p < 2_s^*$, we can conclude that there exists a constant $\rho > 0$ such that $h(\rho) = \max_{t\geq 0} h(t) > 0$. Therefore, take $\beta = \frac{1}{2\nu_2}m(\rho) > 0$, it follows that

$$I(u) \geq \frac{1}{2}\rho m(\rho) =: \alpha > 0$$

wherever $\|u\| = \rho$ and $|g|_2 < \beta$. This completes the proof. □

Lemma 3.2 *Let assumptions (f_1)–(f_3) be satisfied. Then there exists a function $e \in X$ with $\|e\| > \rho$ such that $I(e) < 0$.*

Proof For every $M > 0$, by (f_1)–(f_3), there exists $C(M) > 0$ such that

$$F(x,u) \geq M|u|^4 - C(M)|u|^2 \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.3}$$

Choose $\varphi \in X$ with $|\varphi|_4 = 1$, then

$$\begin{aligned} I(t\varphi) = {} & \frac{t^2}{2}\int_{\mathbb{R}^3}\bigl[\bigl|(-\Delta)^{\frac{s}{2}}\varphi\bigr|^2 + V(x)\varphi^2\bigr]\,dx + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_\varphi^t\varphi^2\,dx - \int_{\mathbb{R}^3}F(x,t\varphi)\,dx \\ & - t\int_{\mathbb{R}^3} g(x)\varphi\,dx, \quad u \in X. \end{aligned}$$

By (2.3) and (2.4), then

$$\int_{\mathbb{R}^3}\phi_\varphi^t\varphi^2\,dx = \bigl|\phi_\varphi^t\bigr|_{D^{t,2}}^2 \leq C_0\|\varphi\|^4,$$

combining with (2.8), (3.3) and Hölder's inequality, one has

$$\begin{aligned} I(t\varphi) &\leq \frac{t^2}{2}\|\varphi\|^2 + \frac{t^4}{4}C\|\varphi\|^4 - M|\varphi|_4^4 t^4 + C(M)|\varphi|_2^2 t^2 - t\int_{\mathbb{R}^3} g(x)\varphi\,dx \\ &\leq \left(\frac{C_0}{4}\|\varphi\|^4 - M\right)t^4 + \left(\frac{1}{2}\|\varphi\|^2 + C(M)|\varphi|_2^2\right)t^2 + |g|_2|\varphi|_2 t, \end{aligned}$$

which implies $I(t\varphi) \to -\infty$ as $t \to +\infty$ by taking $M > \frac{C_0}{4}\|\varphi\|^4$. Hence, there exists $e = t_0\varphi$ with t_0 sufficiently large such that $\|e\| > \rho$ and $I(e) < 0$. The proof is completed. □

Lemma 3.3 *Assume that* $(V_0), (V_1), (f_1), (f_2)$ *hold. Then any bounded Palais–Smale sequence of* I *has a strongly convergent subsequence in* X.

Proof Let $\{u_n\} \subset X$ be any bounded Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in \mathbb{R}$ such that

$$I(u_n) \to c_1, \qquad I'(u_n) \to 0 \quad \text{and} \quad \sup_n \|u_n\| < +\infty. \tag{3.4}$$

Since the embedding $X \hookrightarrow L^p(\mathbb{R}^3)$, $2 \leq p < 2_s^*$, is compact, going if necessary to a subsequence, we can assume that there is $u \in X$ such that

$$\begin{aligned} &u_n \rightharpoonup u, \quad \text{weakly in X;} \\ &u_n \to u, \quad \text{strongly in } L^p(\mathbb{R}^3); \\ &u_n(x) \to u(x), \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

In view of (2.9), then

$$\begin{aligned} \bigl\langle I'(u_n) - I'(u), u_n - u\bigr\rangle = \|u_n - u\|^2 + \int_{\mathbb{R}^3}\bigl(\phi_{u_n}^t u_n - \phi_u^t u\bigr)(u_n - u)\,dx \\ - \int_{\mathbb{R}^3}\bigl[f(x,u_n) - f(x,u)\bigr](u_n - u)\,dx, \end{aligned}$$

and thus

$$\|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)\,dx \\ + \int_{\mathbb{R}^3} \bigl[f(x,u_n) - f(x,u)\bigr](u_n - u)\,dx. \tag{3.5}$$

Clearly, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

By the generalization of Hölder's inequality, Lemma 2.2, and (2.6), it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \phi_{u_n}^t u_n(u_n - u)\,dx\right| &\le |\phi_{u_n}^t|_{2_t^*} |u_n|_{\frac{12}{3+2t}} |u_n - u|_{\frac{12}{3+2t}} \\ &\le C\|\phi_{u_n}^t\|_{D^{t,2}} |u_n|_{\frac{12}{3+2t}} |u_n - u|_{\frac{12}{3+2t}} \\ &\le C|u_n|^3_{\frac{12}{3+2t}} |u_n - u|_{\frac{12}{3+2t}} \\ &\le C\|u_n\|^3 |u_n - u|_{\frac{12}{3+2t}}.\end{aligned}$$

Similarly,

$$\left|\int_{\mathbb{R}^3} \phi_u^t u(u_n - u)\,dx\right| \le C\|u\|^3 |u_n - u|_{\frac{12}{3+2t}}.$$

Then we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)\,dx\right| &\le \left|\int_{\mathbb{R}^3} \phi_{u_n}^t u_n(u_n - u)\,dx\right| + \left|\int_{\mathbb{R}^3} \phi_u^t u(u_n - u)\,dx\right| \\ &\to 0,\end{aligned} \tag{3.7}$$

as $n \to \infty$. By (3.1) and using Hölder's inequality, we can conclude

$$\begin{aligned}&\bigl|\bigl[f(x,u_n) - f(x,u)\bigr](u_n - u)\,dx\bigr| \\ &\quad\le \bigl[\varepsilon + C(\varepsilon)\bigr] \int_{\mathbb{R}^3} \bigl[|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}\bigr]|u_n - u|\,dx \\ &\quad\le \bigl[\varepsilon + C(\varepsilon)\bigr]\bigl(|u_n|_2 + |u|_2\bigr)|u_n - u|_2 + \bigl[\varepsilon + C(\varepsilon)\bigr]\bigl(|u_n|_p^{p-1} + |u|_p^{p-1}\bigr)|u_n - u|_p,\end{aligned}$$

therefore, for $p \in (2, 2_s^*)$, we deduce

$$\int_{\mathbb{R}^3} \bigl[f(x,u_n) - f(x,u)\bigr](u_n - u)\,dx \to 0, \tag{3.8}$$

as $n \to \infty$. Consequently, (3.6)–(3.8) imply that

$$\|u_n - u\| \to 0,$$

as $n \to \infty$. This completes the proof. □

In order to get a negative energy solution for (E), we will use Ekeland's variational principle. We consider a minimization of I constrained in a neighborhood of zero and find a critical point of I which achieves the local minimum of I. Furthermore, the level of this local minimum is negative.

Lemma 3.4 *Assume that* $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, *and* (f_1)–(f_3) *hold. Then*

$$-\infty < \inf\{I(u) : u \in \bar{B}_\rho\} < 0,$$

where $\bar{B}_r := \{u \in X : \|u\| \le r\}$.

Proof By (f_1)–(f_3), it follows from the proof of Lemma 3.2 that

$$F(x,t) \ge C_1|t|^4 - C_2|t|^2, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where C_1 and C_2 are positive constants. Since $g(x) \in L^2(\mathbb{R}^3)$ and $g \not\equiv 0$, we can choose a function $v \in X$ such that

$$\int_{\mathbb{R}^3} g(x)v(x) > 0.$$

Thus,

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\int_{\mathbb{R}^3}\bigl[\bigl|(-\Delta)^{\frac{s}{2}}v\bigr|^2 + V(x)v^2\bigr]\,dx + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_v^t v^2\,dx - \int_{\mathbb{R}^3}F(x,tv)\,dx \\ &\quad - t\int_{\mathbb{R}^3} g(x)v\,dx \\ &\le \frac{t^2}{2}\|v\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_v^t v^2\,dx - C_1|v|_4^4 t^4 + C_2|v|_2^2 t^2 - t\int_{\mathbb{R}^3} g(x)v\,dx \\ &< 0 \end{aligned}$$

for $t > 0$ small enough, which implies $\inf\{I(u) : u \in \bar{B}_\rho\} < 0$. In addition, by (2.1), (2.8), (3.2), and Hölder's inequality,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\bigl[\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2 + V(x)u^2\bigr]\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^t u^2\,dx - \int_{\mathbb{R}^3}F(x,u)\,dx \\ &\quad - \int_{\mathbb{R}^3} g(x)u\,dx \\ &\ge -\int_{\mathbb{R}^3}F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\ &\ge -\bigl(\varepsilon|u|_2^2 + C(\varepsilon)|u|_p^p\bigr) - |g|_2|u|_2 \\ &\ge -\nu_2^2\varepsilon\|u\|^2 - \nu_p^p C(\varepsilon)\|u\|^p - \nu_2|g|_2\|u\|, \end{aligned}$$

which implies I is bounded below in $\bar{B}_\rho$. Therefore, we obtain

$$-\infty < \inf\{I(u) : u \in \bar{B}_\rho\} < 0.$$

The proof is completed. □

In the following, we give the results about the negative energy solution for problem (E).

Lemma 3.5 *Assume that $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, (V_0), (V_1) and (f_1)–(f_3) hold. Then there exists a constant $g_0 > 0$ such that problem (E) has a negative energy solution whenever $|g|_2 < g_0$, that is, there exists a function $u_0 \in X$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.*

Proof By Lemma 3.1 and Lemma 3.4, taking $g_0 = \beta > 0$, we know that

$$-\infty < \inf_{\bar{B}_\rho} I < 0 < \alpha \le \inf_{\partial B_\rho} I,$$

whenever $|g|_2 < g_0$. Set $\frac{1}{n} \in (0, \inf_{\partial B_\rho} I - \inf_{\bar{B}_\rho} I)$, $n \in \mathbb{N}$. Then there is $u_n \in \bar{B}_\rho$ such that

$$I(u_n) \le \inf_{\bar{B}_\rho} I + \frac{1}{n}. \tag{3.9}$$

By Ekeland's variational principle, it follows that

$$I(u_n) \le I(u) + \frac{1}{n}\|u_n - u\|, \quad \forall u \in \bar{B}_\rho. \tag{3.10}$$

Note that $I(u_n) \le \inf_{\bar{B}_\rho} I + \frac{1}{n} < \inf_{\partial B_\rho} I$. Thus, $u_n \in B_\rho$. Define $M_n : X \to \mathbb{R}$ by

$$M_n(u) = I(u) + \frac{1}{n}\|u - u_n\|.$$

By (3.10), we have that $u_n \in B_\rho$ minimizes M_n on $\bar{B}_\rho$. Therefore, for all $\varphi \in X$ with $\|\varphi\| = 1$, taking $t > 0$ small enough such that $u_n + t\varphi \in \bar{B}_\rho$, then

$$\frac{M_n(u_n + t\varphi) - M_n(u_n)}{t} \ge 0,$$

which implies that

$$\frac{I(u_n + t\varphi) - I(u_n)}{t} + \frac{1}{n} \ge 0.$$

Thus, $\langle I'(u_n), \varphi\rangle \ge -\frac{1}{n}$. Hence,

$$\|I'(u_n)\| \le \frac{1}{n}. \tag{3.11}$$

Passing to the limit in (3.9) and (3.11), we deduce that

$$I(u_n) \to \inf_{\bar{B}_\rho} I \quad \text{and} \quad \|I'(u_n)\| \to 0, \tag{3.12}$$

as $n \to \infty$. Note that $\|u_n\| \le \rho$, hence $\{u_n\} \subset X$ is a bounded Palais–Smale sequence of I. By Lemma 3.3, $\{u_n\}$ has a strongly convergent subsequence, still denoted by $\{u_n\}$ and $u_n \to u_0 \in \bar{B}_\rho$, as $n \to \infty$. Consequently, it follows from (3.12) that

$$I(u_0) = \inf_{\bar{B}_\rho} I < 0 \quad \text{and} \quad I'(u_0) = 0.$$

This completes the proof. □

Next, by using the mountain pass theorem, we give the positive energy solution.

Lemma 3.6 *Let assumptions* $(V_0), (V_1)$ *and* (f_1)*–*(f_4) *be satisfied. Then any Palais–Smale sequence of* I *is bounded.*

Proof Let $\{u_n\} \subset X$ be any Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in \mathbb{R}$ such that

$$I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0, \tag{3.13}$$

as $n \to \infty$. Combining (2.1), (2.8), (2.9), (3.13), (V_0), (V_1) with (f_4), we have

$$\begin{aligned}
c_1 + 1 + \|u\| &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^3} \left|(-\Delta)^s u_n\right|^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(x)u_n^2\,dx \\
&\quad + \int_{\mathbb{R}^3} \tilde{F}(x,u_n)\,dx - \frac{3}{4}\int_{\mathbb{R}^3} g(x)u_n\,dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3} \left|(-\Delta)^s u_n\right|^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(x)u_n^2\,dx - \frac{d}{4}\int_{\mathbb{R}^3} u_n^2\,dx \\
&\quad + \int_{A_n} \tilde{F}(x,u_n)\,dx - \frac{3}{4}\nu_2|g|_2\|u_n\| \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3} \left|(-\Delta)^s u_n\right|^2 + \frac{1}{4}\int_{\mathbb{R}^3} V(x)u_n^2\,dx - \frac{1}{8}\int_{\mathbb{R}^3} V_0 u_n^2\,dx \\
&\quad + \int_{A_n} \tilde{F}(x,u_n)\,dx - \frac{3}{4}\nu_2|g|_2\|u_n\| \\
&\geq \frac{1}{16}\|u_n\|^2 + \frac{1}{16}\int_{\mathbb{R}^3} V(x)u_n^2\,dx + \int_{\mathbb{R}^3} \tilde{F}(x,u_n)\,dx - \frac{3}{4}\nu_2|g|_2\|u_n\|,
\end{aligned}$$

where $\tilde{F}(x,u_n) = \frac{1}{4}f(x,u_n)u_n - F(x,u_n)$ and $A_n = \{x \in \mathbb{R}^3 : |u_n| \leq L\}$. Hence

$$c_1 + 1 + \left(1 + \frac{3}{4}\nu_2|g|_2\right)\|u_n\| \geq \frac{1}{16}\|u_n\|^2 + \frac{1}{16}\int_{\mathbb{R}^3} V(x)u_n^2\,dx + \int_{\mathbb{R}^3} \tilde{F}(x,u_n)\,dx. \tag{3.14}$$

For $x \in \mathbb{R}^3$ and $|u_n| \leq L$, by (3.1) and (3.2),

$$\begin{aligned}
\left|\tilde{F}(x,u_n)\right| &\leq \frac{1}{4}\left|f(x,u_n)\right||u_n| + \left|F(x,u_n)\right| \\
&\leq \frac{5}{4}\varepsilon|u_n|^2 + \frac{5}{4}C(\varepsilon)|u_n|^p \\
&= \frac{5}{4}\left[\varepsilon + C(\varepsilon)|u_n|^{p-2}\right]|u_n|^2 \\
&\leq \frac{5}{4}\left[\varepsilon + C(\varepsilon)L^{p-2}\right]|u_n|^2.
\end{aligned}$$

Take $A > \max\{20[\varepsilon + C(\varepsilon)L^{p-2}], V_0\}$, then

$$\tilde{F}(x,u_n) \geq -\frac{A}{16}|u_n|^2, \quad \forall x \in \mathbb{R}^3, |u_n| \leq L. \tag{3.15}$$

Let $\tilde{A} = \{x \in \mathbb{R}^3 : V(x) \leq A\}$. By (V_0), (V_1), and (3.15), one has

$$\begin{aligned}\frac{1}{16}\int_{\mathbb{R}^3} V(x)u_n^2\,dx + \int_{A_n} \tilde{F}(x,u_n)\,dx &\geq \frac{1}{16}\int_{|u_n|\leq L} \big(V(x)-A\big)|u_n|^2\,dx \\ &\geq \frac{1}{16}\int_{\tilde{A}\bigcap A_n} \big(V(x)-A\big)L^2\,dx \\ &\geq \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}\cap A_n) \\ &\geq \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}). \end{aligned} \tag{3.16}$$

Note that $\text{meas}(\tilde{A}) < +\infty$ due to (V_1), it follows from (3.14) and (3.16) that

$$c_1 + 1 + \left(1 + \frac{3}{4}\nu_2|g|_2\right)\|u_n\| \geq \frac{1}{16}\|u_n\|^2 + \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}),$$

which implies $\{u_n\} \subset X$ is bounded in X. Hence, the proof is completed. □

Lemma 3.7 *Assume that $g \in L^2(\mathbb{R}^3)$, $g \not\equiv 0$, (V_0), (V_1) and (f_1)–(f_4) hold. Then problem (E) has a positive energy solution whenever $|g|_2 < g_0$, that is, there exists a function $u_1 \in X$ such that $I'(u_1) = 0$ and $I(u_n) > 0$.*

Proof In order to prove this lemma, we will use Lemma 2.5. In the following, we shall verify that all conditions of Lemma 2.5 are satisfied. By Lemma 3.5, we know that $g_0 = \beta > 0$. Then, by Lemma 3.1, when $|g|_2 < g_0$, I satisfies condition (i). Lemma 3.2 implies that I satisfies condition (ii). By virtue of Lemma 3.3 and Lemma 3.6, I satisfies the (PS) condition. It is easy to verify that $I \in C^1(X,\mathbb{R})$ and $I(0) = 0$. Hence, by Lemma 2.5, there exists a function $u_1 \in X$ such that $I'(u_1) = 0$ and $I(u_1) \geq \alpha > 0$. The proof is completed. □

Proof of Theorem 1.1 By Lemma 3.5 and Lemma 3.7, we can get the conclusion. □

Proof of Theorem 1.3 We only need to prove that (f_3') and (f_4') imply (f_3) and (f_4). By Remark 1.4, we have

$$\frac{F(x,t)}{t^4} \geq c|t|^{\mu-4}, \quad \forall x \in \mathbb{R}^3, |t| \geq 1,$$

which implies (f_3). Moreover, note that $\mu > 4$, then by (f_4') one has

$$\begin{aligned}4F(x,t) - f(x,t)t &= \mu F(x,t) - f(x,t)t + (4-\mu)F(x,t) \\ &\leq (4-\mu)F(x,t) \leq (4-\mu)c|t|^{\mu} \\ &< 0 \leq d|t|^2\end{aligned}$$

for all $x \in \mathbb{R}^3$ and $|t| \geq 1$. This implies that (f_4) holds by taking $L = 1$. That is, (f_3') and (f_4') imply (f_3) and (f_4). Then, similar to the proof of Theorem 1.1, we can get the conclusion. □

Proof of Theorem 1.5 It is sufficient to prove that (f_3''), (f_4'') imply (f_3), (f_4) by applying Theorem 1.1. In fact, for any $(x,r) \in \mathbb{R}^3 \times \mathbb{R}$, define

$$h(t) := F\left(x, \frac{r}{t}\right)t^\mu, \quad \forall t \geq 1.$$

Then, for $|r| \geq L'$ and $t \in [1, \frac{|r|}{L'}]$, (f_4'') implies that

$$h'(t) = f\left(x, \frac{r}{t}\right)\left(-\frac{r}{t^2}\right)t^\mu + \mu F\left(x, \frac{r}{t}\right)t^{\mu-1},$$
$$t^{\mu-1}\left[\mu F\left(x, \frac{r}{t}\right) - f\left(x, \frac{r}{t}\right)\frac{r}{t}\right] \leq d' t^{\mu-1}\left|\frac{r}{t}\right|^2 = d' t^{\mu-3}|r|^2.$$

Thus,

$$h\left(\frac{|r|}{L'}\right) - h(1) \leq \int_1^{\frac{|r|}{L'}} d' t^{\mu-3}|r|^2\,dt = \frac{d'|r|^\mu}{(\mu-2)L'^{\mu-2}} - \frac{d'|r|^2}{\mu-2}.$$

Hence, for any $x \in \mathbb{R}^3$ and $|r| \geq L'$, by (f_3''), one has

$$F(x,r) = h(1) \geq h\left(\frac{|r|}{L'}\right) - \frac{d'|r|^\mu}{(\mu-2)L'^{\mu-2}} + \frac{d'|r|^2}{\mu-2},$$
$$\left[\inf_{x\in\mathbb{R}^3, |t|=L'} F(x,t)\right]\left(\frac{|r|}{L'}\right)^\mu - \frac{d'|r|^\mu}{(\mu-2)L'^{\mu-2}} + \frac{d'|r|^2}{\mu-2} \geq \left(\frac{c'}{L'^\mu} - \frac{d'}{(\mu-2)L'^{\mu-2}}\right)|r|^\mu.$$

By (f_4''), let $C_0 = \frac{c'}{L'^\mu} - \frac{d'}{(\mu-2)L'^{\mu-2}} > 0$, it has $F(x,r) \geq C_0|r|^\mu$ for $x \in \mathbb{R}^3$ and $|r| \geq L'$. Hence,

$$\frac{F(x,r)}{r^4} \geq C_0|r|^{\mu-4}, \quad \forall x \in R^3, |r| \geq L',$$

which implies (f_3) due to $\mu > 4$. Moreover, combined with (f_4''), we also get

$$4F(x,r) - f(x,r)r \leq d'|r|^2 - (\mu-4)C_0|r|^\mu$$

for all $x \in \mathbb{R}^3$ and $|r| \geq L'$. Together with $\mu > 4$, there exists $L > 0$ such that

$$4F(x,r) - f(x,r)r < 0, \quad x \in \mathbb{R}^3, |r| \geq L,$$

which implies (f_4). Hence, the proof is completed. □

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