Xiao et al. *Boundary Value Problems* (2020) 2020:58
https://doi.org/10.1186/s13661-020-01354-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Pullback attraction in H_0^1 for semilinear heat equation in expanding domains

Yanping Xiao[1*], Yuqin Bai[1] and Huanhuan Zhang[1]

*Correspondence:
hardxiao@126.com
[1]College of Mathematics and Computer Science, Northwest Minzu University, Lanzhou, P.R. China

Abstract

In this article, we consider the pullback attraction in H_0^1 of pullback attractor for semilinear heat equation with domains expanding in time. Firstly, we establish higher-order integrability of difference about variational solutions; then, we prove the continuity of variational solution in $H_0^1(O_t)$. As application of continuity, we obtain the pullback $\mathscr{D}_{\lambda_1}$ attraction in H_0^1-norm.

MSC: 35K57; 35L05; 35B40; 35B41

Keywords: Pullback $\mathscr{D}_{\lambda_1}$ attraction; Continuity; Variational solution

1 Introduction

Let $\{\mathcal{O}_t\}_{t\in\mathbb{R}}$ be a family of nonempty bounded open subsets of $\mathbb{R}^N$ such that

$$\mathcal{O}_s \subset \mathcal{O}_t, \quad s < t. \tag{1}$$

Define

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\}, \qquad \tilde{Q}_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_T \times \{t\} \quad \text{for any } T > \tau \tag{2}$$

and

$$Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}, \quad \forall \tau \in \mathbb{R},$$

$$\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\}, \qquad \Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}, \quad \forall \tau < T.$$

We consider the following initial boundary value problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_\tau, \\ u = 0 \quad \text{on } \Sigma_\tau, \\ u(\tau,x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{3}$$

where $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ and $f : Q_\tau \to \mathbb{R}$ are given for $\tau \in \mathbb{R}$, and $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies the conditions: there exist nonnegative constants α_1, α_2, β, l, and $p \geq 2$ such that

$$-\beta + \alpha_1|s|^p \leq g(s)s \leq \beta + \alpha_2|s|^p, \quad \forall s \in \mathbb{R} \tag{4}$$

and

$$g'(s) \geq -l, \quad \forall s \in \mathbb{R}. \tag{5}$$

For later observe that there exist nonnegative constants $\tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\beta}$ such that

$$-\tilde{\beta} + \tilde{\alpha_1}|s|^p \leq G(s) \leq \tilde{\beta} + \tilde{\alpha_2}|s|^p, \quad \forall s \in \mathbb{R}, \tag{6}$$

where

$$G(s) := \int_0^s g(r)\,dr.$$

For each $T > \tau$, consider the auxiliary problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \quad \text{on } \Sigma_{\tau,T}, \\ u(\tau,x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{7}$$

where $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ for $\tau \in \mathbb{R}$, g satisfies (4)–(5) and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathcal{O}_t))$.

The issue of non-cylindrical region usually refers to the problem that spatial region changes with time, also known as the problem of variable region. Variable region problems are applied widely in physics, chemistry, and cybernetics, so have been focused on relevant experts. Compared with the invariant regional system, the study of variable regional problem can vividly describe the actual phenomena. In addition, the problem defined in the variable region is essentially non-autonomous, so the discussion of variable regional problem adds vitality to the development and perfection of theory of non-autonomous system.

Based on the actual requirement, many mathematical researchers began to focus on the variable region problems, for example, see [1, 4–8, 15, 16] and so on. Recently, the existence and uniqueness of variational solution of system (3) have been considered in [6] with monotonic increase region, and then (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor has been established. In 2009, by means of differ-morphism, a similar conclusion of system (3) was obtained in [7]. Later, in [11], by the solution orbit being shifted via a fixed complete orbit, the authors obtained the pullback $\mathscr{D}_{\lambda_1}$ attraction of L^2 pullback attractor in higher-order integrable spaces.

The continuity of solution plays an important role in the study of dynamic systems, especially in pullback attraction, fractal dimension, and so on. For the invariant region, the continuity of strong solution with respect to the initial data in $H^1_0(\mathcal{O})$ was considered for the space dimension $N \leq 2$, and the nonlinear term exponent $p \geq 2$, but $p \leq 4$ for $N = 3$ was required. For an autonomous system, in order to obtain continuity in $H^1_0(\mathcal{O})$ and

$L^p(\mathcal{O})$, the concept of norm-to-weak continuity was given in [12], and then the existence of global attractor was established. Then, the norm-to-weak continuity concept to the case of a non-autonomous system was studied in [10]. However, for a long time, the continuity of solution in $H_0^1(\mathcal{O})$ with respect to initial data has still been an open problem. Until 2008, when the nonlinear term f of autonomous system satisfying (4) and (5) was introduced, the author obtained the uniform boundedness of $tu(t)$ by differentiating equation about time t, then considered the continuity of solution about initial data, see details in [14]. However, for a non-autonomous system, we cannot differentiate equation, so the method in [14] cannot be shifted to solve non-autonomous problems. In order to overcome the difficulties deriving from the non-autonomous character, in 2015, for the case of random equation, [2] discussed the continuity in $H_0^1(\mathcal{O})$ by studying the higher-order integrability of solutions difference near the initial time. Then, a natural problem arose: Does it still hold for variable domains? As far as the author knows, the continuity of solution in $H_0^1(\mathcal{O}_t)$ about initial data is still unknown.

Enlightened by the above, we consider the continuity of variational solution in $H_0^1(\mathcal{O}_t)$ with respect to initial data when the region of system (3) is monotonically increasing. As an application of continuity, we establish the pullback $\mathscr{D}_{\lambda_1}$ attraction in $H_0^1(\mathcal{O}_t)$ for any $t \in \mathbb{R}$.

This paper is organized as follows. In Sect. 2, we recall some concepts and related results about variational solution. In Sect. 3, we prove higher-order integrability of difference of variational solutions near initial data (*Theorem* 3.3) and the continuity in $H_0^1(\mathcal{O}_t)$ (*Theorem* 3.4), then establish the pullback $\mathscr{D}_{\lambda_1}$ attraction in $H_0^1(\mathcal{O}_t)$ (*Theorem* 3.5).

2 Variational solutions

For each $t \in \mathbb{R}$, denoted by $(\cdot,\cdot)_t$ and $|\cdot|_t$ the usual inner product and related norm in $L^2(\mathcal{O}_t)$ and by $((\cdot,\cdot))_t$ and $\|\cdot\|_t$ the usual gradient inner product and associated norm in $H_0^1(\mathcal{O}_t)$. The usual duality product between $H_0^1(\mathcal{O}_t)$ and $H^{-1}(\mathcal{O}_t)$ is denoted by $\langle\cdot,\cdot\rangle_t$. And $(\cdot,\cdot)_t$ and $\|\cdot\|_{L^p(\mathcal{O}_t)}$ represent the duality product between $L^p(\mathcal{O}_t)$ and $L^q(\mathcal{O}_t)$ with $\frac{1}{p}+\frac{1}{q}=1$ and the associated norm.

We consider a process U on a Banach space X, i.e., a family $\{U(t,\tau); -\infty < \tau \le t < +\infty\}$ of continuous mappings $U(t,\tau): X \to X$ such that

$$U(\tau,\tau)x = x \quad \text{and} \quad U(t,\tau) = U(t,s)U(s,\tau) \quad \text{for all } \tau \le s \le t \text{ and } x \in X.$$

Suppose that $\mathscr{D}$ is a nonempty class of parameterized sets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 2.1 ([3]) The family $\hat{\mathscr{A}} = \{\mathscr{A}(t) : \mathscr{A}(t) \in \mathcal{P}(X), t \in \mathbb{R}\}$ is said to be a pullback $\mathscr{D}$-attractor for the process $U(\cdot,\cdot)$ if

(1) $\mathscr{A}(t)$ is compact in X for all $t \in \mathbb{R}$;

(2) $\hat{\mathscr{A}}$ is pullback $\mathscr{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty} \operatorname{dist}_X\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) = 0 \quad \text{for all } \hat{D} \in \mathscr{D} \text{ and all } t \in \mathbb{R};$$

(3) $\hat{\mathscr{A}}$ is invariant, i.e.,

$$U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t) \quad \text{for any } -\infty < \tau \le t < \infty.$$

Fix $T > \tau$, for each $t \in [\tau, T]$ denoted by

$$H_0^1(\mathcal{O}_t)^{\perp} = \{v \in H_0^1(\mathcal{O}_T) : ((v, w))_T = 0, \forall w \in H_0^1(\mathcal{O}_t)\}$$

is the orthogonal subspace of $H_0^1(\mathcal{O}_t)$ with respect to the inner product in $H_0^1(\mathcal{O}_T)$. We may identify w with its null-expansion and by $P(t) \in L(H_0^1(\mathcal{O}_T))$ the orthogonal projection operator from $H_0^1(\mathcal{O}_T)$ to $H_0^1(\mathcal{O}_t)^{\perp}$, which is defined as

$$P(t)v \in H_0^1(\mathcal{O}_t)^{\perp}, \quad v - P(t)v \in H_0^1(\mathcal{O}_t)$$

for each $v \in H_0^1(\mathcal{O}_T)$. Consider the family $p(t; \cdot, \cdot)$ of symmetric bilinear forms on $H_0^1(\mathcal{O}_T)$ defined by

$$p(t; v, w) := ((P(t)v, w))_T, \quad \forall v, w \in H_0^1(\mathcal{O}_T), \forall t \geq \tau.$$

It can be proved that the mapping $[\tau, +\infty) \ni t \to p(t; v, w)$ is measurable for all $v, w \in H_0^1(\mathcal{O}_T)$. For each integer $k \geq 1$ and $t \geq \tau$, define

$$p_k(t; v, w) := k \int_0^{\frac{1}{k}} p(t + r; v, w)\, dr \quad \forall v, w \in H_0^1(\mathcal{O}_T), \forall t \geq \tau,$$

and denote by $P_k(t) \in L(H_0^1(\mathcal{O}_T))$ the associated operator defined by

$$((P_k(t)v, w)) := p_k(t; v, w), \quad \forall v, w \in H_0^1(\mathcal{O}_T), \forall t \geq \tau.$$

Then we know from the above that, for any integers $1 \leq h \leq k$, any $t \geq \tau$, and any $v, w \in H_0^1(\mathcal{O}_T)$,

$$0 \leq p_h(t; v, v) \leq p_k(t; v, v) \leq p(t; v, v) = \|P(t)v\|_T^2 \leq \|v\|_T^2. \tag{8}$$

For each $T > \tau$, denote

$$\begin{aligned} \mathcal{U}_{\tau,T} := \{&\phi \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap L^p(\tau, T; L^p(\mathcal{O}_T)), \phi' \in L^2(\tau, T; L^2(\mathcal{O}_T)) \\ &\text{and } \phi(\tau) = \phi(T) = 0, \phi(t) \in H_0^1(\mathcal{O}_t) \text{ for a.e. } t \in (\tau, T)\}. \end{aligned}$$

Definition 2.2 A variational solution of equation (7) is a function u such that

(C1) $u \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap L^p(\tau, T; L^p(\mathcal{O}_T))$;
(C2) $u(t) \in H_0^1(\mathcal{O}_t)$ a.e. $t \in (\tau, T)$;
(C3) for all $\phi \in \mathcal{U}_{\tau,T}$,

$$\int_\tau^T [-(u(t), \phi'(t))_T + ((u(t), \phi(t)))_T + (g(u(t)), \phi(t))_T]\, dt = \int_\tau^T (f(t), \phi(t))_T\, dt;$$

(C4) $\lim_{t \to \tau} \frac{1}{t-\tau} \int_\tau^t |u(s) - u(\tau)|_T^2\, ds = 0$.

The existence and uniqueness of variational solution for equation (7) have been derived as follows.

Theorem 2.3 ([6]) *Suppose that* (1), (2), (4), *and* (5) *hold; for* $f \in L^2(\tau, T; L^2(\mathcal{O}_T))$ *and* $u_\tau \in L^2(\mathcal{O}_\tau)$, *there exists a unique variational solution* $u \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap L^p(\tau, T; L^p(\mathcal{O}_T))$ *of equation* (7), *which satisfies energy equality a.e.* $t \in [\tau, T]$, *that is,*

$$\big|u(t)\big|_T^2 + 2\int_\tau^t \big\|u(s)\big\|_T^2\,ds + 2\int_\tau^t \big(g\big(u(s)\big), u(s)\big)_T\,ds = |u_\tau|_T^2 + 2\int_\tau^t \big(f(s), u(s)\big)_T\,ds \tag{9}$$

holds for a.e. $t \in [\tau, T]$. *In addition,* $u \in C([\tau, T]; L^2(\mathcal{O}_T))$ *and satisfies the energy equality for all* $t \in [\tau, T]$. *Moreover, if* $u_\tau \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$, *then* u *also satisfies*

$$u \in L^\infty\big(\tau, T; H_0^1(\mathcal{O}_T)\big) \cap L^\infty\big(\tau, T; L^p(\mathcal{O}_T)\big), \quad u' \in L^2\big(\tau, T; L^2(\mathcal{O}_T)\big).$$

Remark 2.4 ([6]) If $T_2 > T_1 > \tau$ and u is a variational solution of (7) with $T = T_2$, then the restriction of u to Q_{τ,T_1} is a variational solution of (7) with $T = T_1$.

We can also obtain the following result.

Theorem 2.5 *Under the assumptions of Theorem* 2.3, *if* $u_\tau \in H_0^1(\mathcal{O}_\tau) \cap L^p(\mathcal{O}_\tau)$, *then* $u \in L^2(\tau, T; H^2(\mathcal{O}_T))$.

Proof One can take an orthonormal Hilbert basis $\{w_j\}$ of L^2 and H_0^1 formed by the elements of $H_0^1 \cap L^p \cap H^3$ such that the vector space generated by $\{w_j\}$ is dense in H_0^1 and L^p. Then one takes a sequence $\{u_{\tau n}\}$ such that $u_{\tau n} \to u_\tau$ in $H_0^1(\mathcal{O}_\tau)$ with $u_{\tau n}$ in the vector space spanned by the n first w_j.

Consider the equality

$$\left(\frac{\partial u_{kn}(t)}{\partial t}, \omega_j\right)_T + \big\langle A_k(t)u_{kn}(t), \omega_j\big\rangle_T + \big(g\big(u_{kn}(t)\big), \omega_j\big)_T = \big(f(t), \omega_j\big)_T \tag{10}$$

for a.e. $t \in [\tau, T]$.

Multiplying (10) by $\lambda_j r_{kn,j}(t)$ and summing from $j = 1$ to n, we know that

$$\begin{aligned}&\left(\frac{\partial u_{kn}(t)}{\partial t}, -\Delta u_{kn}(t)\right)_T + \big\langle A_k(t)u_{kn}, -\Delta u_{kn}(t)\big\rangle_T + \big(g\big(u_{kn}(t)\big), -\Delta u_{kn}(t)\big)_T \\ &\quad = \big(f(t), -\Delta u_{kn}(t)\big)_T\end{aligned}$$

for a.e. $t \in [\tau, T]$.

According to (5), it follows

$$\int_{\mathcal{O}_T} g\big(u_{kn}(t)\big)\big(-\Delta u_{kn}(t)\big)\,dx = \int_{\mathcal{O}_T} g'\big(u_{kn}(t)\big)\big|\nabla u_{kn}(t)\big|^2\,dx \geq -l\big|\nabla u_{kn}(t)\big|_T^2. \tag{11}$$

Combining (11) and Hölder's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\big|\nabla u_{kn}(t)\big|_T^2 + \big|\Delta u_{kn}(t)\big|_T^2 + 2k\big(\big(P_k(t)\nabla u_{kn}(t), \nabla u_{kn}(t)\big)\big)_T \\ &\quad \leq 2l\int_{\mathcal{O}_T} \big|\nabla u_{kn}(t)\big|^2\,dx + \big|f(t)\big|_T^2\end{aligned}$$

a.e. $t \in (\tau, T)$, integrate the inequality above from τ to t,

$$\begin{aligned} &\left|\nabla u_{kn}(t)\right|_T^2 + \int_\tau^t \left|\Delta u_{kn}(s)\right|_T^2 ds + 2k \int_\tau^t \left(\left(P_k(s)\nabla u_{kn}(s), \nabla u_{kn}(s)\right)\right)_T ds \\ &\quad \le 2l \int_\tau^t \left|\nabla u_{kn}(s)\right|_T^2 ds + \int_\tau^t \left|f(s)\right|_T^2 ds + \left|\nabla u_{kn}(\tau)\right|_T^2 \quad \text{a.e. } t \in (\tau, T), \end{aligned} \tag{12}$$

so, we have

$$\left|\nabla u_{kn}(t)\right|_T^2 \le ce^{2l(t-\tau)}\left(\left|\nabla u_{kn}(\tau)\right|_T^2 + \int_\tau^t \left|f(s)\right|_T^2 ds\right) \tag{13}$$

and

$$\int_\tau^t \left|\nabla u_{kn}(s)\right|_T^2 ds \le ce^{l(t-\tau)}\left(\left|\nabla u_{kn}(\tau)\right|_T^2 + \int_\tau^t \left|f(s)\right|_T^2 ds\right). \tag{14}$$

Note (8), we have

$$\int_\tau^t \left(\left(P_k(s)\nabla u_{kn}(s), \nabla u_{kn}(s)\right)\right)_T ds \ge 0.$$

By (12)–(14), we obtain

$$\left|\nabla u_{kn}(t)\right|_T^2 + \int_\tau^t \left|\Delta u_{kn}(s)\right|_T^2 ds \le ce^{l(t-\tau)}\left(\left|\nabla u_{kn}(\tau)\right|_T^2 + \int_\tau^t \left|f(s)\right|_T^2 ds\right)$$

for a.e. $t \in (\tau, T)$, where c may be different from line to line, recall that $u_{kn}(\tau) = u_{\tau n}, u_{\tau n} \to u_\tau$ in $H_0^1(\mathcal{O}_\tau)$ as $n \to \infty$. So, $\{u_{kn}\}$ is bounded in $L^2(\tau, T; H^2(\mathcal{O}_T)) \cap L^\infty(\tau, T; H_0^1(\mathcal{O}_T))$, there exists a subsequence, denoted still by $\{u_{kn}\}$, such that as $n \to \infty$

$$\begin{aligned} &u_{kn} \rightharpoonup u_k \quad \text{weakly in } L^2\big(\tau, T; H^2(\mathcal{O}_T)\big), \\ &u_{kn} \rightharpoonup u_k \quad \text{weakly star in } L^\infty\big(\tau, T; H_0^1(\mathcal{O}_T)\big), \end{aligned}$$

therefore,

$$\left|\nabla u_k(t)\right|_T^2 + \int_\tau^t \left|\Delta u_k(s)\right|_T^2 ds \le ce^{l(t-\tau)}\left(\|u_\tau\|_T^2 + \int_\tau^t \left|f(s)\right|_T^2 ds\right),$$

and also $\{u_k\}$ is bounded in $L^2(\tau, T; H^2(\mathcal{O}_T)) \cap L^\infty(\tau, T; H_0^1(\mathcal{O}_T))$. There exists a subsequence, denoted still by $\{u_k\}$, such that it is convergent weakly, convergent weakly star to the uniqueness variational solution u of (7) in $L^2(\tau, T; H^2(\mathcal{O}_T)) \cap L^\infty(\tau, T; H_0^1(\mathcal{O}_T))$ as $k \to \infty$. □

Theorem 2.6 ([15]) *Suppose that* $u_\tau \in L^\infty(\mathcal{O}_\tau) \cap H_0^1(\mathcal{O}_\tau)$, $f \in L^\infty(\tilde{Q}_{\tau,T})$ *hold and* g *satisfies* (4). *Then there exists a positive constant* K *which depends on* $\|u_\tau\|_{L^\infty(\mathcal{O}_\tau)}$, $\|f\|_{L^\infty(\tilde{Q}_{\tau,T})}$, β *and* α_1 *such that the variational solution* u *of* (7) *satisfies*

$$\|u\|_{L^\infty(\tilde{Q}_{\tau,T})} \le K.$$

3 Pullback $\mathscr{D}_{\lambda_1}$ attraction in $H_0^1(\mathcal{O}_t)$

By *Theorem* 2.3 and *Remark* 2.4, we know that, for any $\tau \in \mathbb{R}$ and any $u_\tau \in L^2(\mathcal{O}_\tau)$, there exists a unique variational solution $u(\cdot;\tau,u_\tau)$ satisfying energy equality for a.e. $t \in (\tau,T)$ and any $T > \tau$. Moreover, $u \in C([\tau,T];L^2(\mathcal{O}_T))$ satisfying energy equality for all $t \in (\tau,T)$ with any $T > \tau$.

Define

$$U(t,\tau)u_\tau := u(t;\tau,u_\tau), \quad -\infty < \tau \le t < \infty, u_\tau \in L^2(\mathcal{O}_\tau). \tag{15}$$

By the uniqueness of variational solution for (7) and $u \in C([\tau,T];L^2(\mathcal{O}_T))$ satisfying energy equality for all $t \in (\tau,T)$ with any $T > \tau$, we know $U(\cdot,\cdot)$ defined by (15) is a process for the family of Hilbert spaces $\{L^2(\mathcal{O}_t), t \in \mathbb{R}\}$.

To obtain main results, the following lemma is necessary.

Lemma 3.1 ([11]) *For any $k > 0$ and any $\phi \in H_0^1(\mathcal{O}_t) \cap L^\infty(\mathcal{O}_t)$, the following equality holds*:

$$\int_{\mathcal{O}_t} \nabla\phi \cdot \nabla\big(|\phi|^k\phi\big)\,dx = (k+1)\left(\frac{2}{k+2}\right)^2 \int_{\mathcal{O}_t} \big|\nabla|\phi|^{\frac{k+2}{2}}\big|^2\,dx, \tag{16}$$

where $\cdot$ stands for the usual inner product in $\mathbb{R}^N$.

In the following, suppose

$$f \in L^2_{\mathrm{loc}}\big(\mathbb{R}^{N+1}\big) \quad \text{and} \quad u_\tau, v_\tau \in L^2(\mathcal{O}_\tau). \tag{17}$$

Due to the density of $L^\infty(\mathcal{O}_t)$ in $L^2(\mathcal{O}_t)$, there exist sequences $\{u_{\tau m}\}$, $\{v_{\tau m}\} \subset L^\infty(\mathcal{O}_\tau)$, $\{f_m\} \subset L^\infty(\tilde{Q}_{\tau,T})$ such that

$$\begin{aligned} &u_{\tau m} \to u_\tau, \qquad v_{\tau m} \to v_\tau \quad \text{in } L^2(\mathcal{O}_\tau), m \to +\infty, \\ &f_m \to f \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}^{N+1}\big), m \to +\infty \end{aligned} \tag{18}$$

and it can be done that, for each $m = 1,2,\ldots,$

$$|u_{\tau m}|_\tau^2 \le 2|u_\tau|_\tau^2 + 1, \qquad |v_{\tau m}|_\tau^2 \le 2|v_\tau|_\tau^2 + 1. \tag{19}$$

Based on the above, applying the interpolation inequality to estimate the L^{2p-2}-norm of approximation solution, we can establish the higher-order integrability near initial time τ for approximation solution as follows.

Theorem 3.2 *Suppose that* (1), (2), (4), *and* (5) *hold*, $f \in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\mathcal{O}_t))$, $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$. *Then, for any $T \ge \tau$, any $k = 1,2,\ldots,$ there exists a positive constant $M_k = M(T-\tau,k,N,l,|u_\tau|_{L^2(\mathcal{O}_\tau)},|v_\tau|_{L^2(\mathcal{O}_\tau)})$ such that*

$$(t-\tau)^{\frac{N}{N-2}} \big\|(t-\tau)^{b_k} w_m(t)\big\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_T)} \le M_k \quad \textit{for all } t \in [\tau,T] \tag{A_k}$$

and

$$\int_\tau^T \left(\int_{\mathcal{O}_T} \left|(t-\tau)^{b_{k+1}} \cdot w_m(t)\right|^{2(\frac{N}{N-2})^{k+1}} dx\right)^{\frac{N-2}{N}} dt \le M_k, \tag{B_k}$$

where $w_m(t) = u_m(t) - v_m(t) = U(t,\tau)u_{\tau m} - U(t,\tau)v_{\tau m}$,

$$b_1 = 1 + \frac{1}{2}, \qquad b_2 = 1 + \frac{1}{2} + 1 \quad \textit{and} \quad b_{k+1} = b_k + \frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}} \quad \textit{for } k = 2,3,\ldots, \tag{20}$$

and all constants $M_k (k = 1,2,\ldots)$ *are independent of* m.

Proof For any fixed $\tau \in (-\infty, T]$, denote

$$w_m(t) := u_m(t) - v_m(t), \quad \tau \le t \le T, \tag{21}$$

where $u_m(t), v_m(t)$ are the variational solutions of equation (7) corresponding to the data $(u_{\tau m}, f_m)$, $(v_{\tau m}, f_m)$ satisfying (18) respectively. By *Theorem* 2.3 and *Theorem* 2.6, we know

$$w_m \in L^2\big(\tau, T; H_0^1(\mathcal{O}_T)\big) \cap L^\infty(\tilde{Q}_{\tau,T})$$

and

$$\int_\tau^T \left[-\big(w_m(t), \phi'(t)\big)_T + \big(\big(w_m(t), \phi(t)\big)\big)_T + \big(g\big(u_m(t)\big) - g\big(v_m(t)\big), \phi(t)\big)_T\right] dt = 0 \tag{22}$$

for any $\phi \in \mathcal{U}_{\tau,T}$.

For any $\theta > 0$, we have

$$|w_m|^\theta w_m \in L^2\big(\tau, T; H_0^1(\mathcal{O}_T)\big) \cap L^\infty(\tilde{Q}_{\tau,T}),$$

and choose any $\eta \in C_c^1(\tau, T)$ to get

$$\begin{cases} \eta|w_m|^\theta w_m \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap L^\infty(\tilde{Q}_{\tau,T}), \\ \frac{d}{dt}(\eta|w_m|^\theta w_m) \in L^2(\tau, T; L^2(\mathcal{O}_T)), \\ \eta(T)|w_m(T)|^\theta w_m(T) = \eta(\tau)|w_m(\tau)|^\theta w_m(\tau) = 0, \\ \eta(t)|w_m(t)|^\theta w_m(t) \in H_0^1(\mathcal{O}_t) \quad \text{a.e. } t \in (\tau, T). \end{cases}$$

Hence, we can choose $\eta|w_m|^\theta w_m$ as a test function to have

$$\begin{aligned} &\int_\tau^T \Big[-\big(w_m(t), \big(\eta(t)\left|w_m(t)\right|^\theta w_m(t)\big)'\big)_T + \big(\big(w_m(t), \eta(t)\left|w_m(t)\right|^\theta w_m(t)\big)\big)_T \\ &\quad + \big(g\big(u_m(t)\big) - g\big(v_m(t)\big), \eta(t)\left|w_m(t)\right|^\theta w_m(t)\big)_T\Big] dt = 0, \end{aligned}$$

note that

$$\int_\tau^T \Big[\big(w_m'(t), \left|w_m(t)\right|^\theta w_m(t)\big)_T + \big(\big(w_m(t), \left|w_m(t)\right|^\theta w_m(t)\big)\big)_T$$

$$+ \bigl(g\bigl(u_m(t)\bigr) - g\bigl(v_m(t)\bigr), \bigl|w_m(t)\bigr|^{\theta} w_m(t)\bigr)_T\bigr]\eta(t)\,dt = 0$$

for any $\eta \in C_c^1(\tau, T)$ holds. Therefore, for a.e. $t \in (\tau, T)$,

$$\begin{aligned}&\bigl(w_m'(t), \bigl|w_m(t)\bigr|^{\theta} w_m(t)\bigr)_T + \bigl(\bigl(w_m(t), \bigl|w_m(t)\bigr|^{\theta} w_m(t)\bigr)\bigr)_T \\ &\quad + \bigl(g\bigl(u_m(t)\bigr) - g\bigl(v_m(t)\bigr), \bigl|w_m(t)\bigr|^{\theta} w_m(t)\bigr)_T = 0.\end{aligned}$$

By (5), for a.e. $t \in (\tau, T)$, we have

$$\begin{aligned}&\frac{1}{\theta+2}\frac{d}{dt}\bigl\|w_m(t)\bigr\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2} - \frac{1}{\theta+2}\int_{\partial\mathcal{O}_T}\bigl|\gamma w_m(t)\bigr|^{\theta+2}\,dS \\ &\quad + (\theta+1)\biggl(\frac{2}{\theta+2}\biggr)^2\int_{\mathcal{O}_T}\bigl|\nabla\bigl|w_m(t)\bigr|^{1+\frac{\theta}{2}}\bigr|^2\,dx \le l\bigl\|w_m(t)\bigr\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2},\end{aligned}$$

thanks to $w_m(t) \in H_0^1(\mathcal{O}_t)$ for a.e. $t \in (\tau, T)$, so $\gamma w_m(t) = 0$ for a.e. $t \in (\tau, T)$, it follows

$$\begin{aligned}&\frac{1}{\theta+2}\frac{d}{dt}\bigl\|w_m(t)\bigr\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2} + (\theta+1)\biggl(\frac{2}{\theta+2}\biggr)^2\int_{\mathcal{O}_T}\bigl|\nabla\bigl|w_m(t)\bigr|^{1+\frac{\theta}{2}}\bigr|^2\,dx \\ &\quad \le l\bigl\|w_m(t)\bigr\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2}.\end{aligned} \tag{23}$$

In the following, we separate our proof into two steps.

Step 1. $k = 1$

Firstly, taking $\phi = w_m$ in (22), from the definition of variational solution and (5), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w_m|_T^2 + \int_{\mathcal{O}_T}\bigl|\nabla w_m(t)\bigr|^2\,dx &= -\int_{\mathcal{O}_T}\bigl(g(u_m) - g(v_m)\bigr)w_m\,dx \\ &\le l\bigl|w_m(t)\bigr|_T^2 \quad \text{a.e. } t \in (\tau, T),\end{aligned}$$

which implies that

$$\bigl|w_m(t)\bigr|_T^2 \le e^{2l(t-\tau)}\bigl|w_m(\tau)\bigr|_T^2,$$

and then,

$$\int_\tau^T\bigl|\nabla w_m(t)\bigr|_T^2\,dt \le l\int_\tau^T\bigl|w_m(s)\bigr|_T^2\,ds + \frac{1}{2}\bigl|w_m(\tau)\bigr|_T^2 \le \frac{1}{2}\bigl(e^{2l(T-\tau)} + 1\bigr)\bigl|w_m(\tau)\bigr|_T^2.$$

Consequently, combining with the embedding

$$\biggl(\int_{\mathcal{O}_s}|v|^{\frac{2N}{N-2}}\,dx\biggr)^{\frac{N-2}{N}} \le c_{N,\tau,T}\int_{\mathcal{O}_s}|\nabla v|^2\,dx, \quad \forall v \in H^1(\mathcal{O}_s), \forall s \in [\tau, T], \tag{24}$$

we can deduce that

$$\int_\tau^T\biggl(\int_{\mathcal{O}_T}\bigl|(t-\tau)^{b_1}w_m(t)\bigr|^{\frac{2N}{N-2}}\,dx\biggr)^{\frac{N-2}{N}}\,dt$$

$$\leq (T-\tau)^{2b_1}\frac{c_{N,\tau,T}}{2}\left(e^{2l(T-\tau)}+1\right)\left|w_m(\tau)\right|_T^2, \tag{25}$$

note that here the embedding constant $c_{N,\tau,T}$ in (24) depends only on the domain $\mathcal{O}_T$.

Secondly, take $\theta = \frac{2N}{N-2} - 2$ in (23), by *Lemma* 3.1, we have that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{N-2}{N}\right)\frac{d}{dt}\left\|w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} + \frac{\frac{2N}{N-2}-1}{(\frac{N}{N-2})^2}\int_{\mathcal{O}_T}\left|\nabla\left|w_m(t)\right|^{\frac{N}{N-2}}\right|^2 dx \\
&\quad\leq l\left\|w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} \qquad \text{a.e. } t\in(\tau,T).
\end{aligned}$$

In the following we denote by c, c_i $(i = 1, 2, \ldots)$ the constants which depend only on N, $T-\tau$, and l, which may differ from line to line. Then the above inequality can be written as

$$\frac{d}{dt}\left\|w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} + c_1\int_{\mathcal{O}_T}\left|\nabla\left|w_m(t)\right|^{\frac{N}{N-2}}\right|^2 dx \leq c_2\left\|w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}}, \tag{26}$$

and by multiplying both sides with $(t-\tau)^{\frac{3N}{N-2}}$, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} + c_1\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_1}w_m(t)\right|^{\frac{N}{N-2}}\right|^2 dx \\
&\quad\leq c_2\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} + c_3(t-\tau)^{\frac{3N}{N-2}-1}\|w_m\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} \\
&\quad\leq c\left(1+\frac{1}{t-\tau}\right)\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}},
\end{aligned} \tag{27}$$

here $b_1 = 1 + \frac{1}{2}$.

One direct result of (27) is that

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} \leq c\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}},$$

and so

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{2} \leq c\frac{N-2}{N}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{2}. \tag{28}$$

Consequently, for any $t\in[\tau,T]$, integrating (28) over $[\tau,t]$, we obtain that

$$\begin{aligned}
(t-\tau)\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{2} &\leq \left(c\frac{N-2}{N}+1\right)\int_\tau^T\left\|(s-\tau)^{b_1}w_m(s)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{2} ds \\
&\leq c\left|w_m(\tau)\right|_T^2 \quad (\text{by } (25)),
\end{aligned}$$

hence,

$$(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} \leq c\left|w_m(\tau)\right|_T^{\frac{2N}{N-2}} \quad \text{for any } t\in[\tau,T]. \tag{29}$$

Then, multiplying (27) by $(t-\tau)^{\frac{2N}{N-2}}$, we obtain that: for a.e. $t\in(\tau,T)$,

$$(t-\tau)^{\frac{2N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}^{\frac{2N}{N-2}} + c_1\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_1+1}w_m(t)\right|^{\frac{N}{N-2}}\right|^2 dx$$

$$\begin{aligned}&\le c(t-\tau)^{\frac{N+2}{N-2}}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_T)}\\&\le c\left|w_m(\tau)\right|^{\frac{2N}{N-2}}_T.\end{aligned}$$

Integrating the above inequality over $[\tau, T]$ with respect to t, we obtain that

$$\int_\tau^T\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_2}w_m(t)\right|^{\frac{N}{N-2}}\right|^2\,dx\,dt\le c\left|w_m(\tau)\right|^{\frac{2N}{N-2}}_T.$$

Consequently, applying embedding (24) again, we can deduce that

$$\int_\tau^T\left(\int_{\mathcal{O}_T}\left|(t-\tau)^{b_2}w_m(t)\right|^{2(\frac{N}{N-2})^2}\,dx\right)^{\frac{N-2}{N}}dt\le c_{N,\tau,T}c_1\left|w_m(\tau)\right|^{\frac{2N}{N-2}}_T. \tag{30}$$

Therefor, noticing (18) and (19), from (29) and (30) we know that there is a positive constant M_1, which depends only on $N, \tau, T, l, |u_\tau|_\tau, |v_\tau|_\tau$ such that (A_1) and (B_1) hold.

Step 2. Assume (A_k) and (B_k) hold for $k \geq 1$, in the following, we will show that (A_{k+1}) and (B_{k+1}) hold.

Take $\theta = 2(\frac{N}{N-2})^{k+1} - 2$ in (23), we obtain that

$$\begin{aligned}&\frac{d}{dt}\left\|w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}+c\int_{\mathcal{O}_T}\left|\nabla\left|w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\&\quad\le c_1\left\|w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\qquad\text{a.e. } t\in(\tau,T).\end{aligned} \tag{31}$$

Multiplying both sides of (31) with $(t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}}$, we deduce that

$$\begin{aligned}&\frac{d}{dt}\left((t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}}\|w_m\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\right)+c\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\&\quad\le c_1\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\\&\qquad+c_2(t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}-1}\left\|w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)},\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}+c\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\&\quad\le\left(c_1+\frac{c_2}{t-\tau}\right)\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}.\end{aligned} \tag{32}$$

At first, from (32) we have

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\le c\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}, \tag{33}$$

and so,

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}$$

$$\leq c\frac{N-2}{N}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}. \tag{34}$$

Integrating (34) over $[\tau,t]$ and applying (B_k), we deduce that

$$\begin{aligned}
&(t-\tau)\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\\
&\leq \left(c\frac{N-2}{N}+1\right)\int_\tau^T\left\|(s-\tau)^{b_{k+1}}\cdot w_m(s)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\,ds\\
&\leq \left(c\frac{N-2}{N}+1\right)M_k \quad \text{for all } t\in[\tau,T],
\end{aligned}$$

which implies that

$$\begin{aligned}
&(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\\
&\leq \left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}} \quad \text{for all } t\in[\tau,T].
\end{aligned} \tag{35}$$

Multiplying both sides of (32) by $(t-\tau)^{1+\frac{N}{N-2}}$, we obtain that

$$\begin{aligned}
&(t-\tau)^{1+\frac{N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\\
&\quad + c\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_{k+1}+\frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\
&\leq c_3(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}.
\end{aligned}$$

Then, from (35) and the definition of b_{k+2}, we obtain that

$$\begin{aligned}
&(t-\tau)^{1+\frac{N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_T)}\\
&\quad + c\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\
&\leq c_3\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}} \quad \text{for all } t\in[\tau,T].
\end{aligned}$$

Integrating the above inequality over $[\tau,T]$ and using (35) again, we deduce that

$$\int_\tau^T\int_{\mathcal{O}_T}\left|\nabla\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\,dt\leq c_4\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}}. \tag{36}$$

Combining (36) with the embedding inequality (24), we obtain that

$$\int_\tau^T\left(\int_{\mathcal{O}_T}\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{2(\frac{N}{N-2})^{k+2}}dx\right)^{\frac{N-2}{N}}dt\leq c_5\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}}. \tag{37}$$

Therefore, by setting

$$M_{k+1} = (1+c_5)\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}},$$

(35) and (37) implies that (A_{k+1}) and (B_{k+1}) hold respectively. □

Next, we start to establish the higher-order integrability near the initial time τ for the variational solution of equation (7). This result shows some decay rate of variational solution in $L^{2(\frac{N}{N-2})^{k+1}}$-norm near the initial time τ.

Theorem 3.3 *Suppose that* (1), (2), (4), *and* (5) *hold*, $f \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathcal{O}_t))$, $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$. *Then, for any* $T \geq \tau$, *any* $k = 1, 2, \ldots$, *there exists a positive constant* $M_k = M(T-\tau, k, N, l, |u_\tau|_{L^2(\mathcal{O}_\tau)}, |v_\tau|_{L^2(\mathcal{O}_\tau)})$ *such that*

$$(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_k}w(t)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_T)} \leq M_k \quad \textit{for all } t \in [\tau, T],$$

where $w(t) = U(t,\tau)u_\tau - U(t,\tau)v_\tau$ *and*

$$b_1 = 1+\frac{1}{2}, \qquad b_2 = 1+\frac{1}{2}+1 \quad \textit{and}$$
$$b_{k+1} = b_k + \frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}} \quad \textit{for } k = 2, 3, \ldots.$$

Proof For any (fixed) $\tau \in \mathbb{R}$ and $T \geq \tau$, choose two sequences $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$ satisfying (18), (19).

Then from *Theorem* 3.2 (A_k) we have that, for any $k = 1, 2, \ldots$, there exists a positive constant $M_k = M(T-\tau, k, N, l, |u_\tau|_{L^2(\mathcal{O}_\tau)}, |v_\tau|_{L^2(\mathcal{O}_\tau)})$ such that

$$(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_k}\big(u_m(t)-v_m(t)\big)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_T)} \leq M_k \quad \text{for all } t \in [\tau, T], \tag{38}$$

where u_m and v_m are the unique variational solutions of (3) corresponding to the regular data $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$ on the interval $[\tau, T]$ respectively.

On the other hand, from [6] Proposition 11, there exist a subsequence $\{u_{mj}\}$ of $\{u_m\}$ and $\{v_{mj}\}$ of $\{v_m\}$ such that

$$u_{m_j}(t) \to u(t) \quad \text{and} \quad v_{m_j}(t) \to v(t) \quad \text{a.e. on } \mathcal{O}_T \text{ as } j \to \infty.$$

Hence, by applying Fatou's lemma,

$$\begin{aligned}
&(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_k}\big(u(t)-v(t)\big)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_T)}\\
&\quad= (t-\tau)^{\frac{N}{N-2}}\int_{\mathcal{O}_T}\liminf_{j\to\infty}\left|(t-\tau)^{b_k}\big(u_{m_j}(t)-v_{m_j}(t)\big)\right|^{2(\frac{N}{N-2})^k}dx\\
&\quad\leq \liminf_{j\to\infty}(t-\tau)^{\frac{N}{N-2}}\int_{\mathcal{O}_T}\left|(t-\tau)^{b_k}\big(u_{m_j}(t)-v_{m_j}(t)\big)\right|^{2(\frac{N}{N-2})^k}dx
\end{aligned}$$

$$\le M_k(t). \qquad \square$$

The following result is the continuity of variational solution in $H_0^1(\mathcal{O}_t)$ w.r.t. initial data in $L^2(\mathcal{O}_\tau)$, which is necessary to deduce (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor in the topology $H_0^1(\mathcal{O}_t)$.

Theorem 3.4 (Continuity) *Assume that* (1), (2), (4), *and* (5) *hold*, $f \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathcal{O}_t))$. *For any* $\tau \in \mathbb{R}$ *and any* $t > \tau$, *if* $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$ *and* $|u_\tau - v_\tau|_{L^2(\mathcal{O}_\tau)} \to 0$, *then*

$$U(t,\tau)u_\tau \to U(t,\tau)v_\tau.$$

More precisely, the following estimate holds:

$$\begin{aligned}&\bigl\|U(t,\tau)u_\tau - U(t,\tau)v_\tau\bigr\|^2_{H_0^1(\mathcal{O}_t)}\\ &\quad\le c_{r_0,t-\tau,l}|u_\tau - v_\tau|_\tau^2 + c_{r_0,M_{k_0},p,M_0,t-\tau,\theta,l}|u_\tau - v_\tau|_\tau^{2\theta},\end{aligned} \tag{39}$$

where $\theta \in (0,1)$ *is the exponent of the interpolation* $\|\cdot\|_{L^{2p-2}} \le \|\cdot\|^{1-\theta}_{L^{2(\frac{N}{N-2})^{k_0}}}\|\cdot\|^{\theta}_{L^2}$ *with some* $k_0 \in \mathbb{N}$ *satisfying* $2(\frac{N}{N-2})^{k_0} > 2p-2$, *and* $r_0 = (\frac{N}{N-2})\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}} + (2-2\theta)b_{k_0}$; *the constant* M_0 *depends only on* $t-\tau$, $\mathcal{O}_t$, $\lambda_{1,t}$, $\int_\tau^t |f(s)|_s^2\,ds$, β, α_1, $|u_\tau|_\tau$, p, *uniform bound of* $\{u_{\tau n}\}_{n=1}^\infty$ *in* $L^2(\mathcal{O}_\tau)$ *and* M_{k_0}.

Proof For any fixed $\tau \in (-\infty, T]$, denote

$$w_n(s) := u_n(s) - v_n(s), \quad \tau \le t \le T,$$

where $u_n(s), v_n(s)$ are the variational solutions of equation (7) corresponding to data $(u_{\tau n}, f_n)$, $(v_{\tau n}, f_n)$ satisfying (18). Then the following holds:

$$\int_\tau^T \bigl[-\bigl(w_n(s), \phi'(s)\bigr)_T + \bigl(\bigl(w_n(s), \phi(s)\bigr)\bigr)_T + \bigl(g\bigl(u_n(s)\bigr) - g\bigl(v_n(s)\bigr), \phi(s)\bigr)_T\bigr]\,ds = 0 \tag{40}$$

for any $\phi \in \mathcal{U}_{\tau,T}$.

Noticing $u_n \in L^2(\tau, T; H_0^1(\mathcal{O}_T)) \cap L^\infty(\tilde{Q}_{\tau,T})$, so $\eta|u_n|^\theta u_n$ ($\eta \in C_c^1(\tau, T), \theta > 0$) can be selected as a test function, hence, for a.e. $s \in (\tau, T)$,

$$\begin{aligned}&\bigl(u_n'(s), \bigl|u_n(s)\bigr|^\theta u_n(s)\bigr)_T + \bigl(\bigl(u_n(s), \bigl|u_n(s)\bigr|^\theta u_n(s)\bigr)\bigr)_T\\ &\quad + \bigl(g\bigl(u_n(s)\bigr), \bigl|u_n(s)\bigr|^\theta u_n(s)\bigr)_T = \bigl(f_n(s), \bigl|u_n(s)\bigr|^\theta u_n(s)\bigr)_T.\end{aligned}$$

By (4) and the standard energy estimate (e.g., see [9]), we have the following a priori estimates:

$$\int_\tau^t \int_{\mathcal{O}_s} \bigl|u_n(s)\bigr|^{2p-2}\,dx\,ds + \int_\tau^t \int_{\mathcal{O}_s} \bigl|v_n(s)\bigr|^{2p-2}\,dx\,ds \le M, \tag{41}$$

where the constant M depends only on g, $T-\tau$, $\mathcal{O}_t$, $\int_\tau^T |f(s)|_s^2\,ds$, α_1, β, p, $u_{\tau n}$, $v_{\tau n}$ and $\lambda_{1,T}$;

$$\bigl|w_n(s)\bigr|_s^2 \le e^{2l(s-\tau)}\bigl|w_n(\tau)\bigr|_\tau^2, \quad \forall t \ge \tau; \tag{42}$$

and

$$\int_\tau^t |\nabla w_n(s)|_s^2\,ds \le \frac{1}{2\lambda_{1,T}}|w_n(\tau)|_\tau^2 + \frac{l}{\lambda_{1,T}}\int_\tau^t |w_n(s)|_s^2\,ds, \quad \forall t \ge \tau, \tag{43}$$

recall that $\lambda_{1,T}$ is the first eigenvalue of $-\Delta$ on $H_0^1(\mathcal{O}_T)$ and the constant l comes from (5).

Noticing $u_n \in L^2(\tau,T;H^2(\mathcal{O}_t))$ in Theorem 2.5, let $\phi = -\eta\Delta w_n$ ($\eta \in C_c^1(\tau,T)$) in (40), then

$$\int_\tau^T \big[\big(w_n'(s),-\Delta w_n(s)\big)_T + |\Delta w_n(s)|_T^2 + \big(g\big(u_n(s)\big)-g\big(v_n(s)\big),-\Delta w_n(s)\big)_T\big]\eta\,ds = 0$$

for any $\eta \in C_c^1(\tau,T)$. Hence,

$$\begin{aligned}
&-\int_{\mathcal{O}_s} w_n'\Delta w_n\,dx + \int_{\mathcal{O}_s}|\Delta w_n(s)|^2\,dx = \int_{\mathcal{O}_s}\big(g\big(u_n(s)\big)-g\big(v_n(s)\big)\big)\Delta w_n(s)\,dx,\\
&-\int_{\mathcal{O}_s} w_n'\Delta w_n\,dx = \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}_s}|\nabla w_n(s)|^2\,dx - \int_{\Gamma_s}|\nabla w_n(s)|^2 w\cdot n_s\,d\sigma,
\end{aligned} \tag{44}$$

where n_s is the outside unit normal vector, w is a velocity field. By trace theory and interpolation, for all $\delta \ge \frac{1}{2}$ (reference [13]),

$$\left|\int_{\Gamma_s}|\nabla w_n(s)|^2 w\cdot n_s\,d\sigma\right| \le C_\delta\left(\int_{\mathcal{O}_s}|\Delta w_n(s)|^2\,dx\right)^\delta\left(\int_{\mathcal{O}_s}|\nabla w_n(s)|^2\,dx\right)^{1-\delta}. \tag{45}$$

In particular, let $\delta = \frac{1}{2}$ and by Cauchy's inequality, for all $s \in [\tau,T]$, we have

$$\left|\int_{\Gamma_s}|\nabla w_n(s)|^2 w\cdot n_s\,d\sigma\right| \le \frac{1}{4}\int_{\mathcal{O}_s}|\Delta w_n(s)|^2\,dx + c_{\frac{1}{2}}^2\int_{\mathcal{O}_s}|\nabla w_n(s)|^2\,dx. \tag{46}$$

On the other hand, by using (4) we have that

$$\begin{aligned}
&\left|\int_{\mathcal{O}_s}\big(g\big(u_n(s)\big)-g\big(v_n(s)\big)\big)\Delta w_n(s)\,dx\right|\\
&\quad\le c\int_{\mathcal{O}_s}\big(1+|u_n(s)|^{p-2}+|v_n(s)|^{p-2}\big)|w_n(s)||\Delta w_n(s)|\,dx\\
&\quad\le c\int_{\mathcal{O}_s}|w_n(s)||\Delta w_n(s)|\,dx + c\int_{\mathcal{O}_s}\big(|u_n(s)|^{p-2}+|v_n(s)|^{p-2}\big)|w_n(s)||\Delta w_n(s)|\,dx\\
&\quad\le \frac{1}{4}\int_{\mathcal{O}_s}|\Delta w_n(s)|^2\,dx + c|w_n(s)|_s^2\\
&\qquad + c\big(\|u_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4} + \|v_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\big)\|w_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^2.
\end{aligned} \tag{47}$$

Combining (44)–(47), we obtain that

$$\begin{aligned}
\frac{d}{ds}|\nabla w_n|_s^2 \le{}& c|w_n(t)|_s^2 + c\big(\|u_n(t)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\\
&+ \|v_n(t)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\big)\|w_n(t)\|_{L^{2p-2}(\mathcal{O}_s)}^2 \quad \text{a.e. } (\tau,t).
\end{aligned}$$

Since $2(\frac{N}{N-2})^k \to \infty$ as $k \to \infty$, there is $k_0 \in \mathbb{N}$ such that

$$2\left(\frac{N}{N-2}\right)^{k_0} > 2p-2.$$

For this k_0, by interpolation, we have

$$\|w_n\|_{L^{2p-2}(\mathcal{O}_s)} \le \|w_n\|^{1-\theta}_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)} \|w_n\|^{\theta}_{L^2(\mathcal{O}_s)},$$

where $\theta \in (0,1)$ depends only on p, k_0.

Therefore, we have that

$$\begin{aligned}\frac{d}{ds}|\nabla w_n|_s^2 \le c|w_n|_s^2 &+ c\big(\|u_n\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)} \\ &+ \|v_n\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\big)\|w_n\|^{2-2\theta}_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)} \cdot |w_n|_s^{2\theta} \quad \text{a.e. } (\tau,t).\end{aligned}$$

Denoting $r_0 = (\frac{N}{N-2})\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}} + (2-2\theta)b_{k_0}$ and multiplying the above inequality by $(s - \frac{t+\tau}{2})^{r_0}$, we obtain that

$$\begin{aligned}&\left(s-\frac{t+\tau}{2}\right)^{r_0}\frac{d}{ds}|\nabla w_n|_s^2 \\ &\quad\le c\left(s-\frac{t+\tau}{2}\right)^{r_0}\big(|\nabla w_n|_s^2 + |w_n|_s^2\big) \\ &\qquad + c\big(\|u_n\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)} + \|v_n\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\big) \\ &\qquad \times \Big((s-\tau)^{\frac{N}{N-2}}\big\|(s-\tau)^{b_{k_0}}w_n\big\|^{2(\frac{N}{N-2})^{k_0}}_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)}\Big)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}} \cdot |w_n|_s^{2\theta},\end{aligned}$$

where b_{k_0} is given by (20).

On the other hand, thanks to *Theorem* 3.2, we know that there is a constant M_{k_0}, which depends only on $t-\tau$, N, k_0, and the $H_0^1 \cap L^p$-bounds of $u_{\tau n}, v_{\tau n}$ such that

$$\Big((s-\tau)^{\frac{N}{N-2}}\big\|(s-\tau)^{b_{k_0}}w_n\big\|^{2(\frac{N}{N-2})^{k_0}}_{L^{2(\frac{N}{N-2})^{k_0}}_{(\mathcal{O}_s)}}\Big)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}} \le M_{k_0}^{2-2\theta} \quad \text{for all } n = 1,2,\dots, s \in [\tau,t].$$

Therefore, we have the following estimate for any $n = 1,2,\dots$:

$$\begin{aligned}&\left(s-\frac{t+\tau}{2}\right)^{r_0}\frac{d}{ds}|\nabla w_n|_s^2 \\ &\quad\le c\left(s-\frac{t+\tau}{2}\right)^{r_0}\big(|\nabla w_n|_s^2 + |w_n|_s^2\big) \\ &\qquad + cM_{k_0}^{2-2\theta}\big(\big\|u_n(s)\big\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)} + \big\|v_n(s)\big\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\big)\cdot\big|w_n(s)\big|_s^{2\theta}, \quad \text{a.e. } s \in \left[\frac{\tau+t}{2}, t\right].\end{aligned}$$

To ensure that the exponent r_0 is strictly larger than 1, we may multiply both sides by $(s - \frac{t+\tau}{2})$, and then we obtain that

$$\begin{aligned} &\left(s-\frac{t+\tau}{2}\right)^{1+r_0}\frac{d}{ds}\left|\nabla w_n(s)\right|_s^2 \\ &\quad\le c\left(s-\frac{t+\tau}{2}\right)^{1+r_0}\left(|\nabla w_n|_s^2+|w_n|_s^2\right) \\ &\qquad + c\left(s-\frac{t+\tau}{2}\right)M_{k_0}^{2-2\theta}\left(\|u_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\cdot\left|w_n(s)\right|_s^{2\theta}, \\ &\quad \text{a.e. } s\in\left[\frac{\tau+t}{2},t\right]. \end{aligned} \tag{48}$$

Integrating the inequality above over $[\frac{\tau+t}{2}, t]$ with respect to s, we finally obtain that, for any $n = 1, 2, \ldots,$

$$\begin{aligned} &\left(\frac{t-\tau}{2}\right)^{1+r_0}\left|\nabla w_n(t)\right|_t^2 \\ &\quad\le (1+r_0)\left(\frac{t-\tau}{2}\right)^{r_0}\int_{\frac{\tau+t}{2}}^{t}\left|\nabla w_n(s)\right|_s^2\,ds \\ &\qquad + c\left(\frac{t-\tau}{2}\right)^{1+r_0}\int_{\frac{\tau+t}{2}}^{t}\left(\left|\nabla w_n(s)\right|_s^2+\left|w_n(s)\right|_s^2\right)ds \\ &\qquad + c\left(\frac{t-\tau}{2}\right)M_{k_0}^{2-2\theta}\int_{\frac{\tau+t}{2}}^{t}\left(\|u_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_n(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\left|w_n(s)\right|_s^{2\theta}\,ds \\ &\quad := I_1+I_2+I_3. \end{aligned} \tag{49}$$

Note that, from (42)–(43), we have that

$$I_1 + I_2 \le c_{r_0,t-\tau,l}\left|w_n(\tau)\right|_\tau^2. \tag{50}$$

For the estimate of I_3, by Hölder's inequality, we have

$$I_3 \le c\frac{t-\tau}{2}M_{k_0}^{2-2\theta}2M_0^{\frac{2p-4}{2p-2}}\left(\int_{\frac{\tau+t}{2}}^{t}\left|w_n(s)\right|_s^{2\theta(p-1)}\,ds\right)^{\frac{2}{2p-2}} \le c_{M_{k_0},p,M_0,t-\tau,\theta,l}\left|w_n(\tau)\right|_\tau^{2\theta}. \tag{51}$$

Combining with (42)–(49), it implies that

$$\left|\nabla w_n(t)\right|_t^2 \le c_{r_0,t-\tau,l}\left|w_n(\tau)\right|_\tau^2 + c_{M_{k_0},p,M_0,t-\tau,\theta,l}\left|w_n(\tau)\right|_\tau^{2\theta}. \tag{52}$$

From (52) we know w_n is bounded in $H_0^1(\mathcal{O}_t)$, so there exists a subsequence $\{w_{nj}\}$ such that

$$w_{nj} \rightharpoonup \chi \quad \text{in } H_0^1(\mathcal{O}_t), \text{ as } j\to\infty. \tag{53}$$

By [6] Proposition 11 again, it follows

$$w_{nj}(t) \to u(t) - v(t) \quad \text{in } L^2(\mathcal{O}_t), \text{ as } j\to\infty,$$

hence, $\chi = u(t) - v(t)$.

Combining (52), (53), (21), and (19), we deduce that

$$\begin{aligned}\left|\nabla\big(u(t)-v(t)\big)\right|_t^2 &\le \liminf_{j\to\infty}\left|\nabla w_{nj}(t)\right|_t^2 \\ &\le c_{r_0,t-\tau,l}|u_\tau - v_\tau|_\tau^2 + c_{r_0,M_{k_0},p,M_0,t-\tau,\theta,l}|u_\tau - v_\tau|_\tau^{2\theta}.\end{aligned}$$

□

In [15], the existence of pullback $\mathscr{D}_{\lambda_1}$ attractor defined in time varying domains has been considered. Then we can establish the regularity attraction of (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor.

Theorem 3.5 *Suppose that $U(t,\tau)$ is the process corresponding to a variational solution of (3), $\hat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ is the (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor associated with $U(t,\tau)$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$. Then $\hat{\mathscr{A}}$ is pullback $\mathscr{D}_{\lambda_1}$ attraction in H^1_0.*

That is, for any $t \in \mathbb{R}$, any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_{\lambda_1}$,

$$\mathrm{dist}_{H^1_0(\mathcal{O}_t)}\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) \to 0, \quad \tau \to -\infty.$$

Proof For each $t \in \mathbb{R}$, from the definition of the (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor $\mathscr{A}$, we know that $\mathscr{A}(t-1)$ is compact in $L^2(\mathcal{O}_t)$.

By $B(t)$ being the 1-neighborhood of $\mathscr{A}(t)$ for each $t \in \mathbb{R}$ under the $L^2(\mathcal{O}_t)$ norm, $B(t)$ is bounded in $L^2(\mathcal{O}_t)$. By (39), let t be fixed, $\tau = t-1$, and $u_{\tau i} \in B(t-1)$ $(i = 1, 2)$, we have

$$\begin{aligned}&\left\|U(t,t-1)u_{\tau 1} - U(t,t-1)u_{\tau 2}\right\|^2_{H^1_0(\mathcal{O}_t)} \\ &\quad\le c_1|u_{\tau 1} - u_{\tau 2}|^2_{t-1} + c_2|u_{\tau 1} - u_{\tau 2}|^{2\theta}_{t-1},\end{aligned}$$

where c_1, c_2 are two constants. Now, for this fixed t and for any $\varepsilon > 0$, by the definition of the (L^2, L^2) pullback $\mathscr{D}_{\lambda_1}$ attractor again, for any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_{\lambda_1}$, there is a time $\tau_0 (< t-1)$ which depends only on t, ε, and $\hat{D}$ such that

$$\begin{aligned}&U(t-1,\tau)D(\tau) \subset B(t-1) \quad \text{for all } \tau \le \tau_0, \\ &\mathrm{dist}_{L^2(\mathcal{O}_t)}\big(U(t-1,\tau)D(\tau), \mathscr{A}(t-1)\big) \le \varepsilon \quad \text{for all } \tau \le \tau_0.\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathrm{dist}_{H^1_0(\mathcal{O}_t)}\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) \\ &\quad= \mathrm{dist}_{H^1_0(\mathcal{O}_t)}\big(U(t,t-1)U(t-1,\tau)D(\tau), U(t,t-1)\mathscr{A}(t-1)\big) \\ &\quad\le c_1\varepsilon^2 + c_2\varepsilon^{2\theta} \quad \text{for all } \tau \le \tau_0.\end{aligned}$$

Noticing the arbitrariness of ε and $\hat{D}$, the conclusion is proved. □

Acknowledgements
We would like to thank the referees for their detailed suggestions that helped to improve the original manuscript.

Funding
This work is supported by introducing talent item for Northwest Minzu University :xbmuyjrc201701.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 21 October 2019 Accepted: 4 March 2020 Published online: 11 March 2020

References

1. Bonaccorsi, S., Guatteri, G.: A variational approach to evolution problems with variable domains. J. Differ. Equ. **175**, 51–70 (2001)
2. Cao, D., Sun, C., Yang, M.: Dynamics for a stochastic reaction-diffusion equation with additive noise. J. Differ. Equ. **259**, 838–872 (2015)
3. Caraballo, T., Łukaszewicz, G., Real, J.: Pullback attractors for asymptotically compact non-autonomous dynamical systems. Nonlinear Anal. **64**, 484–498 (2006)
4. Crauel, H., Kloeden, P.E., Yang, M.: Random attractors of stochastic reaction-diffusion equations on variable domains. Stoch. Dyn. **11**, 301–314 (2011)
5. He, C., Hsiao, L.: Two-dimensional Euler equations in a time dependent domain. J. Differ. Equ. **163**, 265–291 (2000)
6. Kloeden, P.E., Marín-Rubio, P., Real, J.: Pullback attractors for a semilinear heat equation in a non-cylindrical domain. J. Differ. Equ. **244**, 2062–2090 (2008)
7. Kloeden, P.E., Real, J., Sun, C.: Pullback attractors for a semilinear heat equation on time-varying domains. J. Differ. Equ. **246**, 4702–4730 (2009)
8. Límaco, J., Medeiros, L.A., Zuazua, E.: Existence, uniqueness and controllability for parabolic equations in non-cylindrical domains. Mat. Contemp. **23**, 49–70 (2002)
9. Łukaszewicz, G.: On pullback attractors in L^p for nonautonomous reaction-diffusion equations. Nonlinear Anal. **73**, 350–357 (2010)
10. Li, Y., Wang, S., Wu, H.: Pullback attractors for non-autonomous reaction-diffusion equations in L^p. Appl. Math. Comput. **207**, 373–379 (2009)
11. Sun, C., Yuan, Y.: L^p-type pullback attractors for a semilinear heat equation on time-varying domains. Proc. R. Soc. Edinb., Sect. A **145**, 1029–1052 (2015)
12. Zhong, C., Yang, M., Sun, C.: The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations. J. Differ. Equ. **223**, 367–399 (2006)
13. Duvaut, G.: Mécanique des Milieux Continus. Masson, Paris (1990)
14. Trujillo, T., Wang, B.: Continuity of strong solutions of reaction-diffusion equation in initial data. Nonlinear Anal. **69**, 2525–2532 (2008)
15. Xiao, Y., Sun, C.: Higher-order asymptotic attraction of pullback attractors for a reaction-diffusion equation in non-cylindrical domains. Nonlinear Anal. **113**, 309–322 (2015)
16. Xiao, Y.: Pullback attractors for non-autonomous reaction-diffusion equation in non-cylindrical domains. Adv. Differ. Equ. **2016**, 230 (2016)