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Multiple positive solutions for nonlinear high-order Riemann–Liouville fractional differential equations boundary value problems with p-Laplacian operator

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Abstract

In this paper, we study the existence of multiple positive solutions for boundary value problems of high-order Riemann–Liouville fractional differential equations involving the p-Laplacian operator. Not only new existence conclusions of two positive solutions are obtained by employing functional-type cone expansion-compression fixed point theorem, but also some sufficient conditions for existence of at least three positive solutions are established by applying the Leggett–Williams fixed point theorem. In addition, we demonstrate the effectiveness of the main result by using an example.

MSC: 34B18; 34A08; 35J05

Keywords: p-Laplacian operator; Leggett–Williams fixed point theorem; Riemann–Liouville fractional differential equations; Multiple positive solutions

1 Introduction

In this paper, we study the high-order Riemann–Liouville fractional differential equations with p-Laplacian operator as follows:

$$\begin{cases} {}_0^RD_t^\alpha(\varphi_p({}_0^RD_t^\alpha u(t))) = f(t,u(t),{}_0^RD_t^\alpha u(t)), \quad 0 \le t \le 1; \\ u^{(i)}(0) = 0, \qquad [\varphi_p({}_0^RD_t^\alpha u)]^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2; \\ [{}_0^RD_t^\beta u(t)]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1; \\ [{}_0^RD_t^\beta(\varphi_p({}_0^RD_t^\alpha u(t)))]_{t=1} = 0; \end{cases} \tag{1.1}$$

where $n-1 < \alpha \le n$, ${}_0^RD_t^\alpha$ is the standard Riemann–Liouville fractional derivative, φ_p is the p-Laplacian operator, $p > 1$ and $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$. By using some fixed point theorems, we establish sufficient conditions that ensure the existence of multiple positive solutions for system (1.1).

Fractional calculus has been applied to various areas of engineering, physics, chemistry, etc.; it is due to the fact that fractional derivatives provide power tools for describing memory and hereditary characteristics. There are many papers and monographs that deal with all kinds of problems in fractional calculus (see [1–6]).

It is well known that the p-Laplacian operator is also used in analyzing dynamic systems, physics, mechanics, and the related fields of mathematical modeling. For studying the turbulent flow problem in a porous medium, Leibenson [7] introduced the model of a differential equation with the p-Laplacian operator. Since then, p-Laplacian differential equations have been widely applied in different fields of physics and natural phenomena, see [8–11] and the references therein. In recent years, the topic of fractional-order boundary value problems with the p-Laplacian operator has been intensively studied by several researchers, many important results related to the boundary value problems of fractional p-Laplacian equations have been obtained. We refer the reader to [12–14].

In [15], the author is concerned with the existence and uniqueness of positive solutions for the following boundary value problem with p-Laplacian operator:

$$\begin{cases} (\varphi_p(D_{0+}^{\alpha}u(t)))' + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = D_{0+}^{\alpha}u(0) = 0, \qquad {}^{C}D_{0}^{\beta}u(0) = {}^{C}D_{0}^{\beta}u(1) = 0, \end{cases} \tag{1.2}$$

where $0 < \beta \leq 1$, $2 < \alpha \leq 2 + \beta$ are real numbers, D_{0+}^{α} and ${}^{C}D_{0}^{\beta}$ are the Riemann–Liouville fractional derivative and Caputo fractional derivative of order α, β, respectively, $p > 1$, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By using the Banach contraction mapping principle and the Guo–Krasnosel'skii fixed point theorem, respectively, three theorems on the existence and uniqueness of nontrivial positive solutions for fractional boundary value problems (FBVP) (1.2) were obtained.

Chen *et al.* [12] investigated Caputo fractional differential equation boundary value problems with p-Laplacian operator at resonance:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}x(t))) = f(t,x(t),D_{0+}^{\alpha}x(t)), \quad t \in [0,1]; \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0, \end{cases} \tag{1.3}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, D_{0+}^{α} and D_{0+}^{β} are standard Caputo fractional derivatives. The existence of solutions for boundary value problem (1.3) was obtained by means of the coincidence degree theory.

Lu *et al.* [16] studied the following Riemann–Liouville fractional differential equations boundary problems with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) = f(t,u(t)), \quad 0 \leq t \leq 1; \\ u(0) = u'(0) = u'(1) = 0; \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \end{cases} \tag{1.4}$$

where $2 < \alpha \leq 3$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, D_{0+}^{α}, D_{0+}^{β} are standard Riemann–Liouville fractional derivatives. By using the Guo–Krasnosel'skii fixed point theorem and upper-lower solutions method, some new results on the existence of positive solutions were obtained.

Recently, in [17], we studied the high-order conformable differential equations with p-Laplacian operator as follows:

$$\begin{cases} T_\alpha^{0+}(\varphi_p(T_\alpha^{0+}u(t))) = f(t,u(t),T_\alpha^{0+}u(t)), \\ u^{(i)}(0) = 0, \qquad [\varphi_p(T_\alpha^{0+}u)]^{(i)}(0) = 0, \\ [T_\beta^{0+}u(t)]_{t=1} = 0, \qquad [T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1} = 0, \end{cases} \tag{1.5}$$

where $n-1 \le \alpha < n$ and $T_\alpha^{0^+}$ is a new fractional derivative called "the conformable fractional derivative". By means of the Guo–Krasnosel'skii fixed point theorem, we established sufficient conditions that ensure the existence of positive solutions to boundary value problem (1.5).

However, in [17], we only obtained the existence of a positive solution for system (1.5). Similarly, in [16] and [12], the authors only got the existence and uniqueness of a nontrivial solution to systems (1.4) and (1.3), respectively. But so far, we have not investigated the multiplicity of positive solutions for fractional p-Laplacian differential equations. To the best of our knowledge, there are few studies that consider the existence of multiple positive solutions on nonlinear Riemann–Liouville high-order fractional differential equations, especially with the p-Laplacian operator. Motivated greatly by the above mentioned excellent works and in order to fill this gap in the literature, in this paper, we investigate the multiple positive solutions of boundary value problems for high-order Riemann–Liouville fractional differential equations with p-Laplacian operator.

The rest of this paper is organized as follows. In Sect. 2, we briefly introduce some necessary basic definitions and preliminary results which are used to prove our main results. In Sect. 3, by means of the properties of the Green's function, Leggett–Williams fixed point theorem, and functional-type cone expansion-compression fixed point theorem, we investigate multiple positive solutions for boundary value problem of Riemann–Liouville fractional differential p-Laplacian equation systems on $n-1 < \alpha \le n$, our work establishes some novel results on a nonlinear Riemann–Liouville fractional-order boundary value problem. Finally, in Sect. 4, we demonstrate the effectiveness of the main results by one example.

2 Preliminaries

For the convenience of the reader, we give some background material from cone theory and fractional calculus to facilitate the analysis of FBVP (1.1).

A nonempty closed convex set $P \subset E$ is a cone if it satisfies:

(I_1) $x \in P$, $\lambda \ge 0 \Rightarrow \lambda x \in P$;

(I_2) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, then we denote $x < y$ or $y > x$. θ denotes the zero element of E. In addition, if $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 < x < x_2\}$ is called the order interval between x_1 and x_2.

Definition 2.1 ([1]) Putting $\mathring{P} := \{x \in P \mid x \text{ is interior point of } P\}$, P is said to be a solid cone if its interior $\mathring{P}$ is nonempty. Moreover, P is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le M\|y\|$; in this case M is called the normality constant of P.

Definition 2.2 ([16]) The fractional integral of order $\alpha > 0$ of a function $y : [0, +\infty) \to \mathbb{R}$ is given by

$$ {}_0I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, $$

provided the right-hand side is pointwise defined on $[0, +\infty)$.

Definition 2.3 ([12]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : [0, +\infty) \to \mathbb{R}$ is given by

$$ {}_0^RD_t^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds, $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided the right-hand side is pointwise defined on $[0, +\infty)$.

Definition 2.4 ([18]) Let $p > 1$, the p-Laplacian operator is given by

$$ \varphi_p(x) = |x|^{p-2} x. $$

Obviously, φ_p is continuous, increasing, invertible and its inverse operator is φ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1 ([12])

(1) *Let* $h(t) \in C[0,1] \cap L^1[0,1]$, $\alpha > 0$, *then*

$$ {}_0I_t^\alpha {}_0^RD_t^\alpha h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, $$

where $c_i \in R$, $i = 1, 2, 3, \ldots, n$ $(n = [\alpha] + 1)$.

(2) *If* $u \in L^1(0,1)$, $\alpha > \beta > 0$, *then*

$$ {}_0I_t^\alpha {}_0I_t^\beta u(t) = {}_0I_t^{\alpha+\beta} u(t), \qquad {}_0^RD_t^\beta {}_0I_t^\alpha u(t) = {}_0I_t^{\alpha-\beta}, \qquad {}_0^RD_t^\beta {}_0I_t^\beta u(t) = u(t). $$

(3) *If* $\rho > 0$, $\mu > 0$, *then*

$$ {}_0^RD_t^\rho t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} t^{\mu-\rho-1}. $$

Lemma 2.2 *Let* g *be a continuous function on* $(0,1)$. *Then the Riemann–Liouville fractional boundary value problem*

$$ \begin{cases} {}_0^RD_t^\alpha u(t) + g(t) = 0, \quad t \in (0,1), n-1 < \alpha \le n; \\ u^{(i)}(0) = 0, \quad i = 0, 1, 2, \ldots, n-2; \\ [{}_0^RD_t^\beta u(t)]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1; \end{cases} \tag{2.1} $$

has a unique positive solution

$$ u(t) = \int_0^1 G(t,s) g(s)\, ds, \tag{2.2} $$

where

$$G(t,s)=\begin{cases}\frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1;\\ \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1;\end{cases}\tag{2.3}$$

is the Green's function for this problem.

Proof For boundary value problems (2.1), by using Lemma 2.1, we can get

$$\begin{aligned}{}_0I_0^{\alpha}{}_0^RD_t^{\alpha}u(t)&=u(t)+c_1t^{\alpha-1}+c_2^{\alpha-2}+\cdots+c_nt^{\alpha-n}\\&=-{}_0I_0^{\alpha}g(t)\\&=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\,ds.\end{aligned}$$

From $u^{(i)}(0)=0$, $(i=0,1,\dots,n-2)$, it is easy to know $c_n=c_{n-1}=\cdots=c_2=0$. So, we obtain

$$u(t)=-c_1t^{\alpha-1}-{}_0I_0^{\alpha}g(t)=-c_1t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\,ds.$$

For $0<\beta\le\alpha-1$, applying the Riemann–Liouville fractional derivative operator ${}_0^RD_t^{\beta}$ on both sides of the above equation, we can know

$$\begin{aligned}{}_0^RD_t^{\beta}u(t)&=-{}_0^RD_t^{\beta}\bigl(c_1t^{\alpha-1}\bigr)-{}_0^RD_t^{\beta}\bigl({}_0I_t^{\alpha}g(t)\bigr)\\&=-c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}-{}_0^RI_t^{\alpha-\beta}g(t)\\&=-c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}-\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}g(s)\,ds.\end{aligned}$$

Setting $t=1$ in the above equation, by the condition $[{}_0^RD_t^{\beta}u(t)]_{t=1}=0$, we can know

$$c_1=-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}g(s)\,ds.$$

From the above equations, we can get

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}t^{\alpha-1}g(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\,ds\\&=\int_0^1G(t,s)g(s)\,ds,\end{aligned}$$

where

$$G(t,s)=\begin{cases}\frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1;\\ \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1;\end{cases}$$

is the Green's function of FBVP (2.1). □

Lemma 2.3 *For* $(t,s) \in (0,1) \times (0,1)$, *Green's function* (2.3) *has the following properties*:

(1) $G(t,s)$ *is a continuous function*;

(2) $G(t,s) \geq 0$;

(3) $G(t,s) \leq \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}$;

(4) $G(t,s) \geq \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1}$.

Proof It is evident that $G(t,s)$ is a continuous function and inequality (3) holds. So, we only need to prove inequality (4) and $G(t,s) \geq 0$.

If $0 \leq s \leq t \leq 1$, then we have $0 \leq t-s \leq t-ts = (1-s)t$, and thus $(t-s)^{\alpha-1} \leq (1-s)^{\alpha-1}t^{\alpha-1}$. Hence, if $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G(t,s) &= \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1}. \end{aligned}$$

If $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned} G(t,s) &= \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} \\ &\geq \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1}. \end{aligned}$$

In addition, for $\forall s \in (0,1)$, from $0 < \beta \leq \alpha - 1$ and $n-1 \leq \alpha \leq n$, we get $(1-s)^{\alpha-\beta-1} \geq (1-s)^{\alpha-1}$. In other words, for $\forall (t,s) \in (0,1) \times (0,1)$, we obtain

$$\begin{aligned} G(t,s) &\geq \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1} \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\bigl[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\bigr] \\ &\geq 0. \end{aligned}$$

Therefore, the proof is done. □

Lemma 2.4 ([19]) *Let* E *be an ordered Banach space*, $P \subset E$ *be a cone, and suppose that* Ω_1, Ω_2, Ω_3 *are bounded open subsets of* E *with* $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $\overline{\Omega}_2 \subset \Omega_3$, *and let* $\Phi : P \cap (\overline{\Omega}_3 \setminus \Omega_1) \to P$ *be a completely continuous operator such that*

(A_1) $\|\Phi u\| \geq \|u\|$ *for* $\forall u \in P \cap \partial\Omega_1$;

(A_2) $\|\Phi u\| \leq \|u\|$, $\Phi u \neq u$ *for* $\forall u \in P \cap \partial\Omega_2$;

(A_3) $\|\Phi u\| \geq \|u\|$ *for* $\forall u \in P \cap \partial\Omega_3$.

Then Φ *has at least two fixed points* u_1^*, u_2^* *such that* $u_1^* \in P \cap \overline{\Omega}_2 \setminus \Omega_1$ *and* $u_2^* \in P \cap \overline{\Omega}_3 \setminus \Omega_2$.

Let $0 < a < b$ be given, and let γ be a nonnegative continuous concave functional on P. Define the convex sets P_r and $P(\gamma,a,b)$ by $P_r = \{x \in P \mid \|x\| < r\}$ and $P(\gamma,a,b) = \{x \in P \mid \gamma(x) \geq a, \|x\| \leq b\}$.

Proposition 2.5 ([20]) *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator, and let γ be a nonnegative continuous concave functional on P such that $\gamma(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose that there exist $0 < a < b < d \le c$ such that*

(B_1) $\{x \in P(\gamma, b, d) \mid \gamma(x) > b\} \neq \phi$ *and* $\gamma(Ax) > b$ *for* $x \in P(\gamma, b, d)$;

(B_2) $\|Ax\| \le a$ *for* $\|x\| \le a$;

(B_3) $\gamma(Ax) > b$ *for* $x \in P(\gamma, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, and x_3 such that $\|x_1\| < a$, $\gamma(x_2) > b$, and $\|x_3\| > a$ with $\gamma(x_3) < b$.

3 Main results

Lemma 3.1 *If $g \in C[0,1]$ is given, then the Riemann–Liouville fractional boundary value problem*

$$\begin{cases} {}_0^R D_t^\alpha(\varphi_p({}_0^R D_t^\alpha u(t))) = g(t), \quad 0 \le t \le 1, n-1 < \alpha \le n; \\ u^{(i)}(0) = 0, \qquad [\varphi_p({}_0^R D_t^\alpha u)]^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2; \\ [{}_0^R D_t^\beta u(t)]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1; \\ [{}_0^R D_t^\beta(\varphi_p({}_0^R D_t^\alpha u(t)))]_{t=1} = 0 \end{cases} \tag{3.1}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g(\tau)\,d\tau\right) ds, \tag{3.2}$$

where $G(t,s)$ is given in (2.3).

Proof Applying the fractional integral operator ${}_0I_t^\alpha$ on both sides of the first equation of (3.1), we have

$$\begin{aligned} {}_0I_t^\alpha {}_0^R D_t^\alpha\bigl(\varphi_p\bigl({}_0^R D_t^\alpha u(t)\bigr)\bigr) &= \varphi_p\bigl({}_0^R D_t^\alpha u(t)\bigr) + d_1 t^{\alpha-1} + d_2^{\alpha-2} + \cdots + d_n t^{\alpha-n} \\ &= {}_0I_t^\alpha g(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds. \end{aligned}$$

From the boundary value conditions $[\varphi_p({}_0^R D_t^\alpha u)]^{(i)}(0) = 0$ ($i = 0,1,2,\dots,n-2$), we can know that $d_n = d_{n-1} = \cdots = d_3 = d_2 = 0$. So, we obtain

$$\varphi_p\bigl({}_0^R D_t^\alpha u(t)\bigr) = -d_1 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds.$$

For $0 < \beta \le \alpha - 1$, applying the fractional derivative operator ${}_0^R D_t^\beta$ on both sides of the equation above, we have

$$\begin{aligned} {}_0^R D_t^\beta\bigl(\varphi_p\bigl({}_0^R D_t^\alpha u(t)\bigr)\bigr) &= {}_0^R D_t^\beta\bigl(-d_1 t^{\alpha-1}\bigr) + {}_0^R D_t^\beta {}_0I_t^\alpha g(t) \\ &= -d_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + {}_0^R I_t^{\alpha-\beta} g(t) \\ &= -d_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)\,ds. \end{aligned}$$

Letting $t=1$, by the condition $[{}_0^RD_t^\beta(\varphi_p({}_0^RD_t^\alpha u(t)))]_{t=1}=0$, we can get

$$d_1=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}g(s)\,ds.$$

Furthermore, we can obtain

$$\begin{aligned}\varphi_p\bigl({}_0^RD_t^\alpha u(t)\bigr)&=-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}t^{\alpha-1}g(s)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)\,ds\\&=-\int_0^1G(t,s)g(s)\,ds.\end{aligned}$$

By using the Laplacian operator φ_q on both sides of the equation above, we get

$$ {}_0^RD_t^\alpha u(t)+\varphi_q\left(\int_0^1G(t,s)g(s)\,ds\right)=0. \tag{3.3}$$

In addition, setting $\widetilde{g}(t)=\varphi_q(\int_0^1G(t,s)g(s)\,ds)$, thus, the high-order Riemann–Liouville fractional differential systems with p-Laplacian operator (3.1) is equivalent to the problem

$$\begin{cases}{}_0^RD_t^\alpha u(t)+\widetilde{g}(t)=0, & t\in(0,1), n-1<\alpha\le n;\\ u^{(i)}(0)=0, & 0\le i\le n-2;\\ [{}_0^RD_t^\beta u(t)]_{t=1}=0, & 0<\beta\le\alpha-1.\end{cases} \tag{3.4}$$

Applying Lemma 2.2, we know that the Riemann–Liouville fractional differential system (3.4) has a unique integral solution

$$\begin{aligned}u(t)&=\int_0^1G(t,s)\widetilde{g}(s)\,ds\\&=\int_0^1G(t,s)\varphi_q\left(\int_0^1G(s,\tau)g(\tau)\,d\tau\right)ds.\end{aligned} \tag{3.5}$$

Moreover, by (3.3), we can easily know that

$$ {}_0^RD_t^\alpha u(t)=-\varphi_q\left(\int_0^1G(t,s)g(s)\,ds\right). \tag{3.6}$$

This constitutes the complete proof. □

Now, we denote that $E=C^\alpha[0,1]:=\{u\mid u\in C[0,1],{}_0^RD_t^\alpha u\in C[0,1]\}$ and endowed with the norm $\|u\|_\alpha=\max\{\|u\|_\infty,\|{}_0^RD_t^\alpha u\|_\infty\}$, where $\|u\|_\infty=\max_{0\le t\le1}|u(t)|$ and $\|{}_0^RD_t^\alpha u\|_\infty=\max_{0\le t\le1}|{}_0^RD_t^\alpha u(t)|$. Then $(E,\|\cdot\|_\alpha)$ is a Banach space. Let $P=\{u\in E\mid u(t)\ge0,{}_0^RD_t^\alpha u(t)\le0\}$. Then P is a cone on the space E.

In addition, we define the operator $\Phi:P\to E$ by

$$(\Phi u)(t)=\int_0^1G(t,s)\varphi_q\left(\int_0^1G(s,\tau)f\bigl(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\bigr)\,d\tau\right)ds, \tag{3.7}$$

and for any $u \in E$, it is easy to show that $\Phi u \in E$ and

$$ {}_0^R D_t^\alpha (\Phi u)(t) = -\varphi_q\left(\int_0^1 G(t,s) f\left(s, u(s), {}_0^R D_t^\alpha u(s)\right) ds\right). \tag{3.8}$$

Obviously, the function u is a positive solution of boundary value problem (1.1) if and only if u is a fixed point of the operator Φ in P.

Lemma 3.2 *Suppose that $f \in C([0,1] \times [0,+\infty) \times [-\infty,0),[0,+\infty))$. Then $\Phi : P \to P$ is a completely continuous operator.*

Proof Firstly, for $\forall u \in P$, by (3.7) and Lemma 2.3, it is easy to know that $\Phi : P \to P$. Let $\{u_j\} \subset P$ and $\lim_{j\to\infty} u_j = u \in P$, so there exists a constant $\gamma_0 > 0$ such that $\|u_j\|_\alpha \le \gamma_0$ and $\|u\|_\alpha \le \gamma_0$ for $j = 1,2,\ldots$.

Letting $M_0 = \max_{(t,u,v)\in[0,1]\times[-\gamma_0,\gamma_0]\times[-\gamma_0,\gamma_0]} f(t,u,v)$, for $(t,u,v) \in [0,1] \times [-\gamma_0,\gamma_0] \times [-\gamma_0,\gamma_0]$, we can get $0 \le f(t,u,v) \le M_0$ and

$$\lim_{j\to\infty} f\left(t, u_j, {}_0^R D_t^\alpha u_j(t)\right) = f\left(t, u, {}_0^R D_t^\alpha u(t)\right), \quad \text{for } t \in [0,1].$$

In addition, for $\forall(t,s) \in (0,1) \times (0,1)$, from Lemma 2.3 we can get that

$$\frac{(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} t^{\alpha-1} \le G(t,s) \le \frac{(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha)}.$$

Then, from the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{j\to\infty} (\Phi u_j)(t) &= \lim_{j\to\infty} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) f\left(\tau, u_j(\tau), {}_0^R D_t^\alpha u_j(\tau)\right) d\tau\right) ds \\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 \lim_{j\to\infty} G(s,\tau) f\left(\tau, u_j(\tau), {}_0^R D_t^\alpha u_j(\tau)\right) d\tau\right) ds \\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) f\left(\tau, u(\tau), {}_0^R D_t^\alpha u(\tau)\right) d\tau\right) ds \\
&= (\Phi u)(t)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\lim_{j\to\infty} {}_0^R D_t^\alpha (\Phi u_j)(t) &= -\varphi_q\left(\lim_{j\to\infty} \int_0^1 G(t,s) f\left(\tau, u_j(s), {}_0^R D_t^\alpha u_j(s)\right) ds\right) \\
&= -\varphi_q\left(\int_0^1 G(t,s) f\left(\tau, u(s), {}_0^R D_t^\alpha u(s)\right) ds\right) \\
&= {}_0^R D_t^\alpha (\Phi u)(t).
\end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) imply that $\lim_{j\to\infty} (\Phi u_j)(t) = (\Phi u)(t)$ uniformly on $[0,1]$. Hence, Φ is continuous.

Secondly, let $A \subset P$ be any bounded set. Then there exists a constant $\gamma_1 > 0$ such that $\|u\|_\alpha \le \gamma_1$ for each $u \in A$, which implies that $|u(t)| \le \gamma_1$ and $|{}_0^R D_t^\alpha u(t)| \le \gamma_1$ for $t \in [0,1]$.

Because f is continuous, there exists $M_1 > 0$ such that $0 \le f(t,u(t),{}_0^RD_t^\alpha u(t)) \le M_1$ for $t \in [0,1]$. Let $L = \frac{M_1}{\Gamma(\alpha)(\alpha-\beta)}$. Then

$$\begin{aligned}
0 \le \left|(\Phi u)(t)\right| &= \left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\right)d\tau\right)ds\right| \\
&\le \int_0^1 G(t,s)\varphi_q\left(\int_0^1 \frac{(1-\tau)^{\alpha-\beta-1}}{\Gamma(\alpha)}M_1\,d\tau\right)ds \\
&\le \varphi_q\left(\frac{M_1}{\Gamma(\alpha)(\alpha-\beta)}\right)\int_0^1 G(t,s)\,ds \\
&\le \frac{1}{\Gamma(\alpha)(\alpha-\beta)}\varphi_q(L) \\
&= \frac{L}{M_1}\varphi_q(L)
\end{aligned}$$

and

$$\begin{aligned}
0 \le \left|{}_0^RD_t^\alpha(\Phi u)(t)\right| &= \left|-\varphi_q\left(\int_0^1 G(t,s)f\left(s,u(s),{}_0^RD_t^\alpha u(s)\right)ds\right)\right| \\
&\le \varphi_q\left(\frac{M_1}{\Gamma(\alpha)(\alpha-\beta)}\right) \\
&= \varphi_q(L),
\end{aligned} \tag{3.11}$$

which implies that $\Phi(A)$ is uniformly bounded in P.

Finally, because $G(t,s)$ is continuous on $[0,1]\times[0,1]$, then $G(t,s)$ is uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, whenever $t_1,t_2 \in [0,1]$ and $|t_2 - t_1| < \delta_1$,

$$\left|G(t_2,s) - G(t_1,s)\right| < \frac{\varepsilon}{\varphi_q(L)+1}.$$

Furthermore, for any $u \in P$, we have

$$\begin{aligned}
&\left|(\Phi u)(t_2) - (\Phi u)(t_1)\right| \\
&\quad= \left|\int_0^1 G(t_2,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\right)d\tau\right)ds\right. \\
&\qquad\left.-\int_0^1 G(t_1,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\right)d\tau\right)ds\right| \\
&\quad\le \int_0^1 \left|G(t_2,s)-G(t_1,s)\right|\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\right)d\tau\right)ds \\
&\quad\le \varphi_q(L)\int_0^1 \left|G(t_2,s)-G(t_1,s)\right|ds \\
&\quad< \varepsilon.
\end{aligned}$$

In addition, setting $(Fu)(t) = \int_0^1 G(t,s)f(s,u(s),{}_0^RD_t^\alpha u(s))\,ds$, so we have

$$0 \le (Fu)(t) = \int_0^1 G(t,s)f\left(s,u(s),{}_0^RD_t^\alpha u(s)\right)ds \le L.$$

Because $\varphi_q(x)$ is continuous on $[0,L]$, we can get $\varphi_q(x)$ is uniformly continuous on $[0,L]$. For $\varepsilon > 0$ above, there exists $\eta > 0$ such that

$$\left|\varphi_q(x_2) - \varphi_q(x_1)\right| < \varepsilon, \quad \text{whenever } x_1, x_2 \in [0,L] \text{ and } |x_2 - x_1| < \eta. \tag{3.12}$$

In view of that $G(t,s)$ is uniformly continuous, for $\eta > 0$, there exists $\delta_2 > 0$, whenever $t_1, t_2 \in [0,1]$, $s \in [0,1]$ and $|t_2 - t_1| < \delta_2$, we have $|G(t_2,s) - G(t_1,s)| < \frac{\eta}{M+1}$. Furthermore,

$$\begin{aligned}
\left|(Fu)(t_2) - (Fu)(t_1)\right| &= \left|\int_0^1 G(t_2,s) f\left(s,u(s),{}_0^R D_t^\alpha u(s)\right) ds\right. \\
&\quad \left. - \int_0^1 G(t_1,s) f\left(s,u(s),{}_0^R D_t^\alpha u(s)\right) ds\right| \\
&\le \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| f\left(s,u(s),{}_0^R D_t^\alpha u(s)\right) ds \\
&\le M_1 \frac{\eta}{M_1+1} < \eta.
\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), it is easy to see that

$$\begin{aligned}
\left|\left({}_0^R D_t^\alpha \Phi u\right)(t_2) - \left({}_0^R D_t^\alpha \Phi u\right)(t_1)\right| &= \left|\varphi_q\left(\int_0^1 G(t_2,s) f\left(s,u(s),{}_0^R D_t^\alpha u(s)\right) ds\right)\right. \\
&\quad \left. - \varphi_q\left(\int_0^1 G(t_1,s) f\left(s,u(s),{}_0^R D_t^\alpha u(s)\right) ds\right)\right| \\
&= \left|\varphi_q\left(Fu(t_2)\right) - \varphi_q\left(Fu(t_1)\right)\right| \\
&< \varepsilon.
\end{aligned}$$

Thus, $\Phi(A)$ is equicontinuous. By Arzela–Ascoli theorem, we can show that Φ is completely continuous. □

For convenience, we introduce the following notations:

$$\begin{aligned}
A^{-1} &= \max\left\{\max_{0\le t\le 1} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds,\right. \\
&\qquad \left.\max_{0\le t\le 1} \varphi_q\left(\int_0^1 G(t,s)\,ds\right)\right\}, \\
B^{-1} &= \min\left\{\max_{0\le t\le 1} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \left((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\right) \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds,\right. \\
&\qquad \left.\max_{0\le t\le 1} \varphi_q\left(\int_0^1 G(t,s)\,ds\right)\right\}, \\
C^{-1} &= \min_{0\le t\le 1} \int_0^1 \left(\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)} t^{\alpha-1}\right) \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds, \\
\phi(l) &= \max\left\{f(t,u,v), (t,u,v) \in [0,1] \times [0,l] \times [-l,0]\right\}, \\
\psi(l) &= \min\left\{f(t,u,v), (t,u,v) \in [0,1] \times [0,l] \times [-l,0]\right\}.
\end{aligned}$$

Theorem 3.1 *Assume that the following assumptions hold*:

(H_1) $f \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$;

(H_2) *There exist three positive constants* $a < b < c$ *such that* $\psi(a) \geq \varphi_p(aB), \phi(b) \leq \varphi_p(bA)$, *and* $\psi(c) \geq \varphi_p(cB)$ *for any* $t \in [0,1]$.

Then problem (1.1) *has at least two positive solutions* $u_1^*, u_2^* \in P$ *such that* $a \leq \|u_1^*\| \leq b$ *and* $b \leq \|u_2^*\| \leq c$.

Proof We know that $\Phi : P \to P$ is completely continuous by Lemma 3.2, we only need to consider the existence of a fixed point of operator Φ in P. Now, we divide the proof into the following three steps.

Step 1. Let $\Omega_a := \{u \in P \mid \|u\|_\alpha < a\}$. For any $u \in \partial\Omega_a$, we have $\|u\|_\alpha = a$ and $f(t,u(t), {}_0^RD_t^\alpha u(t)) \geq \psi(a) \geq \varphi_p(aB) \geq 0$ for $(t,u,v) \in [0,1] \times [0,a] \times [-a,0]$. Hence, we can know

$$\begin{aligned}
\|\Phi u\|_\infty &= \max_{0\leq t\leq 1}\bigl|(\Phi u)(t)\bigr| = \max_{0\leq t\leq 1}(\Phi u)(t) \\
&= \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\geq \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)\varphi_p(aB)\,d\tau\biggr)\,ds \\
&\geq aB\max_{0\leq t\leq 1}\int_0^1\biggl(\frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1}\biggr)\varphi_q\biggl(\int_0^1 G(s,\tau)\,d\tau\biggr)\,ds \\
&\geq a
\end{aligned}$$

and

$$\begin{aligned}
\bigl\|{}_0^RD_t^\alpha \Phi u\bigr\|_\infty &= \max_{0\leq t\leq 1}\bigl|\bigl({}_0^RD_t^\alpha \Phi u\bigr)(t)\bigr| \\
&= \max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)f\bigl(\tau,u(s),{}_0^RD_t^\alpha u(s)\bigr)\,ds\biggr) \\
&\geq \max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)\varphi_p(aB)\,ds\biggr) \\
&= aB\max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)\,ds\biggr) \\
&\geq a.
\end{aligned}$$

So

$$\|\Phi u\|_\alpha \geq \|u\|_\alpha, \quad \forall u \in \partial\Omega_a.$$

Step 2. Let $\Omega_b := \{u \in P \mid \|u\|_\alpha < b\}$. For any $u \in \partial\Omega_b$, we have $\|u\|_\alpha = b$ and $f(t,u(t), {}_0^RD_t^\alpha u(t)) \leq \phi(b) \leq \varphi_p(bA)$ for $(t,u,v) \in [0,1] \times [0,b] \times [-b,0]$. So, we get

$$\begin{aligned}
\|\Phi u\|_\infty &= \max_{0\leq t\leq 1}\bigl|(\Phi u)(t)\bigr| = \max_{0\leq t\leq 1}(\Phi u)(t) \\
&= \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),{}_0^RD_t^\alpha u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)\varphi_p(bA)\,d\tau\biggr)\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq bA \max_{0\leq t\leq 1} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\
&\leq b
\end{aligned}$$

and

$$\begin{aligned}
\left\|{}_0^R D_t^\alpha \Phi u\right\|_\infty &= \max_{0\leq t\leq 1}\left|\left({}_0^R D_t^\alpha \Phi u\right)(t)\right| \\
&= \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s) f\left(\tau, u(s), {}_0^R D_t^\alpha u(s)\right) ds\right) \\
&\leq \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s)\varphi_p(bA)\,ds\right) \\
&= bA \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s)\,ds\right) \\
&\leq b.
\end{aligned}$$

So

$$\|\Phi u\|_\alpha \leq \|u\|_\alpha, \quad \forall u \in \partial\Omega_a.$$

Step 3. Let $\Omega_c := \{u \in P \mid \|u\|_\alpha < c\}$, if $u \in \partial\Omega_c$, we have $\|u\|_\alpha = c$ and $f(t, u(t), {}_0^R D_t^\alpha u(t)) \geq \psi(c) \geq \varphi_p(cB) \geq 0$ for $(t,u,v) \in [0,1] \times [0,c] \times [-c,0]$. Then we have

$$\begin{aligned}
\|\Phi u\|_\infty &= \max_{0\leq t\leq 1}\left|(\Phi u)(t)\right| = \max_{0\leq t\leq 1}(\Phi u)(t) \\
&= \max_{0\leq t\leq 1} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau) f\left(\tau, u(\tau), {}_0^R D_t^\alpha u(\tau)\right) d\tau\right) ds \\
&\geq \max_{0\leq t\leq 1} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\varphi_p(cB)\,d\tau\right) ds \\
&\geq cB \max_{0\leq t\leq 1} \int_0^1 \left(\frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} t^{\alpha-1}\right)\varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\
&\geq c
\end{aligned}$$

and

$$\begin{aligned}
\left\|{}_0^R D_t^\alpha \Phi u\right\|_\infty &= \max_{0\leq t\leq 1}\left|\left({}_0^R D_t^\alpha \Phi u\right)(t)\right| \\
&= \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s) f\left(\tau, u(s), {}_0^R D_t^\alpha u(s)\right) ds\right) \\
&\geq \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s)\varphi_p(cB)\,ds\right) \\
&= cB \max_{0\leq t\leq 1} \varphi_q\left(\int_0^1 G(t,s)\,ds\right) \\
&\geq c.
\end{aligned}$$

So

$$\|\Phi u\|_\alpha \geq \|u\|_\alpha, \quad \forall u \in \partial\Omega_c.$$

By Lemma 2.4, Φ has at least two fixed points u_1^*, u_2^* in $P \cap \overline{\Omega}_c \backslash \Omega_a$, i.e., system (1.1) has at least two positive solutions u_1^*, u_2^* such that $a \leq \|u_1^*\| \leq b$ and $b \leq \|u_2^*\| \leq c$. □

Now, we define a nonnegative continuous concave function on a cone P by $\gamma(u) = \min_{0\leq t\leq 1}(u(t))$. It is obvious that, for each $u \in P$, $\gamma(u) \leq \|u\|_\alpha$.

Theorem 3.2 *Assume that condition* (H_1) *holds and that there exist nonnegative numbers* a, b, c, *and* $0 < \theta < 1$ *such that* $0 < a < b \leq \frac{b}{\theta} \leq c$, *and* $\gamma(u) \geq \theta\|u\|_\alpha$ *for* $\forall u \in P_c$. *In addition, the following assumptions hold*:

(H_3) $f(t, u(t), {}_0^R D_t^\alpha u(t)) \leq \varphi_p(cA)$ *for* $(t, u, v) \in [0, 1] \times [0, c] \times [-c, 0]$;

(H_4) $f(t, u(t), {}_0^R D_t^\alpha u(t)) \leq \varphi_p(aA)$ *for* $(t, u, v) \in [0, 1] \times [0, a] \times [-a, 0]$;

(H_5) $f(t, u(t), {}_0^R D_t^\alpha u(t)) \geq \varphi_p(bC)$ *for* $(t, u, v) \in [0, 1] \times [b, \frac{b}{\theta}] \times [-\frac{b}{\theta}, -b]$.

Then problem (1.1) *has at least three positive solutions* u_1^*, u_2^*, *and* u_3^* *such that* $\|u_1^*\|_\alpha \leq a$, $\gamma(u_2^*) \geq b$, *and* $\|u_3^*\|_\alpha \geq a$ *with* $\gamma(u_3^*) < b$.

Proof Firstly, we show that $\Phi : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. In fact, if $u \in \overline{P_c}$, by condition (H_3), we get

$$\begin{aligned}
\|\Phi u\|_\infty &= \max_{0\leq t\leq 1}\bigl|(\Phi u)(t)\bigr| = \max_{0\leq t\leq 1}(\Phi u)(t) \\
&= \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau, u(\tau), {}_0^R D_t^\alpha u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\leq \max_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)\varphi_p(cA)\,d\tau\biggr)\,ds \\
&\leq cA\max_{0\leq t\leq 1}\int_0^1 \frac{(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha)}\varphi_q\biggl(\int_0^1 G(s,\tau)\,d\tau\biggr)\,ds \\
&\leq c
\end{aligned}$$

and

$$\begin{aligned}
\bigl\|{}_0^R D_t^\alpha \Phi u\bigr\|_\infty &= \max_{0\leq t\leq 1}\bigl|\bigl({}_0^R D_t^\alpha \Phi u\bigr)(t)\bigr| = \max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)f\bigl(\tau, u(s), {}_0^R D_t^\alpha u(s)\bigr)\,ds\biggr) \\
&\leq \max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)\varphi_p(cA)\,ds\biggr) \\
&= cA\max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(\tau,s)\,ds\biggr) \\
&\leq c.
\end{aligned}$$

Therefore, $\|\Phi u\|_\alpha \leq c$, that is, $\Phi : \overline{P_c} \to \overline{P_c}$. The operator Φ is completely continuous by an application of the Ascoli–Arzela theorem.

In a completely analogous way, condition (H_4) implies that condition (B_2) of Proposition 2.5 is satisfied.

Secondly, we show that condition (B_1) of Proposition 2.5 is satisfied. It is clear that $\{u \in P(\gamma, b, \frac{b}{\theta}) \mid \gamma(u) \geq b\} \neq \phi$. If $u \in P(\gamma, b, \frac{b}{\theta})$, then $b \leq u(t) \leq \frac{b}{\theta}$ for $t \in (0,1)$. By condition (H_5), we have

$$
\begin{aligned}
\gamma\bigl((\Phi u)(t)\bigr) &= \min_{0\leq t\leq 1}\bigl((\Phi u)(t)\bigr) \\
&= \min_{0\leq t\leq 1}\biggl(\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),{}_0^RD_t^{\alpha}u(\tau)\bigr)\,d\tau\biggr)\,ds\biggr) \\
&\geq \min_{0\leq t\leq 1}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)\varphi_p(bC)\,d\tau\biggr)\,ds \\
&\geq bC\min_{0\leq t\leq 1}\int_0^1\biggl(\frac{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}t^{\alpha-1}\biggr)\varphi_q\biggl(\int_0^1 G(s,\tau)\,d\tau\biggr)\,ds \\
&\geq b.
\end{aligned}
$$

Therefore, condition (B_1) of Proposition 2.5 is satisfied.

Finally, we show that condition (B_3) of Proposition 2.5 is also satisfied. If $u \in P(\gamma, b, c)$ and $\|\Phi u\|_\alpha > \frac{b}{\theta}$, then we get $\gamma((\Phi u)(t)) = \min_{0\leq t\leq 1}(\Phi u)(t) \geq \theta\|\Phi u\|_\alpha > b$. Therefore, condition (B_3) of Proposition 2.5 also holds. By Proposition 2.5, there exist three positive solutions u_1^*, u_2^*, and u_3^* such that $\|u_1^*\|_\alpha < a$, $\gamma(u_2^*) > b$, and $\|u_3^*\| > a$ with $\gamma(u_3^*) < b$. So we get the conclusion. □

4 Applications

Example 4.1

$$
\begin{cases}
{}_0^RD_t^{1.5}(\varphi_2({}_0^RD_t^{1.5}u(t))) = (\frac{3}{2})^u + \frac{t^2}{8}, \quad 0 \leq t \leq 1, \\
u(0) = [\varphi_2({}_0^RD_t^{1.5}u)](0) = 0, \\
[{}_0^RD_t^{0.5}u](1) = {}_0^RD_t^{0.5}(\varphi_2({}_0^RD_t^{1.5}u))(1) = 0.
\end{cases}
\tag{4.1}
$$

In system (4.1), we see that $\alpha = 1.5$, $\beta = 0.5$, $p = 2$, $q = 2$, $n = 2$. Moreover, we can calculate that

$$
\begin{aligned}
&\max_{0\leq t\leq 1}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}\varphi_q\biggl(\int_0^1 G(s,\tau)\,d\tau\biggr)\,ds \\
&\quad= \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-\beta-1}\biggl(\frac{s^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}-\frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\biggr)\,ds \\
&\quad= 0.5093, \\
&\max_{0\leq t\leq 1}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1\bigl((1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}\bigr)\varphi_q\biggl(\int_0^1 G(s,\tau)\,d\tau\biggr)\,ds \\
&\quad= \frac{1}{\Gamma(\alpha)}\int_0^1\bigl((1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}\bigr)\biggl(\frac{s^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}-\frac{s^{\alpha}}{\alpha\Gamma(\alpha)}\biggr)\,ds \\
&\quad= 0.176,
\end{aligned}
$$

and

$$
\max_{0\leq t\leq 1}\varphi_q\biggl(\int_0^1 G(t,s)\,ds\biggr)
$$

$$
\begin{aligned}
&= \max_{0\le t\le 1} \varphi_2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds\right) \\
&= \max_{0\le t\le 1}\left(\frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)} - \frac{t^{\alpha}}{\alpha\Gamma(\alpha)}\right) \\
&= 0.5319.
\end{aligned}
$$

Then we obtain $A^{-1} = 0.5319$ and $B^{-1} = 0.176$. It is easy to know that $A = 1.8801$ and $B = 5.6818$.

Besides, let $f(t,u) = (\frac{3}{2})^u + \frac{t^2}{8} \ge 0$ for $\forall t \in [0,1]$, and choose $a = \frac{1}{10}$, $b = 2$, $c = 12$. We get

$$
\begin{aligned}
&\min\left\{\left(\frac{3}{2}\right)^u + \frac{t^2}{8}\right\} > 1 \ge \varphi_2(aB) = 0.56818, \quad \text{for } \forall t \in [0,1] \text{ and } \|u\| = \frac{1}{10}; \\
&\max\left\{\left(\frac{3}{2}\right)^u + \frac{t^2}{8}\right\} \le 2.375 \le \varphi_2(bA) = 3.7602, \quad \text{for } \forall t \in [0,1] \text{ and } \|u\| = 2; \\
&\min\left\{\left(\frac{3}{2}\right)^u + \frac{t^2}{8}\right\} \ge 129.7 \ge \varphi_2(cB) = 68.1816, \quad \text{for } \forall t \in [0,1] \text{ and } \|u\| = 12.
\end{aligned}
$$

From the definitions of ϕ and ψ, we get $\psi(a) \ge \varphi_p(aB)$, $\phi(b) \le \varphi_p(bA)$, and $\psi(c) \ge \varphi_p(cB)$. So, all conditions of Theorem 3.1 are satisfied. Then system (4.1) has at least two positive solutions $u_1^*, u_2^* \in P$ such that $\frac{1}{10} \le \|u_1^*\| \le 2$ and $2 \le \|u_2^*\| \le 12$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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