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Existence of a random attractor for non-autonomous stochastic plate equations with additive noise and nonlinear damping on $\mathbb{R}^n$

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Abstract

Based on the abstract theory of pullback attractors of non-autonomous non-compact dynamical systems by differential equations with both dependent-time deterministic and stochastic forcing terms, introduced by Wang in (J. Differ. Equ. 253:1544–1583, 2012), we investigate the existence of pullback attractors for the non-autonomous stochastic plate equations with additive noise and nonlinear damping on $\mathbb{R}^n$.

MSC: 35B40; 35B41

Keywords: Pullback attractors; Pate equation; Unbounded domains; The splitting technique; Additive noise

1 Introduction

Plate equations have been studied for many years because of their worth in certain physical areas such as vibration and elasticity theories of solid mechanics. The research of the long-time dynamical behavior of plate equations has become an important area in the field of the infinite-dimensional dynamical system.

The purpose of this paper is to investigate the following non-autonomous stochastic plate equations with additive noise and nonlinear damping defined in the entire space $\mathbb{R}^n$:

$$u_{tt} + h(u_t) + \Delta^2 u + \lambda u + f(x,u) = g(x,t) + \phi(x)\frac{dW}{dt}, \tag{1.1}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \qquad u_t(x,\tau) = u_1(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, λ is a positive constant, f is a nonlinearity that satisfies certain growth and dissipative conditions, $g(x,\cdot)$ and ϕ are given functions in $L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n) \cap H^3(\mathbb{R}^n)$, respectively, $W(t)$ is a two-sided real-valued Wiener process on a probability space.

As we know, the attractor is regarded as a proper notation describing the long-time dynamics of solutions, and many classical literature works and monographs have appeared for both the deterministic and stochastic dynamical systems over the last decades, see [1, 5, 7, 8, 10, 11, 14, 19, 23, 25] and the references therein. However, in reality, a system is always affected by some random factors such as external noise. In order to scrutinize the large-time behavior and characterization of solution for the stochastic partial differential equations driven by noise, Crauel and Flandoi [7, 8], Flandoi and Schmalfuss [10], and Schmalfuss [19] introduced the concept of pullback attractors and established some abstract results for the existence of such attractors about compact dynamical system [1, 8, 10, 14, 15]. Since these methods required the compactness of a pullback absorbing set for systems, they could not be used to deal with the stochastic PDEs on unbounded domains. Therefore, in [3], Bates, Lisei, and Lu presented the concept of asymptotic compactness for random dynamical systems, which is an extension of deterministic systems. And then, using these abstract results, they proved the existence of random attractors for reaction-diffusion equations on unbounded domain in [4]. Wang in [25] further extended the concept of asymptotic compactness to the case of partial differential equations with both random and time-dependent forcing terms; moreover, he applied these criteria into the stochastic reaction-diffusion equation with additive noise on $\mathbb{R}^n$ and obtained the existence of a unique pullback attractor. For most of works on stochastic PDEs, please refer to [9, 22, 27–29, 32] and the references therein.

Just for problem (1.1)–(1.2) and the corresponding plate equations, in the deterministic case (i.e., $\varepsilon = 0$), existence of global attractors has been studied by several authors, see for instance [2, 12–14, 30, 31, 33, 34, 37]. As far as the stochastic case driven by additive noise goes, when the deterministic forcing term g is independent of time, that is, $g(x,t) \equiv g(x)$, the existence of a random pullback attractor on bounded domain has been obtained in [17, 20, 21]. Recently, on the unbounded domain, the authors investigated the existence and upper semi-continuity of random attractors for stochastic plate equation with rotational inertia and Kelvin–Voigt dissipative term as well as dependent-on-time terms (see [36] for details) and asymptotic behavior for non-autonomous stochastic plate equation on unbounded domains [35]. To the best of our knowledge, it has not been considered by any predecessors for the stochastic plate equation with additive noise and nonlinear damping on unbounded domain. It is well known that nonlinear damping makes the problem more complex and interesting even to the case of bounded domain. Besides, the theory and applications of Wang in [24–26] gave us the idea of solving this problem and inspired us, so we decided to study the existence of pullback attractors for problem (1.1)–(1.2).

Notice that (1.1) is a non-autonomous stochastic equation, i.e., the external term g is time-dependent. In this case, as in [25], we introduce two parametric spaces to present its dynamics: one is for the deterministic non-autonomous perturbations, while the other for the stochastic perturbations. In addition, since Sobolev embeddings are not compact on $\mathbb{R}^n$, we cannot get the asymptotic compactness directly from the regularity of solutions. We conquer the difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$ and the splitting technique [27] and the compactness methods introduced in [16].

The organization of this paper is as follows: In Sect. 2, we present some notations and a proposition about random dynamical systems. In Sect. 3, we establish a continuous cocycle for Eq. (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In Sect. 4, we obtain all necessary uniform estimates of solutions. Finally, in Sect. 5, we show the existence and uniqueness of a random attractor for (1.1)–(1.2), denoted by $\mathbb{R}^n$.

Throughout the paper, the letters c and c_i ($i = 1, 2, \ldots$) are positive constants which may change their values from line to line or even in the same line.

2 Preliminaries

In order to state and prove our main results, we introduce some notations and a proposition related to random attractors for stochastic dynamical systems.

Let X be a separable Banach space and $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$ which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t) \quad \text{for all } \omega \in \Omega, t \in \mathbb{R}. \tag{2.1}$$

We often say that $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a parametric dynamical system.

Definition 2.1 ([6]) A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if, for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $t, s \in \mathbb{R}^+$, the following conditions (1)–(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;
(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Definition 2.2 ([6]) Assume that Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of all tempered families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \bigl\{D = \bigl\{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\bigr\}\bigr\}.$$

Definition 2.3 ([6]) $\mathcal{D}$ is said to be tempered if there exists $x_0 \in X$ such that, for every $c > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, the following holds:

$$\lim_{t\to-\infty} e^{ct} d\bigl(D(\tau + t, \theta_t\omega), x_0\bigr) = 0. \tag{2.2}$$

Definition 2.4 ([6]) Given $D \in \mathcal{D}$, the family $\Omega(D) = \{\Omega(D, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of D where

$$\Omega(D, \tau, \omega) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \Phi\bigl(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)\bigr)}. \tag{2.3}$$

Definition 2.5 ([6]) The cocycle Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\bigl\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\bigr\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \tag{2.4}$$

whenever $t_n \to \infty$, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$ with $\{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.6 ([6]) A family $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for Φ if, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D,\tau,\omega) > 0$ such that

$$\Phi\big(t,\tau - t,\theta_{-t}\omega, D(\tau - t,\theta_{-t}\omega)\big) \subseteq K(\tau,\omega) \quad \text{for all } t \geq T. \tag{2.5}$$

Definition 2.7 ([6]) K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ if $K(\tau,\omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$.

Definition 2.8 ([6]) A family $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)–(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

(1) $\mathcal{A}(\tau,\omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$;

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi\big(t,\tau,\omega,\mathcal{A}(\tau,\omega)\big) = \mathcal{A}(\tau + t,\theta_t\omega); \tag{2.6}$$

(3) For every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H\big(\Phi\big(t,\tau - t,\theta_{-t}\omega, D(\tau - t,\theta_{-t}\omega)\big),\mathcal{A}(\tau,\omega)\big) = 0, \tag{2.7}$$

where d_H is the Hausdorff semi-distance given by $d_H(F,G) = \sup_{u\in F} \inf_{v\in G} \|u - v\|_X$ for any $F, G \subset X$.

Definition 2.9 ([6]) A mapping $\Psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a random complete solution of Φ if, for every $\tau \in \mathbb{R}^+$, $s,\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$\Phi\big(t,\tau + s,\theta_s\omega,\Psi(s,\tau,\omega)\big) = \Psi(t + s,\tau,\omega). \tag{2.8}$$

Definition 2.10 ([6]) Ψ is called a tempered random complete solution of Φ, if there exists a tempered family $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\Psi(t,\tau,\omega)$ belongs to $D(\tau + t,\theta_t\omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Proposition 2.1 ([25]) *Suppose that Φ is $\mathcal{D}$-pullback asymptotically compact in X and has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. Then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \bigcup_{D\in\mathcal{D}} \Omega(D,\tau,\omega) \tag{2.9}$$

$$= \big\{\Psi(0,\tau,\omega) : \Psi \text{ is a tempered random complete solution of } \Phi\big\}. \tag{2.10}$$

3 Cocycles for stochastic plate equation

In this section, we firstly present the precise hypotheses on problem (1.1)–(1.2), then show that it generates a continuous cocycle in $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ with the domain $D(A) = H^4(\mathbb{R}^n)$. We can also define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm:

$$(u,v)_\nu = \bigl(A^{\frac{\nu}{4}}u, A^{\frac{\nu}{4}}v\bigr), \qquad \|\cdot\|_\nu = \bigl\|A^{\frac{\nu}{4}}\cdot\bigr\|.$$

As usual, $(\cdot,\cdot)$ denotes L^2-inner product and $\|\cdot\|$ denotes the L^2-norm.

Let $E = H^2 \times L^2$, with the Sobolev norm

$$\|y\|_{H^2\times L^2} = \bigl(\|v\|^2 + \|u\|^2 + \|\Delta u\|^2\bigr)^{\frac{1}{2}} \quad \text{for } y = (u,v)^\top \in E. \tag{3.1}$$

Let $\xi = u_t + \delta u$, where δ is a small positive constant whose value will be determined later, then (1.1)–(1.2) can be reduced to the equivalent system

$$\begin{cases} \frac{du}{dt} + \delta u = \xi, \\ \frac{d\xi}{dt} - \delta\xi + (\lambda + \delta^2 + A)u + h(\xi - \delta u) + f(x,u) = g(x,t) + \phi(x)\frac{dW}{dt}, \end{cases} \tag{3.2}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \qquad \xi(x,\tau) = \xi_0(x), \tag{3.3}$$

where $\xi_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$.

Assume that the functions h, f satisfy the following conditions:

(1) Let $F(x,u) = \int_0^u f(x,s)\,ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, there exist positive constants c_i ($i = 1,2,3,4$) such that

$$\bigl|f(x,u)\bigr| \le c_1|u|^p + \eta_1(x), \quad \eta_1 \in L^2\bigl(\mathbb{R}^n\bigr), \tag{3.4}$$

$$f(x,u)u - c_2F(x,u) \ge \eta_2(x), \quad \eta_2 \in L^1\bigl(\mathbb{R}^n\bigr), \tag{3.5}$$

$$F(x,u) \ge c_3|u|^{p+1} - \eta_3(x), \quad \eta_3 \in L^1\bigl(\mathbb{R}^n\bigr), \tag{3.6}$$

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \le \beta, \qquad \left|\frac{\partial f}{\partial x}(x,u)\right| \le \eta_4(x), \quad \eta_4 \in L^2\bigl(\mathbb{R}^n\bigr), \tag{3.7}$$

where $\beta > 0$, $1 \le p \le \frac{n+4}{n-4}$. Note that (3.4) and (3.5) imply

$$F(x,u) \le c\bigl(|u|^2 + |u|^{p+1} + \eta_1^2 + \eta_2\bigr). \tag{3.8}$$

(2) There exist two constants β_1, β_2 such that

$$h(0) = 0, \quad 0 < \beta_1 \le h'(v) \le \beta_2 < \infty. \tag{3.9}$$

We identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t) = W(t,x)$, $t \in \mathbb{R}$. To study the dynamical behavior of problem (3.2)–(3.3), we need to convert the stochastic system into a deterministic one with a random parameter. To this end, we set $v(t) = \xi(t) - \phi\omega(t)$, we obtain the equivalent system of (3.2)–(3.3):

$$\begin{cases} \frac{du}{dt} + \delta u = v + \phi\omega(t), \\ \frac{dv}{dt} - \delta v + (\lambda + \delta^2 + A)u + f(x,u) = g(x,t) - h(v + \phi\omega(t) - \delta u) + \delta\phi\omega(t), \end{cases} \tag{3.10}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \qquad v(x,\tau) = v_0(x), \tag{3.11}$$

where $v_0(x) = \xi_0(x) - \phi\omega(t)$, $x \in \mathbb{R}^n$.

By a standard method as in [5, 18, 23, 36], one may show the following lemma under conditions (3.4)–(3.9).

Lemma 3.1 *Put $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0) = (u(t+\tau,\tau,\theta_{-\tau}\omega,u_0), v(t+\tau,\tau,\theta_{-\tau}\omega,v_0))^\top$, where $\varphi_0 = (u_0,v_0)^\top$, and let (3.4)–(3.9) hold. Then, for every $\omega \in \Omega$, $\tau \in \mathbb{R}$, and $\varphi_0 \in E(\mathbb{R}^n)$, problem (3.10)–(3.11) has a unique $(\mathcal{F}, \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable solution $\varphi(\cdot,\tau,\omega,\varphi_0) \in C([\tau,\infty), E(\mathbb{R}^n))$ with $\varphi(\tau,\tau,\omega,\varphi_0) = \varphi_0$, $\varphi(t,\tau,\omega,\varphi_0) \in E(\mathbb{R}^n)$ being continuous in φ_0 with respect to the usual norm of $E(\mathbb{R}^n)$ for each $t > \tau$. Moreover, for every $(t,\tau,\omega,\varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$, the mapping*

$$\Phi(t,\tau,\omega,\varphi_0) = \varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_0) \tag{3.12}$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$ to $E(\mathbb{R}^n)$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Introducing the homeomorphism $P(\theta_t\omega)(u,v)^\top = (u, v + z(\theta_t\omega))^\top$, $(u,v)^\top \in E(\mathbb{R}^n)$ whose inverse homeomorphism $P^{-1}(\theta_t\omega)(u,v)^\top = (u, v - z(\theta_t\omega))^\top$. Then the transformation

$$\widetilde{\Phi}\big(t,\tau,\omega,(u_0,\xi_0)\big) = P(\theta_t\omega)\Phi\big(t,\tau,\omega,(u_0,v_0)\big)P^{-1}(\theta_t\omega) \tag{3.13}$$

also generates a continuous cocycle with (3.2)–(3.3) over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.13), it is easy to check that $\widetilde{\Phi}$ has a random attractor provided Φ possesses a random attractor. Then we only need to consider the continuous cocycle Φ.

Next we make another assumption:

Assume that σ, δ, and g satisfy the following conditions:

$$\sigma = \min\left\{\delta, \frac{\delta c_2}{2}\right\}, \tag{3.14}$$

$$\lambda + \delta^2 - \beta_2\delta > 0 \quad \text{and} \quad \beta_1 > 4\delta + \frac{3\beta^2}{\delta(\lambda + \delta^2 - \beta_2\delta)}. \tag{3.15}$$

Moreover,

$$\int_{-\infty}^0 e^{\sigma s}\big\|g(\cdot,\tau+s)\big\|_1^2\,ds < \infty, \quad \forall\tau \in \mathbb{R}, \tag{3.16}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^0 e^{\sigma s}\int_{|x|\geq k}\big|g(x,\tau+s)\big|^2\,dx\,ds = 0, \quad \forall\tau \in \mathbb{R}, \tag{3.17}$$

where $|\cdot|$ denotes the absolute value of a real number in $\mathbb{R}$.

Given a bounded nonempty subset B of E, we write $\|B\| = \sup_{\phi\in B}\|\phi\|_E$. Let $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of E such that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{s\to-\infty} e^{\sigma s}\big\|D(\tau + s, \theta_s\omega)\big\|_E^2 = 0. \tag{3.18}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D} = \big\{D = \big\{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\big\} : D \text{ satisfies (3.18)}\big\}. \tag{3.19}$$

4 Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of the stochastic plate equations (3.2)–(3.3) defined on $\mathbb{R}^n$.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = \big(\|v\|^2 + \big(\lambda + \delta^2 - \beta_2\delta\big)\|u\|^2 + \|\Delta u\|^2\big)^{\frac{1}{2}} \quad \text{for } Y = (u,v) \in E. \tag{4.1}$$

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2\times L^2}$ in (3.1).

The next lemma shows that the cocycle Φ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

Lemma 4.1 *Under (3.4)–(3.9) and (3.14)–(3.17), for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that, for all $t \geq T$, the solution of problem (3.10)–(3.11) satisfies*

$$\begin{aligned}
&\big\|Y\big(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)\big)\big\|_E^2 \leq R_1(\tau,\omega),\\
&e^{-\sigma t}\int_{\tau-t}^{t} e^{\sigma s}\big\|Y\big(s, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)\big)\big\|_E^2\,ds \leq R_1(\tau,\omega),
\end{aligned}$$

and $R_1(\tau,\omega)$ is given by

$$R_1(\tau,\omega) = M + M\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\big\|g(x,s)\big\|^2\,ds + c\int_{-\infty}^{0} e^{\sigma s}\big(\big|\omega(s)\big|^2 + \big|\omega(s)\big|^{p+1}\big)\,ds, \tag{4.2}$$

where M is a positive constant independent of τ, ω, D.

Proof Taking the inner product of the second equation of (3.10) with v in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 - \delta\|v\|^2 + \big(\lambda + \delta^2\big)(u,v) + (Au,v) + \big(f(x,u),v\big)\\
&\quad= \big(g(x,t),v\big) - \big(h\big(v + \phi\omega(t) - \delta u\big),v\big) + \delta(\phi,v)\omega(t).
\end{aligned} \tag{4.3}$$

By the first equation of (3.10), we have

$$v = u_t - \phi\omega(t) + \delta u. \tag{4.4}$$

By Lagrange's mean value theorem and (3.9), we get

$$\begin{aligned}
&-\big(h\big(v+\phi\omega(t)-\delta u\big),v\big)\\
&\quad=-\big(h\big(v+\phi\omega(t)-\delta u\big)-h(0),v\big)\\
&\quad=-\big(h'(\vartheta)\big(v+\phi\omega(t)-\delta u\big),v\big)\\
&\quad\le-\beta_1\|v\|^2-\big(h'(\vartheta)\big(\phi\omega(t)-\delta u\big),v\big)\\
&\quad\le-\beta_1\|v\|^2+\beta_2\big|\omega(t)\big|\|\phi\|\|v\|+h'(\vartheta)\delta(u,v)\\
&\quad\le-\beta_1\|v\|^2+\frac{\beta_1-\delta}{6}\|v\|^2+\frac{3\beta_2^2}{2(\beta_1-\delta)}\big|\omega(t)\big|^2\|\phi\|^2+h'(\vartheta)\delta(u,v),
\end{aligned}\tag{4.5}$$

where ϑ is between 0 and $v+\phi\omega(t)-\delta u$.

By (3.9) and (4.4), we get

$$\begin{aligned}
&h'(\vartheta)\delta(u,v)\\
&\quad=h'(\vartheta)\delta\big(u,u_t-\phi\omega(t)+\delta u\big)\\
&\quad\le\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\beta_2\delta^2\|u\|^2+\beta_2\delta\big|\omega(t)\big|\|\phi\|\|u\|\\
&\quad\le\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|u\|^2+\beta_2\delta^2\|u\|^2+\frac{1}{4}\delta\big(\lambda+\delta^2-\beta_2\delta\big)\|u\|^2+c\big|\omega(t)\big|^2\|\phi\|^2.
\end{aligned}\tag{4.6}$$

Then substituting the v in (4.4) into the third and fourth terms on the left-hand side of (4.3), we find that

$$\begin{aligned}
&\big(\lambda+\delta^2\big)(u,v)\\
&\quad=\big(\lambda+\delta^2\big)\big(u,u_t-\phi\omega(t)+\delta u\big)\\
&\quad\ge\frac{1}{2}\big(\lambda+\delta^2\big)\frac{d}{dt}\|u\|^2+\delta\big(\lambda+\delta^2\big)\|u\|^2-\big(\lambda+\delta^2\big)\big|\omega(t)\big|\|\phi\|\|u\|\\
&\quad\ge\frac{1}{2}\big(\lambda+\delta^2\big)\frac{d}{dt}\|u\|^2+\delta\big(\lambda+\delta^2\big)\|u\|^2-\frac{1}{4}\delta\big(\lambda+\delta^2-\beta_2\delta\big)\|u\|^2-c\big|\omega(t)\big|^2\|\phi\|^2,
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
(Au,v)=(\Delta u,\Delta v)&=\big(\Delta u,\Delta u_t-\omega(t)\Delta\phi+\delta\Delta u\big)\\
&\ge\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\delta\|\Delta u\|^2-\big|\omega(t)\big|\|\Delta\phi\|\|\Delta u\|\\
&\ge\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\frac{\delta}{2}\|\Delta u\|^2-\frac{1}{2\delta}\big|\omega(t)\big|^2\|\Delta\phi\|^2.
\end{aligned}\tag{4.8}$$

Using the Cauchy–Schwarz inequality and Young's inequality, we have

$$\delta\big(\phi\omega(t),v\big)\le\delta\big|\omega(t)\big|\|\phi\|\|v\|\le\frac{3\delta^2}{2(\beta_1-\delta)}\|\phi\|^2\big|\omega(t)\big|^2+\frac{\beta_1-\delta}{6}\|v\|^2\tag{4.9}$$

and

$$(g,v)\le\|g\|\|v\|\le\frac{3}{2(\beta_1-\delta)}\|g\|^2+\frac{\beta_1-\delta}{6}\|v\|^2.\tag{4.10}$$

Let $\widetilde{F}(x,u) = \int_{\mathbb{R}^n} F(x,u)\,dx$. Then, for the last term on the left-hand side of (4.3), we have

$$\begin{aligned}\big(f(x,u),v\big) &= \big(f(x,u),u_t - \phi\omega(t) + \delta u\big)\\ &= \frac{d}{dt}\widetilde{F}(x,u) + \delta\big(f(x,u),u\big) - \big(f(x,u),\phi\omega(t)\big).\end{aligned} \tag{4.11}$$

By condition (3.5) we get

$$\big(f(x,u),u\big) \geq c_2\widetilde{F}(x,u) + \int_{\mathbb{R}^n}\eta_2(x)\,dx. \tag{4.12}$$

Using condition (3.4) and (3.6), we obtain

$$\begin{aligned}&\big(f(x,u),\phi\omega(t)\big)\\ &\quad\leq \int_{\mathbb{R}^n}\big(c_1|u|^p + \eta_1(x)\big)\big|\phi\omega(t)\big|\,dx\\ &\quad\leq \big\|\eta_1(x)\big\|\|\phi\|\big|\omega(t)\big| + c_1\left(\int_{\mathbb{R}^n}|u|^{p+1}\,dx\right)^{\frac{p}{p+1}}\|\phi\|_{p+1}\big|\omega(t)\big|\\ &\quad\leq \big\|\eta_1(x)\big\|\|\phi\|\big|\omega(t)\big| + c_1\left(\int_{\mathbb{R}^n}\big(F(x,u) + \eta_3(x)\big)\,dx\right)^{\frac{p}{p+1}}\|\phi\|_{p+1}\big|\omega(t)\big|\\ &\quad\leq \frac{1}{2}\big\|\eta_1(x)\big\|^2 + \frac{1}{2}\|\phi\|^2\big|\omega(t)\big|^2 + \frac{\delta c_2}{2}\widetilde{F}(x,u)\\ &\qquad + \frac{\delta c_2}{2}\int_{\mathbb{R}^n}\eta_3(x)\,dx + c\|\phi\|_{H^2}^{p+1}\big|\omega(t)\big|^{p+1}.\end{aligned} \tag{4.13}$$

By (4.11)–(4.13), we get

$$\begin{aligned}&\delta\big(f(x,u),u\big) - \big(f(x,u),\phi\omega(t)\big)\\ &\quad\geq \frac{\delta c_2}{2}\widetilde{F}(x,u) + \delta\int_{\mathbb{R}^n}\eta_2(x)\,dx - \frac{1}{2}\big\|\eta_1(x)\big\|^2 - \frac{1}{2}\|\phi\|^2\big|\omega(t)\big|^2\\ &\qquad - \frac{\delta c_2}{2}\int_{\mathbb{R}^n}\eta_3(x)\,dx - c\|\phi\|_{H^2}^{p+1}\big|\omega(t)\big|^{p+1}.\end{aligned} \tag{4.14}$$

Substitute (4.5)–(4.14) into (4.3) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|v\|^2 + \big(\lambda + \delta^2 - \beta_2\delta\big)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u)\big)\\ &\qquad + \delta\big(\|v\|^2 + \big(\lambda + \delta^2 - \beta_2\delta\big)\|u\|^2 + \|\Delta u\|^2\big) + \frac{\delta c_2}{2}\widetilde{F}(x,u)\\ &\quad\leq \frac{\delta}{2}\|v\|^2 + \frac{2\delta - \beta_1}{2}\|v\|^2 + \frac{\delta}{2}\big(\lambda + \delta^2 - \beta_2\delta\big)\|u\|^2 + \frac{\delta}{2}\|\Delta u\|^2\\ &\qquad + c\big(1 + \big|\omega(t)\big|^2 + \big|\omega(t)\big|^{p+1}\big) + \frac{3}{2(\beta_1 - \delta)}\|g\|^2.\end{aligned} \tag{4.15}$$

Let $\sigma = \min\{\delta, \frac{\delta c_2}{2}\}$, then

$$\begin{aligned}
&\frac{d}{dt}\bigl(\|v\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u)\bigr) \\
&\quad + \sigma\bigl(\|v\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x,u)\bigr) \\
&\leq \frac{3}{(\beta_1 - \delta)}\|g\|^2 + c\bigl(1 + \bigl|\omega(t)\bigr|^2 + \bigl|\omega(t)\bigr|^{p+1}\bigr).
\end{aligned} \tag{4.16}$$

Multiplying (4.16) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have

$$\begin{aligned}
&e^{\sigma\tau}\bigl(\bigl\|v(\tau,\tau - t,\omega,v_0)\bigr\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\bigl\|u(\tau,\tau - t,\omega,u_0)\bigr\|^2 \\
&\quad + \bigl\|\Delta u(\tau,\tau - t,\omega,u_0)\bigr\|^2 + 2\widetilde{F}\bigl(x,(\tau,\tau - t,\omega,u_0)\bigr)\bigr) \\
&\leq e^{\sigma(\tau - t)}\bigl(\|v_0\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\|u_0\|^2 + \|\Delta u_0\|^2 + 2\widetilde{F}(x,u_0)\bigr) \\
&\quad + \frac{3}{(\beta_1 - \delta)}\int_{\tau - t}^{\tau} e^{\sigma s}\bigl\|g(x,s)\bigr\|^2\,ds + c\int_{\tau - t}^{\tau} e^{\sigma s}\bigl(1 + \bigl|\omega(s)\bigr|^2 + \bigl|\omega(s)\bigr|^{p+1}\bigr)\,ds.
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above, we obtain, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$\begin{aligned}
&\bigl\|v(\tau,\tau - t,\theta_{-\tau}\omega,v_0)\bigr\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\bigl\|u(\tau,\tau - t,\theta_{-\tau}\omega,u_0)\bigr\|^2 \\
&\quad + \bigl\|\Delta u(\tau,\tau - t,\omega,u_0)\bigr\|^2 + 2\widetilde{F}\bigl(x,u(\tau,\tau - t,\theta_{-\tau}\omega,u_0)\bigr) \\
&\leq e^{-\sigma t}\bigl(\|v_0\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\|u_0\|^2 + \|\Delta u_0\|^2 + 2\widetilde{F}(x,u_0)\bigr) \\
&\quad + \frac{3}{(\beta_1 - \delta)}\int_{\tau - t}^{\tau} e^{\sigma(s-\tau)}\bigl\|g(x,s)\bigr\|^2\,ds \\
&\quad + c\int_{\tau - t}^{\tau} e^{\sigma(s-\tau)}\bigl(1 + \bigl|\theta_{-\tau}\omega(s)\bigr|^2 + \bigl|\theta_{-\tau}\omega(s)\bigr|^{p+1}\bigr)\,ds.
\end{aligned} \tag{4.17}$$

Again, by (3.9), we get

$$\widetilde{F}(x,u_0) \leq c\bigl(1 + \|u_0\|^2 + \|u_0\|^{p+1}\bigr).$$

Thus, for the first term on the right-hand side of (4.17), we have

$$\begin{aligned}
&e^{-\sigma t}\bigl(\|v_0\|^2 + \bigl(\lambda + \delta^2 - \beta_2\delta\bigr)\|u_0\|^2 + \|\Delta u_0\|^2 + 2\widetilde{F}(x,u_0)\bigr) \\
&\quad \leq ce^{-\sigma t}\bigl(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{p+1}\bigr).
\end{aligned}$$

Since $(u_0,v_0)^\top \in D(\tau - t,\theta_{-t}\omega)$ and $D \in \mathcal{D}$, then we find

$$\lim_{t\to+\infty} e^{-\sigma t}\bigl(\|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{p+1}\bigr) = 0.$$

Therefore, there exists $T = T(\tau,\omega,D) > 0$ such that, for all $t \geq T$,

$$e^{-\sigma t}\bigl(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{p+1}\bigr) \leq 1. \tag{4.18}$$

For the last term on the right-hand side of (4.17), we find

$$\begin{aligned}
&c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\bigl(1+\bigl|\theta_{-\tau}\omega(s)\bigr|^2+\bigl|\theta_{-\tau}\omega(s)\bigr|^{p+1}\bigr)\,ds\\
&\quad\le c\int_{-t}^{0} e^{\sigma s}\bigl(1+\bigl|\omega(s)\bigr|^2+\bigl|\omega(s)\bigr|^{p+1}\bigr)\,ds\\
&\quad\le c\int_{-\infty}^{0} e^{\sigma s}\bigl(1+\bigl|\omega(s)\bigr|^2+\bigl|\omega(s)\bigr|^{p+1}\bigr)\,ds\\
&\quad\le \frac{c}{\sigma}+c\int_{-\infty}^{0} e^{\sigma s}\bigl(\bigl|\omega(s)\bigr|^2+\bigl|\omega(s)\bigr|^{p+1}\bigr)\,ds. \qquad (4.19)
\end{aligned}$$

Notice that $\omega(s)$ has at most linear growth at $|s|\to\infty$, which combines (3.19), we can have

$$c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\bigl(1+\bigl|\theta_{-\tau}\omega(s)\bigr|^2+\bigl|\theta_{-\tau}\omega(s)\bigr|^{p+1}\bigr)\,ds\to\frac{c}{\sigma}\quad(t\to\infty). \qquad (4.20)$$

Finally, we estimate the fourth term on the left-hand side of (4.17). Thanks to (3.6), we obtain that, for all $t\ge 0$,

$$-2\widetilde{F}\bigl(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\bigr)\le 2\int_{\mathbb{R}^n}\eta_3\,dx. \qquad (4.21)$$

It follows from (4.18)–(4.21) that

$$\begin{aligned}
&\bigl\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\bigr\|^2+\bigl(\lambda+\delta^2-\beta_2\delta\bigr)\bigl\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\bigr\|^2+\bigl\|\Delta u(\tau,\tau-t,\omega,u_0)\bigr\|^2\\
&\quad\le c+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\bigl\|g(x,s)\bigr\|^2\,ds+c\int_{-\infty}^{0} e^{\sigma s}\bigl(\bigl|\omega(s)\bigr|^2+\bigl|\omega(s)\bigr|^{p+1}\bigr)\,ds. \qquad (4.22)
\end{aligned}$$

Thus the proof is completed. □

Lemma 4.2 *Under (3.4)–(3.9) and (3.14)–(3.17), for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, there exists $T=T(\tau,\omega,D)>0$ such that, for all $t\ge T$, the solution of problem (3.10)–(3.11) satisfies*

$$\bigl\|A^{\frac14}Y\bigl(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega)\bigr)\bigr\|_E^2\le R_2(\tau,\omega),$$

and $R_2(\tau,\omega)$ is given by

$$\begin{aligned}
R_2(\tau,\omega)={}&ce^{-\sigma t}\bigl(\bigl\|A^{\frac14}v_0\bigr\|^2+\bigl\|A^{\frac14}u_0\bigr\|^2+\bigl\|A^{\frac34}u_0\bigr\|^2\bigr)\\
&+c\int_{-\infty}^{\tau} e^{\sigma(s-\tau)}\bigl\|g(x,s)\bigr\|_1^2\,ds+c\int_{-\infty}^{0} e^{\sigma s}\bigl(1+\bigl|\omega(s)\bigr|^2\bigr)\,ds. \qquad (4.23)
\end{aligned}$$

Proof Taking the inner product of the second equation of (3.10) with $A^{\frac12}v$ in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}
&\frac12\frac{d}{dt}\bigl\|A^{\frac14}v\bigr\|^2-\delta\bigl\|A^{\frac14}v\bigr\|^2+\bigl(\lambda+\delta^2\bigr)\bigl(u,A^{\frac12}v\bigr)+\bigl(Au,A^{\frac12}v\bigr)+\bigl(f(x,u),A^{\frac12}v\bigr)\\
&\quad=\bigl(g(x,t),A^{\frac12}v\bigr)-\bigl(h\bigl(v+\phi\omega(t)-\delta u\bigr),A^{\frac12}v\bigr)+\delta\bigl(\phi,A^{\frac12}v\bigr)\omega(t). \qquad (4.24)
\end{aligned}$$

Similar to the proof of Lemma 4.1, we have the following estimates:

$$\begin{aligned}
&-\big(h\big(v+\phi\omega(t)-\delta u\big),A^{\frac{1}{2}}v\big)\\
&\quad=-\big(h\big(v+\phi\omega(t)-\delta u\big)-h(0),A^{\frac{1}{2}}v\big)\\
&\quad=-\big(h'(\vartheta)\big(v+\phi\omega(t)-\delta u\big),A^{\frac{1}{2}}v\big)\\
&\quad\le-\beta_1\big\|A^{\frac{1}{4}}v\big\|^2-\big(h'(\vartheta)\big(\phi\omega(t)-\delta u\big),A^{\frac{1}{2}}v\big)\\
&\quad\le-\beta_1\big\|A^{\frac{1}{4}}v\big\|^2+\beta_2\big|\omega(t)\big|\big\|A^{\frac{1}{4}}\phi\big\|\big\|A^{\frac{1}{4}}v\big\|+h'(\vartheta)\delta\big(u,A^{\frac{1}{2}}v\big)\\
&\quad\le-\beta_1\big\|A^{\frac{1}{4}}v\big\|^2+\frac{\beta_1-\delta}{6}\big\|A^{\frac{1}{4}}v\big\|^2+c\big|\omega(t)\big|^2\big\|A^{\frac{1}{4}}\phi\big\|^2+h'(\vartheta)\delta\big(u,A^{\frac{1}{2}}v\big),
\end{aligned}\tag{4.25}$$

$$\begin{aligned}
&h'(\vartheta)\delta\big(u,A^{\frac{1}{2}}v\big)\\
&\quad=h'(\vartheta)\delta\big(u,A^{\frac{1}{2}}u_t-\omega(t)A^{\frac{1}{2}}\phi+\delta A^{\frac{1}{2}}u\big)\\
&\quad\le\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\big\|A^{\frac{1}{4}}u\big\|^2+\beta_2\delta^2\big\|A^{\frac{1}{4}}u\big\|^2+\beta_2\delta\big|\omega(t)\big|\big\|A^{\frac{1}{4}}\phi\big\|\big\|A^{\frac{1}{4}}u\big\|\\
&\quad\le\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\big\|A^{\frac{1}{4}}u\big\|^2+\beta_2\delta^2\big\|A^{\frac{1}{4}}u\big\|^2\\
&\qquad+\frac{1}{6}\delta\big(\lambda+\delta^2-\beta_2\delta\big)\big\|A^{\frac{1}{4}}u\big\|^2+c\big|\omega(t)\big|^2\big\|A^{\frac{1}{4}}\phi\big\|^2,
\end{aligned}\tag{4.26}$$

$$\begin{aligned}
&\big(\lambda+\delta^2\big)\big(u,A^{\frac{1}{2}}v\big)\\
&\quad=\big(\lambda+\delta^2\big)\big(u,A^{\frac{1}{2}}u_t-\omega(t)A^{\frac{1}{2}}\phi+\delta A^{\frac{1}{2}}u\big)\\
&\quad\ge\frac{1}{2}\big(\lambda+\delta^2\big)\frac{d}{dt}\big\|A^{\frac{1}{4}}u\big\|^2+\delta\big(\lambda+\delta^2\big)\big\|A^{\frac{1}{4}}u\big\|^2-\big(\lambda+\delta^2\big)\big|\omega(t)\big|\big\|A^{\frac{1}{4}}\phi\big\|\big\|A^{\frac{1}{4}}u\big\|\\
&\quad\ge\frac{1}{2}\big(\lambda+\delta^2\big)\frac{d}{dt}\big\|A^{\frac{1}{4}}u\big\|^2+\delta\big(\lambda+\delta^2\big)\big\|A^{\frac{1}{4}}u\big\|^2\\
&\qquad-\frac{1}{6}\delta\big(\lambda+\delta^2-\beta_2\delta\big)\big\|A^{\frac{1}{4}}u\big\|^2-c\big|\omega(t)\big|^2\big\|A^{\frac{1}{4}}\phi\big\|^2,
\end{aligned}\tag{4.27}$$

$$\begin{aligned}
\big(Au,A^{\frac{1}{2}}v\big)&=\big(Au,A^{\frac{1}{2}}u_t-\omega(t)A^{\frac{1}{2}}\phi+\delta A^{\frac{1}{2}}u\big)\\
&\ge\frac{1}{2}\frac{d}{dt}\big\|A^{\frac{3}{4}}u\big\|^2+\delta\big\|A^{\frac{3}{4}}u\big\|^2-\big|\omega(t)\big|\big\|A^{\frac{3}{4}}\phi\big\|\big\|A^{\frac{3}{4}}u\big\|\\
&\ge\frac{1}{2}\frac{d}{dt}\big\|A^{\frac{3}{4}}u\big\|^2+\frac{\delta}{2}\big\|A^{\frac{3}{4}}u\big\|^2-\frac{1}{2\delta}\big|\omega(t)\big|^2\big\|A^{\frac{3}{4}}\phi\big\|^2,
\end{aligned}\tag{4.28}$$

$$\delta\big(\phi\omega(t),A^{\frac{1}{2}}v\big)\le\delta\big|\omega(t)\big|\big\|A^{\frac{1}{4}}\phi\big\|\big\|A^{\frac{1}{4}}v\big\|\le c\big\|A^{\frac{1}{4}}\phi\big\|^2\big|\omega(t)\big|^2+\frac{\beta_1-\delta}{6}\big\|A^{\frac{1}{4}}v\big\|^2,\tag{4.29}$$

$$\big(g,A^{\frac{1}{2}}v\big)\le\|g\|_1\big\|A^{\frac{1}{4}}v\big\|\le\frac{3}{2(\beta_1-\delta)}\|g\|_1^2+\frac{\beta_1-\delta}{6}\big\|A^{\frac{1}{4}}v\big\|^2.\tag{4.30}$$

For the last term on the left-hand side of (4.24), thanks to (3.7), we have

$$\begin{aligned}
-\big(f(x,u),A^{\frac{1}{2}}v\big)&\le\big|\big(f(x,u),A^{\frac{1}{2}}v\big)\big|\\
&=\bigg|\int_{\mathbb{R}^n}\frac{\partial}{\partial x}f(x,u)\cdot A^{\frac{1}{4}}v\,dx+\int_{\mathbb{R}^n}\frac{\partial}{\partial u}f(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}v\,dx\bigg|\\
&\le\int_{\mathbb{R}^n}\bigg|\frac{\partial}{\partial x}f(x,u)\bigg|\cdot\big|A^{\frac{1}{4}}v\big|\,dx+\int_{\mathbb{R}^n}\bigg|\frac{\partial}{\partial u}f(x,u)\bigg|\cdot\big|A^{\frac{1}{4}}u\big|\cdot\big|A^{\frac{1}{4}}v\big|\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^n} |\eta_4| \cdot \left|A^{\frac{1}{4}} v\right| dx + \beta \int_{\mathbb{R}^n} \left|A^{\frac{1}{4}} u\right| \cdot \left|A^{\frac{1}{4}} v\right| dx \\
&\le \|\eta_4\| \left\|A^{\frac{1}{4}} v\right\| + \beta \left\|A^{\frac{1}{4}} u\right\| \left\|A^{\frac{1}{4}} v\right\| \\
&\le c + \left(\delta + \frac{3\beta^2}{2\delta(\lambda + \delta^2 - \beta_2\delta)}\right) \left\|A^{\frac{1}{4}} v\right\|^2 \\
&\quad + \frac{1}{6}\delta\left(\lambda + \delta^2 - \beta_2\delta\right) \left\|A^{\frac{1}{4}} u\right\|^2. \qquad (4.31)
\end{aligned}
$$

Plugging (4.25)–(4.31) into (4.24) and together with (3.15), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|A^{\frac{1}{4}} v\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u\right\|^2 + \left\|A^{\frac{3}{4}} u\right\|^2\right) \\
&\qquad + \delta\left(\left\|A^{\frac{1}{4}} v\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u\right\|^2 + \left\|A^{\frac{3}{4}} u\right\|^2\right) \\
&\le \frac{\delta}{2}\left\|A^{\frac{1}{4}} v\right\|^2 + \frac{\delta}{2}\left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u\right\|^2 + \frac{\delta}{2}\left\|A^{\frac{3}{4}} u\right\|^2 + c\left(1 + \left|\omega(t)\right|^2 + \|g\|_1^2\right), \qquad (4.32)
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|A^{\frac{1}{4}} v\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u\right\|^2 + \left\|A^{\frac{3}{4}} u\right\|^2\right) \\
&\qquad + \sigma\left(\left\|A^{\frac{1}{4}} v\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u\right\|^2 + \left\|A^{\frac{3}{4}} u\right\|^2\right) \\
&\le c\left(1 + \left|\omega(t)\right|^2 + \|g\|_1^2\right). \qquad (4.33)
\end{aligned}
$$

Multiplying (4.33) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have

$$
\begin{aligned}
&e^{\sigma\tau}\left(\left\|A^{\frac{1}{4}} v(\tau, \tau - t, \omega, v_0)\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u(\tau, \tau - t, \omega, u_0)\right\|^2\right. \\
&\qquad \left. + \left\|A^{\frac{3}{4}} u(\tau, \tau - t, \omega, u_0)\right\|^2\right) \\
&\le e^{\sigma(\tau - t)}\left(\left\|A^{\frac{1}{4}} v_0\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u_0\right\|^2 + \left\|A^{\frac{3}{4}} u_0\right\|^2\right) \\
&\qquad + c\int_{\tau - t}^{\tau} e^{\sigma s}\left(1 + \left|\omega(s)\right|^2 + \left\|g(x, s)\right\|_1^2\right) ds.
\end{aligned}
$$

Replacing ω by $\theta_{-\tau}\omega$ in the above, we obtain, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$
\begin{aligned}
&\left\|A^{\frac{1}{4}} v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\right\|^2 \\
&\qquad + \left\|A^{\frac{3}{4}} u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\right\|^2 \\
&\le e^{-\sigma t}\left(\left\|A^{\frac{1}{4}} v_0\right\|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left\|A^{\frac{1}{4}} u_0\right\|^2 + \left\|A^{\frac{3}{4}} u_0\right\|^2\right) \\
&\qquad + \frac{3}{(\beta_1 - \delta)}\int_{\tau - t}^{\tau} e^{\sigma(s - \tau)}\left\|g(x, s)\right\|_1^2 ds + c\int_{\tau - t}^{\tau} e^{\sigma(s - \tau)}\left(1 + \left|\theta_{-\tau}\omega(s)\right|^2\right) ds \\
&\le ce^{-\sigma t}\left(\left\|A^{\frac{1}{4}} v_0\right\|^2 + \left\|A^{\frac{1}{4}} u_0\right\|^2 + \left\|A^{\frac{3}{4}} u_0\right\|^2\right) \\
&\qquad + c\int_{-\infty}^{\tau} e^{\sigma(s - \tau)}\left\|g(x, s)\right\|_1^2 ds + c\int_{-\infty}^{0} e^{\sigma s}\left(1 + \left|\omega(s)\right|^2\right) ds. \qquad (4.34)
\end{aligned}
$$

Thus the proof is completed. □

Lemma 4.3 *Under* (3.4)–(3.9) *and* (3.14)–(3.17), *for every* $\eta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exist* $T = T(\tau,\omega,D,\eta) > 0$, $K = K(\tau,\omega,\eta) \geq 1$ *such that, for all* $t \geq T$, $k \geq K$, *the solution of problem* (3.10)–(3.11) *satisfies*

$$\left\| Y\big(\tau,\tau - t,\theta_{-\tau}\omega, D(\tau - t,\theta_{-t}\omega)\big)\right\|^2_{E(\mathbb{R}^n\setminus\mathbb{B}_k)} \leq \eta, \tag{4.35}$$

where for $k \geq 1$, $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ *and* $\mathbb{R}^n \setminus \mathbb{B}_k$ *is the complement of* $\mathbb{B}_k$.

Proof Choose a smooth function ρ such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases} \tag{4.36}$$

and there exist constants μ_1, μ_2, μ_3, μ_4 such that $|\rho'(s)| \leq \mu_1$, $|\rho''(s)| \leq \mu_2$, $|\rho'''(s)| \leq \mu_3$, $|\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of the second equation of (3.10) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx - \delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx \\
&\quad + \left(\lambda + \delta^2\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\,dx + \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)v\,dx \\
&= \delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)v\,dx - \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)h\big(v + \phi\omega(t) - \delta u\big)v\,dx \\
&\quad + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)v\,dx.
\end{aligned} \tag{4.37}$$

Similar to (4.5), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)h\big(v + \phi\omega(t) - \delta u\big)v\,dx \\
&= -\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\big(h\big(v + \phi\omega(t) - \delta u\big) - h(0)\big)v\,dx \\
&\leq -\beta_1\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + h'(\vartheta)\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\,dx \\
&\quad + \beta_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|\big|\omega(t)\big||v|\,dx.
\end{aligned} \tag{4.38}$$

Taking (4.38) into (4.37), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx - (\delta - \beta_1)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx \\
&\quad + \left(\lambda + \delta^2 - h'(\vartheta)\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\,dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)v\,dx \\
&\leq (1 + \delta + \beta_2)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|\big|\omega(t)\big||v|\,dx + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)v\,dx
\end{aligned}$$

$$\begin{aligned} &\leq \frac{\beta_1-\delta}{3}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|^2\left|\omega(t)\right|^2\,dx \\ &\quad + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)v\,dx. \end{aligned} \tag{4.39}$$

For the fourth term on the left-hand side of (4.39), we have

$$\begin{aligned} &\left(\lambda+\delta^2-h'(\vartheta)\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\,dx \\ &\quad = \left(\lambda+\delta^2-h'(\vartheta)\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)u\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)dx \\ &\quad = \left(\lambda+\delta^2-h'(\vartheta)\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{2}\frac{d}{dt}u^2+\delta u^2-\phi\omega(t)u\right)dx \\ &\quad \geq \left(\lambda+\delta^2-\beta_2\delta\right)\left(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx\right) \\ &\qquad - \left(\lambda+\delta^2-\beta_1\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|\left|\omega(t)\right||u|\,dx \\ &\quad \geq \left(\lambda+\delta^2-\beta_2\delta\right)\left(\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx\right) \\ &\qquad - \frac{\delta}{2}\left(\lambda+\delta^2-\beta_2\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx - c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|^2\left|\omega(t)\right|^2\,dx. \end{aligned} \tag{4.40}$$

For the third term on the left-hand side of (4.39), we have

$$\begin{aligned} &\int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)v\,dx \\ &\quad = \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)dx \\ &\quad = \int_{\mathbb{R}^n}(\Delta^2 u)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)dx \\ &\quad = \int_{\mathbb{R}^n}(\Delta u)\Delta\left(\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)\right)dx \\ &\quad = \int_{\mathbb{R}^n}(\Delta u)\left(\left(\frac{2}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)+\frac{4x^2}{k^4}\rho''\left(\frac{|x|^2}{k^2}\right)\right)\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)\right. \\ &\qquad \left. + 2\cdot\frac{2|x|}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)\nabla\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)+\rho\left(\frac{|x|^2}{k^2}\right)\Delta\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)\right)dx \\ &\quad \geq -\int_{k<x<\sqrt{2}k}\left(\frac{2\mu_1}{k^2}+\frac{4\mu_2x^2}{k^4}\right)\left|(\Delta u)v\right|\,dx - \int_{k<x<\sqrt{2}k}\frac{4\mu_1 x}{k^2}\left|(\Delta u)(\nabla v)\right|\,dx \\ &\qquad + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx \\ &\qquad - \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||\Delta\phi|\left|\omega(t)\right|\,dx \\ &\quad \geq -\int_{\mathbb{R}^n}\left(\frac{2\mu_1+8\mu_2}{k^2}\right)\left|(\Delta u)v\right|\,dx - \int_{\mathbb{R}^n}\frac{4\sqrt{2}\mu_1}{k}\left|(\Delta u)(\nabla v)\right|\,dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx\\
&-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||\Delta\phi|\left|\omega(t)\right|dx\\
\geq{}&-\frac{\mu_1+4\mu_2}{k^2}\left(\|\Delta u\|^2+\|v\|^2\right)-\frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\|+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx\\
&+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||\Delta\phi|\left|\omega(t)\right|dx\\
\geq{}&-\frac{\mu_1+4\mu_2}{k^2}\left(\|\Delta u\|^2+\|v\|^2\right)-\frac{2\sqrt{2}\mu_1}{k}\left(\|\Delta u\|^2+\|\nabla v\|^2\right)\\
&+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx\\
&-\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx-c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta\phi|^2\left|\omega(t)\right|^2dx,
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)v\,dx\\
&=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)\left(\frac{du}{dt}+\delta u-\phi\omega(t)\right)dx\\
&=\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\,dx+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\,dx\\
&\quad-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)\phi\omega(t)\,dx.
\end{aligned}
\tag{4.42}
$$

Similar to (4.12) and (4.13) in Lemma 4.1, we have

$$
\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\,dx\geq c_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\,dx+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\eta_2\,dx,
\tag{4.43}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)\phi\omega(t)\,dx\\
&\leq\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta_1|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|^2\left|\omega(t)\right|^2dx\\
&\quad+\frac{\delta c_2}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(F(x,u)+\eta_3\right)dx+c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\phi|^{p+1}\left|\omega(t)\right|^{p+1}dx,
\end{aligned}
\tag{4.44}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)v\,dx\\
&\leq\frac{3}{2(\beta_1-\delta)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left|g(x,t)\right|^2dx+\frac{\beta_1-\delta}{6}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx.
\end{aligned}
\tag{4.45}
$$

By (4.38)–(4.45), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\left(\lambda+\delta^2-\beta_2\delta\right)|u|^2+|\Delta u|^2+2F(x,u)\right)dx\\
&\quad+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\left(\lambda+\delta^2-\beta_2\delta\right)|u|^2+|\Delta u|^2\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta c_2}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\,dx\\
&\le\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+\frac{2\delta-\beta_1}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx\\
&+\frac{\delta}{2}\left(\lambda+\delta^2-\beta_2\delta\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx\\
&+\frac{\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\,dx+\frac{\mu_1+4\mu_2}{k^2}\left(\|\Delta u\|^2+\|v\|^2\right)\\
&+\frac{2\sqrt{2}\mu_1}{k}\left(\|\Delta u\|^2+\|\nabla v\|^2\right)\\
&+c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\eta_1|^2+|\eta_2|+|\eta_3|+|g|^2+\left|\omega(t)\right|^{p+1}|\phi|^{p+1}\right)dx\\
&+c\left|\omega(t)\right|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\phi|^2+|\Delta\phi|^2\right)dx.
\end{aligned}
\tag{4.46}
$$

Since $\phi\in H^2(\mathbb{R}^n)$, $\eta_1\in L^2(\mathbb{R}^n)$, $\eta_2\in L^1(\mathbb{R}^n)$, $\eta_3\in L^1(\mathbb{R}^n)$, we obtain that there exists $K_1=K_1(\tau,\eta)\ge 1$ such that, for all $k\ge K_1$,

$$
\begin{aligned}
&c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\eta_1|^2+|\eta_2|+|\eta_3|+\left|\omega(t)\right|^{p+1}|\phi|^{p+1}\right)dx\\
&\quad+c\left|\omega(t)\right|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\phi|^2+|\Delta\phi|^2\right)dx\\
&=c\int_{|x|\ge k}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\eta_1|^2+|\eta_2|+|\eta_3|+\left|\omega(t)\right|^{p+1}|\phi|^{p+1}\right)dx\\
&\quad+c\left|\omega(t)\right|^2\int_{|x|\ge k}\rho\left(\frac{|x|^2}{k^2}\right)\left(|\phi|^2+|\Delta\phi|^2\right)dx\\
&\le c\int_{|x|\ge k}\left(|\eta_1|^2+|\eta_2|+|\eta_3|+\left|\omega(t)\right|^{p+1}|\phi|^{p+1}\right)dx\\
&\quad+c\left|\omega(t)\right|^2\int_{|x|\ge k}\left(|\phi|^2+|\Delta\phi|^2\right)dx\\
&\le c\eta\left(1+\left|\omega(t)\right|^2+\left|\omega(t)\right|^{p+1}\right),
\end{aligned}
\tag{4.47}
$$

along with

$$
c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g^2(x,t)\,dx\le c\int_{|x|\ge k}g^2(x,t)\,dx,
\tag{4.48}
$$

we have that, for all $k\ge K_1$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\left(\lambda+\delta^2-\beta_2\delta\right)|u|^2+|\Delta u|^2+2F(x,u)\right)dx\\
&\quad+\sigma\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+\left(\lambda+\delta^2-\beta_2\delta\right)|u|^2+|\Delta u|^2+2F(x,u)\right)dx
\end{aligned}
$$

$$\leq \frac{2\mu_1 + 8\mu_2}{k^2}\left(\|\Delta u\|^2 + \|v\|^2\right) + \frac{4\sqrt{2}\mu_1}{k}\left(\|\Delta u\|^2 + \|\nabla v\|^2\right)$$
$$+ c\eta\left(1 + \left|\omega(t)\right|^2 + \left|\omega(t)\right|^{p+1}\right) + c\int_{|x|\geq k} g^2(x,t)\,dx. \tag{4.49}$$

Multiplying (4.49) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we find

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(\left|v(\tau,\tau-t,\omega,v_0)\right|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left|u(\tau,\tau-t,\omega,u_0)\right|^2\right.$$
$$\left.+ \left|\Delta u(\tau,\tau-t,\omega,u_0)\right|^2 + 2F\left(x, u(\tau,\tau-t,\omega,u_0)\right)\right)dx$$
$$\leq e^{-\sigma t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(|v_0|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)|u_0|^2 + |\Delta u_0|^2 + 2F(x,u_0)\right)dx$$
$$+ \frac{2\mu_1 + 8\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\omega,u_0)\right|^2 + \left|v(s,\tau-t,\omega,v_0)\right|^2\right)ds$$
$$+ \frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\omega,u_0)\right|^2 + \left|\nabla v(s,\tau-t,\omega,v_0)\right|^2\right)ds$$
$$+ c\frac{\eta}{\sigma} + c\eta\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\omega(s)\right|^2 + \left|\omega(s)\right|^{p+1}\right)ds$$
$$+ c\int_{\tau-t}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds. \tag{4.50}$$

Replacing ω by $\theta_{-\tau}\omega$, it then follows from (4.50) that

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(\left|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\right|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)\left|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2\right.$$
$$\left.+ \left|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 + 2F\left(x, u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right)\right)dx$$
$$\leq c\eta + e^{-\sigma t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(|v_0|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)|u_0|^2 + |\Delta u_0|^2 + 2F(x,u_0)\right)dx$$
$$+ \frac{2\mu_1 + 8\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 + \left|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right|^2\right)ds$$
$$+ \frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 + \left|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right|^2\right)ds$$
$$+ c\eta\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\theta_{-\tau}\omega(s)\right|^2 + \left|\theta_{-\tau}\omega(s)\right|^{p+1}\right)ds + c\int_{\tau-t}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds$$
$$\leq c\eta + e^{-\sigma t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(|v_0|^2 + \left(\lambda + \delta^2 - \beta_2\delta\right)|u_0|^2 + |\Delta u_0|^2 + 2F(x,u_0)\right)dx$$
$$+ \frac{2\mu_1 + 8\mu_2}{k^2}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 + \left|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right|^2\right)ds$$
$$+ \frac{4\sqrt{2}\mu_1}{k}\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 + \left|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right|^2\right)ds$$
$$+ c\eta\int_{-\infty}^{0} e^{\sigma s}\left(\left|\omega(s)\right|^2 + \left|\omega(s)\right|^{p+1}\right)ds + c\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds. \tag{4.51}$$

By (3.17), we see that there exists $K_2 = K_2(\tau,\eta) \geq K_1$ such that, for all $k \geq K_2$,

$$c\int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds \leq \eta. \tag{4.52}$$

It follows from (4.51)–(4.52), Lemma 4.1, and Lemma 4.2 that there exists $T_1 = T_1(\tau,\omega, D,\eta) > 0$ such that, for all $t \geq T_1$, $k \geq K_2$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\Big(\big|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\big|^2 + \big(\lambda+\delta^2-\beta_2\delta\big)\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&\qquad + \big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 + 2F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\Big)\,dx \\
&\quad \leq c\eta\left(1+\int_{-\infty}^{0} e^{\sigma s}\big(|\omega(s)|^2+|\omega(s)|^{p+1}\big)\,ds + \int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds\right),
\end{aligned} \tag{4.53}$$

where $(u_0,v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$.

Note that (3.6) holds, then we find

$$-2\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\,dx \leq 2\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\eta_3\,dx \leq 2\int_{|x|\geq k} \rho\left(\frac{|x|^2}{k^2}\right)\eta_3\,dx,$$

from which along with $\eta_3 \in L^1(\mathbb{R}^n)$, we see that there exists $K_3 = K_3(\tau,\eta) \geq K_2$ such that, for all $k \geq K_3$,

$$-2\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\,dx \leq \eta. \tag{4.54}$$

Then from (4.53)–(4.54) we know that there exists $T_2 = T_2(\tau,\omega,D,\eta) > T_1$ such that, for all $t \geq T_2$ and $k \geq K_3$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\Big(\big|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\big|^2 + \big(\lambda+\delta^2-\beta_2\delta\big)\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&\qquad + \big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2\Big)\,dx \\
&\quad \leq c\eta\left(1+\int_{-\infty}^{0} e^{\sigma s}\big(|\omega(s)|^2+|\omega(s)|^{p+1}\big)\,ds + \int_{-\infty}^{\tau}\int_{|x|\geq k} e^{\sigma(s-\tau)} g^2(x,s)\,dx\,ds\right),
\end{aligned} \tag{4.55}$$

which completes the proof. □

Let $\widehat{\rho} = 1-\rho$ with ρ given by (4.36). Fix $k \geq 1$, and set

$$\begin{cases} \widehat{u}(t,\tau,\omega,\widehat{u}_0) = \widehat{\rho}(\frac{|x|^2}{k^2})u(t,\tau,\omega,u_0), \\ \widehat{v}(t,\tau,\omega,\widehat{v}_0) = \widehat{\rho}(\frac{|x|^2}{k^2})v(t,\tau,\omega,v_0). \end{cases} \tag{4.56}$$

By (3.10)–(3.11) we find that $\widehat{u}$ and $\widehat{v}$ satisfy the following system in $\mathbb{B}_{2k} = \{x \in \mathbb{R}^n : |x| < 2k\}$:

$$\frac{d\widehat{u}}{dt} = \widehat{v} + \widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t) - \delta\widehat{u}, \tag{4.57}$$

$$
\begin{aligned}
&\frac{d\widehat{v}}{dt} - \delta\widehat{v} + \left(\delta^2 + \lambda + A\right)\widehat{u} + \widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u) \\
&\quad = \widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t) - \widehat{\rho}\left(\frac{|x|^2}{k^2}\right)h\left(v + \phi\omega(t) - \delta u\right) + (1+\delta)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t) \\
&\qquad + 4\Delta\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\nabla u + 6\Delta\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\Delta u \\
&\qquad + 4\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\Delta\nabla u + u\Delta^2\widehat{\rho}\left(\frac{|x|^2}{k^2}\right),
\end{aligned} \tag{4.58}
$$

with boundary conditions

$$
\widehat{u} = \widehat{v} = 0 \quad \text{for } |x| = 2k. \tag{4.59}
$$

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{B}_{2k})$ such that $Ae_n = \lambda_n e_n$ with zero boundary condition in $\mathbb{B}_{2k}$. Given n, let $X_n = \operatorname{span}\{e_1, \ldots, e_n\}$ and $P_n : L^2(\mathbb{B}_{2k}) \to X_n$ be the projection operator.

Lemma 4.4 *Under* (3.4)–(3.9) *and* (3.14)–(3.17), *for every* $\eta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exist* $T = T(\tau,\omega,D,\eta) > 0$, $K = K(\tau,\omega,\eta) \geq 1$, *and* $N = N(\tau,\omega,\eta) \geq 1$ *such that, for all* $t \geq T$, $k \geq K$, *and* $n \geq N$, *the solution of problem* (4.57)–(4.59) *satisfies*

$$
\left\|(I - P_n)\widehat{Y}\left(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega)\right)\right\|_{E(\mathbb{B}_{2k})}^2 \leq \eta.
$$

Proof Let $\widehat{u}_{n,1} = P_n\widehat{u}$, $\widehat{u}_{n,2} = (I - P_n)\widehat{u}$, $\widehat{v}_{n,1} = P_n\widehat{v}$, $\widehat{v}_{n,2} = (I - P_n)\widehat{v}$. Applying $I - P_n$ to (4.57), we obtain

$$
\widehat{v}_{n,2} = \frac{d\widehat{u}_{n,2}}{dt} + \delta\widehat{u}_{n,2} - (I - P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t). \tag{4.60}
$$

Applying $I - P_n$ to (4.58) and taking the inner product with $\widehat{v}_{n,2}$ in $L^2(\mathbb{B}_{2k})$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{v}_{n,2}\|^2 - \delta\|\widehat{v}_{n,2}\|^2 + \left(\lambda + \delta^2 + A\right)(\widehat{u}_{n,2}, \widehat{v}_{n,2}) + \left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u), \widehat{v}_{n,2}\right) \\
&\quad = \left((I - P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t), \widehat{v}_{n,2}\right) + (1+\delta)\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t), \widehat{v}_{n,2}\right) \\
&\qquad - (I - P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\left(h\left(v + \phi\omega(t) - \delta u\right), \widehat{v}_{n,2}\right) \\
&\qquad + \left(4\Delta\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\nabla u + 6\Delta\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\Delta u\right. \\
&\qquad \left. + 4\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\Delta\nabla u + u\Delta^2\widehat{\rho}\left(\frac{|x|^2}{k^2}\right), \widehat{v}_{n,2}\right).
\end{aligned} \tag{4.61}
$$

Substituting $\widehat{v}_{n,2}$ in (4.60) into the third term on the left-hand side of (4.61), we have

$$\begin{aligned}\left(\lambda+\delta^2\right)(\widehat{u}_{n,2},\widehat{v}_{n,2}) &= \left(\widehat{u}_{n,2},\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right)\\ &\geq \frac{1}{2}\left(\lambda+\delta^2\right)\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\delta\left(\lambda+\delta^2\right)\|\widehat{u}_{n,2}\|^2-\frac{1}{4}\delta\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2\\ &\quad -c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2,\end{aligned}\tag{4.62}$$

and then

$$\begin{aligned}(A\widehat{u}_{n,2},\widehat{v}_{n,2}) &= \left(\Delta\widehat{u}_{n,2},\Delta\left(\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right)\right)\\ &\geq \frac{1}{2}\frac{d}{dt}\|\Delta\widehat{u}_{n,2}\|^2+\frac{3\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2\\ &\quad -c\left\|(I-P_n)\Delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right)\right\|^2\left|\omega(t)\right|^2.\end{aligned}\tag{4.63}$$

For the fourth term on the left-hand side of (4.61), we have

$$\begin{aligned}&\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{v}_{n,2}\right)\\ &= \left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right)\\ &= \frac{d}{dt}\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)-\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f_u'(x,u)u_t,\widehat{u}_{n,2}\right)\\ &\quad +\delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)-\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right).\end{aligned}\tag{4.64}$$

For the third term on the right-hand side of (4.61), we have

$$\begin{aligned}&-(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\left(h\left(v+\phi\omega(t)-\delta u\right),\widehat{v}_{n,2}\right)\\ &= -(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\left(h\left(v+\phi\omega(t)-\delta u\right)-h(0),\widehat{v}_{n,2}\right)\\ &\leq -\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta(\widehat{u}_{n,2},\widehat{v}_{n,2})+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2\\ &\leq -\beta_1\|\widehat{v}_{n,2}\|^2+h'(\vartheta)\delta\left(\widehat{u}_{n,2},\frac{d\widehat{u}_{n,2}}{dt}+\delta\widehat{u}_{n,2}-(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right)\\ &\quad +\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2\\ &\leq -\beta_1\|\widehat{v}_{n,2}\|^2+\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\beta_2\delta^2\|\widehat{u}_{n,2}\|^2\\ &\quad +\beta_2\delta\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|\left|\omega(t)\right|\|\widehat{u}_{n,2}\|\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2\\
&\le -\beta_1\|\widehat{v}_{n,2}\|^2+\beta_2\delta\cdot\frac{1}{2}\frac{d}{dt}\|\widehat{u}_{n,2}\|^2+\beta_2\delta^2\|\widehat{u}_{n,2}\|^2+\frac{1}{4}\delta\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2\\
&\quad+\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2.
\end{aligned}
\tag{4.65}
$$

Using the Cauchy–Schwarz inequality and Young's inequality, we get

$$
\begin{aligned}
&\delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t),\widehat{v}_{n,2}\right)\\
&\quad\le\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2,
\end{aligned}
\tag{4.66}
$$

$$
\begin{aligned}
&\left((I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t),\widehat{v}_{n,2}\right)\\
&\quad\le\frac{\beta_1-\delta}{6}\|\widehat{v}_{n,2}\|^2+c\left\|(I-P_n)\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t)\right)\right\|^2.
\end{aligned}
\tag{4.67}
$$

Now, we estimate the last term in (4.61)

$$
\begin{aligned}
&\left(4\Delta\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\cdot\nabla u+6\Delta\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\cdot\Delta u+4\nabla\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\cdot\Delta\nabla u+u\Delta^2\widehat{\rho}\left(\frac{|x|^2}{k^2}\right),\widehat{v}_{n,2}\right)\\
&\quad=\left(4\nabla u\cdot\left(\frac{12|x|}{k^4}\widehat{\rho}''\left(\frac{|x|^2}{k^2}\right)+\frac{8|x|^3}{k^6}\widehat{\rho}'''\left(\frac{|x|^2}{k^2}\right)\right)\right.\\
&\qquad+6\Delta u\cdot\left(\frac{2}{k^2}\widehat{\rho}'\left(\frac{|x|^2}{r^2}\right)+\frac{4x^2}{k^4}\widehat{\rho}''\left(\frac{|x|^2}{k^2}\right)\right)+\frac{8|x|}{k^2}\Delta\nabla u\cdot\widehat{\rho}'\left(\frac{|x|^2}{k^2}\right)\\
&\qquad\left.+u\left(\frac{12}{k^4}\widehat{\rho}''\left(\frac{|x|^2}{k^2}\right)+\frac{48x^2}{k^6}\widehat{\rho}'''\left(\frac{|x|^2}{k^2}\right)+\frac{16x^4}{k^8}\widehat{\rho}''''\left(\frac{|x|^2}{k^2}\right)\right),\widehat{v}_{n,2}\right)\\
&\quad\le\frac{16\sqrt{2}(3\mu_2+4\mu_3)}{k^3}\|\nabla u\|\cdot\|\widehat{v}_{n,2}\|+\frac{12(\mu_1+4\mu_2)}{k^2}\|\Delta u\|\cdot\|\widehat{v}_{n,2}\|\\
&\qquad+\frac{8\sqrt{2}\mu_1}{k}\left\|A^{\frac{3}{4}}u\right\|\cdot\|\widehat{v}_{n,2}\|+\frac{4(3\mu_2+24\mu_3+16\mu_4)}{k^4}\|u\|\cdot\|\widehat{v}_{n,2}\|\\
&\quad\le\frac{8(48\mu_2+64\mu_3)^2}{(\beta_1-\delta)k^6}\|\nabla u\|^2+\frac{4(12\mu_1+48\mu_2)^2}{(\beta_1-\delta)k^4}\|\Delta u\|^2+\frac{512\mu_1^2}{(\beta_1-\delta)k^2}\left\|A^{\frac{3}{4}}u\right\|^2\\
&\qquad+\frac{4(12\mu_2+96\mu_3+64\mu_4)^2}{(\beta_1-\delta)k^8}\|u\|^2+\frac{\beta_1-\delta}{4}\|\widehat{v}_{n,2}\|^2.
\end{aligned}
\tag{4.68}
$$

Assemble together (4.61)–(4.68) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\widehat{v}_{n,2}\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)\right]\\
&\qquad+\delta\left[\|\widehat{v}_{n,2}\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2\right]+\delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)\\
&\quad\le\frac{\delta}{2}\|\widehat{v}_{n,2}\|^2+\frac{3\delta-\beta_1}{4}\|\widehat{v}_{n,2}\|^2+\frac{\delta}{2}\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2+\frac{\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2
\end{aligned}
$$

$$+\frac{2}{\beta_1-\delta}\left(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\left\|A^{\frac{3}{4}}u\right\|^2\right.$$
$$\left.+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\right)+c\left\|(I-P_n)\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t)\right)\right\|^2$$
$$+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2+c\left\|(I-P_n)\Delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right)\right\|^2\left|\omega(t)\right|^2$$
$$+\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f'_u(x,u)u_t,\widehat{u}_{n,2}\right)+\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right). \quad (4.69)$$

For the nonlinear terms in (4.69), by (3.7), using Cauchy's inequality and Young's inequality, we obtain

$$\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f'_u(x,u)u_t,\widehat{u}_{n,2}\right)\le\frac{\delta}{4}\|\Delta\widehat{u}_{n,2}\|^2+c\lambda_{n+1}^{-1}\|u_t\|^2. \quad (4.70)$$

By (3.4), we know

$$\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\omega(t)\right)$$
$$\le c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|\left|\omega(t)\right|+c\|u\|^p_{H^2(\mathbb{R}^n)}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|\left|\omega(t)\right|. \quad (4.71)$$

Since $1\le p\le\frac{n+4}{n-4}$ and $\lambda_n\to\infty$, by Lemmas 4.1 and 4.2, there are $N_1=N(\eta)$, $K_1=K(\eta)$ such that, for all $n\ge N_1$, $k\ge K_1$,

$$\frac{2}{\beta_1-\delta}\left(\frac{4(48\mu_2+64\mu_3)^2}{k^6}\|\nabla u\|^2+\frac{2(12\mu_1+48\mu_2)^2}{k^4}\|\Delta u\|^2+\frac{256\mu_1^2}{k^2}\left\|A^{\frac{3}{4}}u\right\|^2\right.$$
$$\left.+\frac{2(12\mu_2+96\mu_3+64\mu_4)^2}{k^8}\|u\|^2\right)+c\lambda_{n+1}^{-\frac{1}{2}}\|u_t\|^2$$
$$+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|\left|\omega(t)\right|+c\|u\|^p_{H^2(\mathbb{R}^n)}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|\left|\omega(t)\right|$$
$$+c\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right\|^2\left|\omega(t)\right|^2+c\left\|(I-P_n)\Delta\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)\phi\right)\right\|^2\left|\omega(t)\right|^2$$
$$\le c\eta\left(1+\left|\omega(t)\right|^2+\|u_t\|^2+\|u\|^{18}_{H^2(\mathbb{R}^n)}\right). \quad (4.72)$$

Then, by (4.69)–(4.72), we obtain

$$\frac{d}{dt}\left[\|\widehat{v}_{n,2}\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)\right]$$
$$+\sigma\left[\|\widehat{v}_{n,2}\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\widehat{u}_{n,2}\|^2+\|\Delta\widehat{u}_{n,2}\|^2+2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}\right)\right]$$
$$\le c\eta\left(1+\left|\omega(t)\right|^2+\|u_t\|^2+\|u\|^{18}_{H^2(\mathbb{R}^n)}\right)+c\left\|(I-P_n)\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,t)\right)\right\|^2. \quad (4.73)$$

Multiplying (4.73) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have for all $n > N_1$ and $k > K_1$

$$\begin{aligned}
&\left\|\widehat{v}_{n,2}(\tau,\tau-t,\omega)\right\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\left\|\widehat{u}_{n,2}(\tau,\tau-t,\omega)\right\|^2 + \left\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\omega)\right\|^2 \\
&\quad + 2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\omega)\right) \\
&\le e^{-\sigma t}\left(\|v_0\|^2 + \|u_0\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\|\Delta u_0\|^2 + 2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),u_0\right)\right) \\
&\quad + c\eta\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(1+\left|\omega(s)\right|^2 + \left\|u_t(s,\tau-t,\omega,u_0)\right\|^2 + \left\|u(s,\tau-t,\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^{18}\right)ds \\
&\quad + c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x,s)\right\|^2 ds. \qquad (4.74)
\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$, by a similar process as in Lemma 4.1, we get

$$\begin{aligned}
&\left\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\left\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 \\
&\quad + \left\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 + 2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u),\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right) \\
&\le ce^{-\sigma t}\left(1+\|v_0\|^2 + \|u_0\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\|\Delta u_0\|^2 + \|u_0\|_{H^2(\mathbb{R}^n)}^2 + \|u_0\|_{H^2(\mathbb{R}^n)}^{p+1}\right) \\
&\quad + c\eta\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(1+\left|\theta_{-\tau}\omega(s)\right|^2 + \left\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\right. \\
&\quad \left.+ \left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^{18}\right)ds + c\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x,s)\right\|^2 ds \\
&\le ce^{-\sigma t}\left(1+\|v_0\|^2 + \|u_0\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\|\Delta u_0\|^2 + \|u_0\|_{H^2(\mathbb{R}^n)}^2 + \|u_0\|_{H^2(\mathbb{R}^n)}^{p+1}\right) \\
&\quad + c\eta\int_{-\infty}^{0} e^{\sigma s}\left(1+\left|\omega(s)\right|^2\right)ds + c\eta\int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}\left(\left\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\right. \\
&\quad \left.+ \left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^{18}\right)ds \\
&\quad + c\int_{-\infty}^{0} e^{\sigma s}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x,s+\tau)\right\|^2 ds. \qquad (4.75)
\end{aligned}$$

Using the first equation of (3.10) as well as the Minkowski inequality, we can obtain

$$\begin{aligned}
&\left\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\
&= \left\|-\delta u(s,\tau-t,\theta_{-\tau}\omega,u_0) + v(s,\tau-t,\theta_{-\tau}\omega,v_0) + \phi\theta_{-\tau}\omega\right\|^2 \\
&\le c\left(\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 + \left\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2 + |\theta_{-\tau}\omega|^2\right) \\
&\le cR_1(\tau,\omega) + c|\theta_{-\tau}\omega|^2 \qquad (4.76)
\end{aligned}$$

and

$$\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^{18} \le cR_1^9(\tau,\omega), \qquad (4.77)$$

where $c = \max\{\delta, \|\phi\|^2, 1\}$ and $R_1(\tau,\omega)$ is given in Lemma 4.1. Hence, it follows from (4.75)–(4.77) that

$$\begin{aligned}
&\left\|\widehat{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 + \left(\lambda+\delta^2-\beta_2\delta\right)\left\|\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 \\
&\quad + \left\|\Delta\widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right\|^2 + 2\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)f(x,u), \widehat{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega)\right) \\
&\le e^{-\sigma t}\left(1+\|v_0\|^2+\|u_0\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\Delta u_0\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)}\right) \\
&\quad + c\eta R_1^9(\tau,\omega) + c\eta\int_{-\infty}^0 e^{\sigma s}\left(1+\left|\omega(s)\right|^2\right)ds \\
&\quad + c\int_{-\infty}^0 e^{\sigma s}\left\|(I-P_n)\widehat{\rho}\left(\frac{|x|^2}{r^2}\right)g(x,s+\tau)\right\|^2 ds. \qquad (4.78)
\end{aligned}$$

Since $(u_0,v_0)^\top \in D(\tau-t,\theta_{-t}\omega)$ and $D \in \mathcal{D}$, then

$$\begin{aligned}
&e^{-\sigma t}\left(1+\|v_0\|^2+\|u_0\|^2+\left(\lambda+\delta^2-\beta_2\delta\right)\|\Delta u_0\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_0\|^{p+1}_{H^2(\mathbb{R}^n)}\right) \\
&\quad \to 0, \quad t\to\infty. \qquad (4.79)
\end{aligned}$$

For the last term on the right-hand side of (4.78), by (3.16), there exists $N_2 = N_2(\tau,\omega,\eta) \ge N_1$ such that, for all $n \ge N_2$,

$$\int_{-\infty}^0 e^{\sigma s}\left\|(I-P_n)\left(\widehat{\rho}\left(\frac{|x|^2}{k^2}\right)g(x,s+\tau)\right)\right\|^2 ds < \eta. \qquad (4.80)$$

The proof is completed by (3.4), (4.79)–(4.80), and Lemma 4.1. □

5 Random attractors

In this section, we prove the existence of a $\mathcal{D}$-attractor for the stochastic system (3.10)–(3.11). We firstly apply the lemmas shown in Sect. 4 to derive the asymptotic compactness of solutions of (3.10)–(3.11).

Lemma 5.1 *Under (3.4)–(3.9) and (3.14)–(3.17), for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, the sequence of weak solutions of (3.10)–(3.11) $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega))\}_{m=1}^\infty$ has a convergent subsequence in E whenever $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$.*

Proof Let $t_m\to\infty$ and $Y_0(\theta_{-t_m}\omega)\in D(\tau-t_m,\theta_{-t_m}\omega)$ with $D\in\mathcal{D}$. By Lemma 4.1, there exists $m_1 = m_1(\tau,\omega,D) > 0$ such that, for all $m\ge m_1$, we have

$$\left\|Y\left(\tau,\tau-t_m,\theta_{-\tau}\omega, Y_0(\theta_{-t_m}\omega)\right)\right\|_E^2 \le R_1(\tau,\omega). \qquad (5.1)$$

By Lemma 4.3, for every $\eta>0$, there exist $k_0 = r_0(\tau,\omega,\eta)\ge 1$ and $m_2 = m_2(\tau,\omega,D,\eta)\ge m_1$ such that, for all $m\ge m_2$,

$$\left\|Y\left(\tau,\tau-t,\theta_{-\tau}\omega, D(\tau-t,\theta_{-t}\omega)\right)\right\|^2_{E(\mathbb{R}^n\setminus\mathbb{B}_{k_0})} \le \eta. \qquad (5.2)$$

By Lemma 4.4, there exist $k_1 = k_1(\tau,\omega,\eta) \geq k_0$ and $m_3 = m_3(\tau,\omega,D,\eta) \geq m_2$ and $n_1 = n_1(\tau,\omega,\eta) \geq 0$ such that, for all $m \geq m_3$,

$$\big\|(I-P_n)\widehat{Y}\big(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega)\big)\big\|^2_{E(\mathbb{B}_{2k_1})} \leq \eta. \tag{5.3}$$

Using (4.56) and (5.1), we get

$$\big\|P_n\widehat{Y}\big(\tau,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega)\big)\big\|^2_{P_nE(\mathbb{B}_{2k_1})} \leq cR_1(\tau,\omega), \tag{5.4}$$

which together with (5.3) implies that $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{2k_1})$. Note that $\widehat{\rho}(\frac{|x|^2}{k_1^2}) = 1$ for $|x| \leq k_1$. Therefore, $\{Y(\tau,\tau-t_m,\theta_{-\tau}\omega,Y_0(\theta_{-t_m}\omega))\}$ is precompact in $E(\mathbb{B}_{k_1})$, which along with (5.2) shows the precompactness of this sequence in E. □

Theorem 5.1 *Under (3.4)–(3.9) and (3.14)–(3.17), the random dynamical system Φ generated by the stochastic plate equation (3.10)–(3.11) has a unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in the space E.*

Proof Notice that Φ is pullback $\mathcal{D}$-asymptotically compact in E by Lemma 5.1 and has a pullback $\mathcal{D}$-absorbing set by Lemma 4.1. Thus the existence of a unique $\mathcal{D}$-attractor follows from Proposition 2.1 immediately. □

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Authors' contributions
This paper completed by XY deals with the existence of pullback attractors for the non-autonomous stochastic plate equations with additive noise and nonlinear damping on unbounded domains. The author read and approved the final manuscript.

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