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Finite time blow-up for the nonlinear Schrödinger equation in trapped dipolar quantum gases with arbitrarily positive initial energy

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Abstract

In this paper, we study the blow-up criterion for the following nonlinear Schrödinger equation arising in trapped dipolar quantum gases:

$$i\partial_t u = -\frac{1}{2}\Delta u + a^2(x_1^2 + x_2^2 + x_3^2)u + \lambda_1|u|^2u + \lambda_2(K * |u|^2)u, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3.$$

When $a = 0$ or $a \neq 0$, by constructing an invariant set, we establish a new blow-up criterion, which implies the existence of blow-up solutions with arbitrarily large initial energy. This result gives a positive answer to the problem left by Carles, Markowich, and Sparber (Nonlinearity 21:2569–2590, 2008).

Keywords: Nonlinear Schrödinger equation; Trapped dipolar quantum gases; Blow-up criterion

1 Introduction

In the recent years the so-called dipolar Bose–Einstein condensate, i.e., a condensate made out of particles possessing a permanent electric or magnetic dipole moment, has attracted much attention; see, e.g., [2, 3, 17, 22–24]. At a temperature much smaller than the critical temperature, it is well described by the wave function $u(t, x)$ whose evolution is governed by the three-dimensional (3D) Schrödinger equation, see, e.g., [2, 3, 24, 35]:

$$i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u + W(x)u + U_0|u|^2u + (V_{\text{dip}} * |u|^2)u, \quad x \in \mathbb{R}^3, t > 0, \quad (1.1)$$

where t is time, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is the Cartesian coordinates, $*$ denotes the convolution, \hbar is the Planck constant, m is the mass of a dipolar particle, and $W(x) = a^2(x_1^2 + x_2^2 + x_3^2)$ is an external trapping potential, where a is the trapping frequency. $U_0 = 4\pi\hbar^2 a_s/m$ describes the local interaction between dipoles in the condensate with a_s the s-wave scattering length (positive for repulsive interaction and negative for attractive interaction). The

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long-range dipolar interaction potential between two dipoles is given by

$$V_{\text{dip}}(x) = \frac{\mu_0 \mu_{\text{dip}}^2}{4\pi} \frac{1 - 3 \cos^2 \theta}{|x|^3}, \quad x \in \mathbb{R}^3, \quad (1.2)$$

where μ_0 is the vacuum magnetic permeability, μ_{dip} is the permanent magnetic dipole moment, and θ is the angle between $x \in \mathbb{R}^3$ and the dipole axis $\mathbf{n} \in \mathbb{R}^3$, with $|\mathbf{n}| = 1$.

In order to simplify the mathematical analysis, we rescale (1.1) into the following dimensionless Schrödinger equation:

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + W(x)u + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2)u, & (t, x) \in [0, T^*) \times \mathbb{R}^3, \\ u(0, x) = u_0(x). \end{cases} \quad (1.3)$$

To simplify notation, we assume $\mathbf{n} = (0, 0, 1)$. The dimensionless long-range dipolar interaction potential $K(x)$ then reads as follows:

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}, \quad x \in \mathbb{R}^3.$$

We focus on the case when λ_1 and $\lambda_2 \in \mathbb{R}$ fulfill the following conditions:

$$\lambda_1 < \begin{cases} \frac{4}{3}\pi\lambda_2, & \text{if } \lambda_2 > 0, \\ -\frac{8}{3}\pi\lambda_2, & \text{if } \lambda_2 < 0. \end{cases} \quad (1.4)$$

These conditions, following the terminology introduced in [8], define the unstable regime.

Because of important applications of equation (1.3) in physics, it has received much attention both from physics (see [11, 26]) and mathematics (see [1–5, 7, 8, 18, 20, 21, 28]). Carles, Markowich, and Sparber in [8] first studied the local well-posedness and proved that the solution $u(t)$ of (1.3) blows up in finite time in the unstable regime if the initial energy is small. Ma, Cao, and Wang in [20, 21] studied the sharp thresholds of global existence and blow-up, and proved that blow-up may occur if the initial energy $E(u_0) < d$ for some $d > 0$. When $W(x) = 0$, Huang in [18] discussed the exact value of d by using the precise characterization of ground states of (2.5). Similar sharp thresholds of global existence and blow-up for other kinds of nonlinear Schrödinger equations are pursued strongly in [9, 12–16, 32, 36–41].

To our knowledge, the key to showing the existence of blow-up solutions is to prove $J''(t) < 0$ for all $t \in [0, T^*)$, where $J(t) = \int_{\mathbb{R}^3} |x|^2 |u(t, x)|^2 dx$. If the solution $u(t)$ blows up in finite time T^* , i.e., $\|\nabla u(t)\|_{L^2} \rightarrow \infty$ as $t \rightarrow T^*$, it follows from Lemma 2.2 that, for any large $E(u_0) > 0$, there exists $\delta > 0$ such that $J''(t) \leq 6E(u_0) - \int_{\mathbb{R}^3} |\nabla u(t, x)|^2 dx < 0$ for all $t \in [T^* - \delta, T^*)$. Therefore, a natural and interesting question is whether there exist blow-up solutions with arbitrarily large initial energy. In particular, this is an open problem presented by Carles, Markowich, and Sparber, see Remark 5.3 in [8]. This will be the focus of the present paper. In fact, similar problems have been studied for other kinds of nonlinear evolutionary equations, e.g., semilinear pseudo-parabolic equations and nonlinear Klein–Gordon equations, see [6, 19, 25, 27, 29–31, 33, 34].

Motivated by the above paper, in this paper, by establishing a blow-up criterion and a rather delicate analysis, we derive the existence of blow-up solutions with arbitrarily large initial energy. In particular, our method holds in the cases of both $a = 0$ and $a \neq 0$.

Theorem 1.1 *Let $a \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and satisfy assumption (1.4). Then, for any $\mu > 0$, there exists $u_0 \in \Sigma := \{v \in H^1 \text{ and } |x|v \in L^2\}$ such that $E(u_0) = \mu$ and the corresponding solution $u(t)$ of (1.3) blows up in finite time.*

Notation. Throughout this paper, for notational convenience, we use the following notation:

$$F(u) := -\lambda_1 \int_{\mathbb{R}^3} |u(x)|^4 dx - \lambda_2 \int_{\mathbb{R}^3} (K * |u|^2)(x) |u(x)|^2 dx \quad \text{for all } u \in H^1.$$

Let $u(t)$ be the solution of (1.3), we denote

$$J(t) := \int_{\mathbb{R}^3} |x|^2 |u(t, x)|^2 dx, \quad \theta(t) := \sqrt{J(t)}.$$

2 Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for Cauchy problem (1.3) established in [8].

Lemma 2.1 ([8]) *For $a, \lambda_1, \lambda_2 \in \mathbb{R}$, $u_0 \in X := \{v \in H^1, \text{ and } \int_{\mathbb{R}^3} W(x)|v(x)|^2 dx < \infty\}$, there exists $T = T(\|u_0\|_X)$ such that (1.3) admits a unique solution $u \in C([0, T], X)$. Let $[0, T^*)$ be the maximal time interval on which the solution $u(t)$ is well defined, if $T^* < \infty$, then $\|u(t)\|_X \rightarrow \infty$ as $t \rightarrow T^*$. Moreover, the solution $u(t)$ enjoys conservation of mass and energy, i.e., $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $E(u(t)) = E(u_0)$ for all $t \in [0, T^*)$, where $E(u(t))$ is defined by*

$$E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx - \frac{1}{2} F(u(t)). \quad (2.1)$$

In order to prove the existence of blow-up solution, we need the following virial identity, which can be proved by a similar argument as that in [9].

Lemma 2.2 ([8]) *Let $u_0 \in \Sigma$ and $u(t)$ be the solution of (1.3). Then function $J(t)$ belongs to $C^2[0, T^*)$, and*

$$J'(t) = 2 \operatorname{Im} \int_{\mathbb{R}^3} x \bar{u}(t, x) \nabla u(t, x) dx, \quad (2.2)$$

$$J''(t) = 2 \|\nabla u(t)\|_{L^2}^2 - 4 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx - 3F(u(t)). \quad (2.3)$$

Next, we recall the sharp Gagliardo–Nirenberg type inequality established in [1].

Lemma 2.3 ([1]) *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and satisfy assumption (1.4). Then, for all $u \in H^1$,*

$$F(u) = -\lambda_1 \int_{\mathbb{R}^3} |u(x)|^4 dx - \lambda_2 \int_{\mathbb{R}^3} K * |u(x)|^2 |u(x)|^2 dx \leq C_{\text{opt}}^{\frac{3}{2}} \|\nabla u\|_{L^2}^3 \|u\|_{L^2}, \quad (2.4)$$

where the sharp constant $C_{\text{opt}} = \frac{F(Q)^{\frac{2}{3}}}{\|Q\|_{L^2}^{\frac{2}{3}} \|\nabla Q\|_{L^2}^2}$ and Q is the ground state of the following elliptic equation:

$$-\frac{1}{2} \Delta Q + Q + \lambda_1 |Q|^2 Q + \lambda_2 (K * |Q|^2) Q = 0. \quad (2.5)$$

Moreover, the following Pohozaev's identities hold true:

$$\|\nabla Q\|_{L^2}^2 = \frac{3}{2}F(Q) = 6\|Q\|_{L^2}^2. \quad (2.6)$$

From (2.6) and (2.1), we can obtain the following useful result:

$$E(Q) = \frac{1}{4}F(Q) + \int_{\mathbb{R}^3} W(x)|Q|^2 dx. \quad (2.7)$$

Thus, C_{opt} can be rewritten as

$$C_{\text{opt}} = \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{(6E(Q) - 6 \int_{\mathbb{R}^3} W(x)|Q|^2 dx)^{-\frac{1}{3}}}{\|Q\|_{L^2}^{\frac{2}{3}}}. \quad (2.8)$$

Lemma 2.4 ([10]) *Let $a \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and satisfy assumption (1.4). Then, for any $u \in \Sigma$, it holds*

$$\left(\text{Im} \int_{\mathbb{R}^3} x \bar{u} \nabla u dx\right)^2 \leq \int_{\mathbb{R}^3} |x|^2 |u|^2 dx \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{F(u)^{\frac{2}{3}}}{C_{\text{opt}} \|u\|_{L^2}^{\frac{2}{3}}} \right). \quad (2.9)$$

The proof is similar to the one in [10], so we omit it.

3 Invariant evolution flow

In this section, we construct a set which is invariant under the flow generated by Cauchy problem (1.3).

Proposition 3.1 *Let $a \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and satisfy assumption (1.4). Taking an initial data $u_0 \in \Sigma$ such that $4E(u_0) > \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}$ and $\theta'(0) \leq -\sqrt{\frac{y_m}{2}}$, where $y_m = 4E(u_0) - \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}$. Define the set K by*

$$K = \left\{ v \in H^1, F(v) > \frac{8}{27C_{\text{opt}}^3 \|v\|_{L^2}^2} \right\}.$$

If $u_0 \in K$, then the corresponding solution $u(t) \in K$ for all $t \in [0, T^)$.*

Proof Let $u_0 \in K$ and $u(t)$ be the corresponding solution of (1.3). We deduce from $u_0 \in K$ and (2.1) that

$$2\|\nabla u_0\|_{L^2}^2 + 4 \int_{\mathbb{R}^3} W(x)|u_0(x)|^2 dx - 3F(u_0) < 4E(u_0) - \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}. \quad (3.1)$$

In fact, with the notation of y_m and $J''(0)$, formula (3.1) implies that $\frac{J''(0)}{2} < \frac{y_m}{2}$. This combined with the assumption $\theta'(0) \leq -\sqrt{\frac{y_m}{2}}$ implies that

$$\theta''(0) = \frac{1}{\theta(0)} \left(\frac{J''(0)}{2} - (\theta'(0))^2 \right) < 0. \quad (3.2)$$

This implies that there exists $t_0 > 0$ such that $\theta''(t) < 0$ for all $t \in [0, t_0]$. We then deduce that $\theta'(t) < \theta'(0) < -\sqrt{\frac{y_m}{2}}$, and so

$$(\theta'(t))^2 > \frac{y_m}{2}, \quad t \in [0, t_0]. \quad (3.3)$$

Next, using (2.1), (2.3), and the conservation of energy, we derive that

$$F(u(t)) = 4E(u_0) - J''(t) - 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx, \quad (3.4)$$

$$\|\nabla u(t)\|_{L^2}^2 = 6E(u_0) - J''(t) - 10 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx. \quad (3.5)$$

Applying (3.4), (3.5), conservation of mass, and the definition of $J'(t)$ in (2.2), we deduce from Lemma 2.4 that

$$\begin{aligned} (J'(t))^2 &\leq 4J(t) \left\{ 6E(u_0) - J''(t) - 10 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right. \\ &\quad \left. - \frac{1}{C_{\text{opt}} \|u_0\|_{L^2}^{2/3}} \left(4E(u_0) - J''(t) - 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right)^{\frac{2}{3}} \right\} \\ &< 4J(t) \left\{ 6E(u_0) - J''(t) - 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right. \\ &\quad \left. - \frac{1}{C_{\text{opt}} \|u_0\|_{L^2}^{2/3}} \left(4E(u_0) - J''(t) - 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right)^{\frac{2}{3}} \right\}. \end{aligned}$$

We consequently obtain

$$(\theta'(t))^2 = \frac{(J'(t))^2}{4J(t)} \leq \eta \left(J''(t) + 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right), \quad (3.6)$$

where

$$\eta(y) = 6E(u_0) - y - \frac{(4E(u_0) - y)^{\frac{2}{3}}}{C_{\text{opt}} \|u_0\|_{L^2}^{2/3}}. \quad (3.7)$$

Now, we deduce from (3.3) and (3.6) that

$$\eta \left(J''(t) + 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \right) > \frac{y_m}{2} = \eta(y_m)$$

for all $t \in [0, t_0]$. Thus $J''(t) + 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx \neq y_m$ for all $t \in [0, t_0]$. By continuity, it follows that $J''(t) + 8 \int_{\mathbb{R}^3} W(x) |u(t, x)|^2 dx < y_m$ for all $t \in [0, t_0]$, which, together with (2.3) and the definition of y_m , implies $F(u(t)) > \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}$ for all $t \in [0, t_0]$. Moreover,

$\theta''(t_0) = \frac{1}{\theta(t_0)} \left(\frac{J''(t_0)}{2} - (\theta'(t_0))^2 \right) < \frac{-4 \int_{\mathbb{R}^3} W(x) |u(t_0, x)|^2 dx}{\theta(t_0)} \leq 0$. Therefore, applying an elementary Bootstrap argument, we can obtain

$$F(u(t)) > \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2} = \frac{8}{27C_{\text{opt}}^3 \|u(t)\|_{L^2}^2}$$

for all $t \in [0, T^*)$. This completes the proof. \square

4 Existence of blow-up solutions

In this section, we use the same notations as in Sect. 3 and prove Theorem 1.1. Firstly, we establish the following blow-up criterion for (1.3).

Theorem 4.1 *Let $a \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and satisfy assumption (1.4). Assume that $u_0 \in \Sigma$ such that $J'(0) \leq 0$. If*

$$4\|u_0\|_{L^2}^2 E(u_0) > \|Q\|_{L^2}^2 F(Q), \quad (4.1)$$

$$\|u_0\|_{L^2}^2 F(u_0) > \|Q\|_{L^2}^2 F(Q), \quad (4.2)$$

$$E(u_0)\|u_0\|_{L^2}^2 \left(1 - \frac{(J'(0))^2}{8E(u_0)J(0)}\right) \leq \|Q\|_{L^2}^2 \left(E(Q) - \int_{\mathbb{R}^3} W(x)|Q|^2 dx\right), \quad (4.3)$$

then the solution $u(t)$ of (1.3) blows up in finite time.

Proof Firstly, it follows from (2.7) and (4.1) that $\|u_0\|_{L^2}^2 E(u_0) > \|Q\|_{L^2}^2 (E(Q) - \int_{\mathbb{R}^3} W(x) \times |Q|^2 dx)$. On the other hand, by (2.8) and the definition of y_m , we can obtain

$$\|u_0\|_{L^2}^2 \left(E(u_0) - \frac{y_m}{4}\right) = \|Q\|_{L^2}^2 \left(E(Q) - \int_{\mathbb{R}^3} W(x)|Q|^2 dx\right). \quad (4.4)$$

These imply that (4.1) is equivalent to $y_m > 0$, and (4.3) is equivalent to

$$(\theta'(0))^2 \geq \eta(y_m) = \frac{y_m}{2}. \quad (4.5)$$

In addition, the assumption $J'(0) \leq 0$ implies $\theta'(0) \leq 0$. In view of (2.7) and (4.4), assumption (4.2) yields the estimate

$$\|u_0\|_{L^2}^2 F(u_0) > 4\|Q\|_{L^2}^2 \left(E(Q) - \int_{\mathbb{R}^3} W(x)|Q|^2 dx\right) = 4\|u_0\|_{L^2}^2 \left(E(u_0) - \frac{y_m}{4}\right),$$

which, together with (3.4), implies

$$J''(0) + 8 \int_{\mathbb{R}^3} W(x)|u_0(x)|^2 dx < y_m. \quad (4.6)$$

Therefore, $F(u_0) > \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}$. It follows from Proposition 3.1 that $F(u(t)) > \frac{8}{27C_{\text{opt}}^3 \|u_0\|_{L^2}^2}$ for all $t \in [0, T^*)$. Moreover, we know from the proof of Proposition 3.1 that $(\theta'(t))^2 > \frac{y_m}{2}$ for all $t \in [0, T^*)$. We consequently obtain

$$\theta''(t) = \frac{1}{\theta(t)} \left(\frac{J''(t)}{2} - (\theta'(t))^2 \right) < 0 \quad \text{for all } t \in [0, T^*). \quad (4.7)$$

This implies that the solution $u(t)$ of (1.3) blows up in finite time. If not, i.e., $T^* = +\infty$, it follows from $\theta'(0) \leq 0$ and (4.7) that

$$\theta(t) = \theta(1) + \int_1^t \theta'(s) ds < \theta(1) + \theta'(1)(t-1) < 0,$$

for sufficiently large t , which is a contradiction with $\theta(t) > 0$ for all $t \in [0, T^*)$. \square

Finally, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 We claim that the result in Theorem 4.1 can be valid for arbitrarily large energy. Let $v_0 = \gamma^2 Q(\gamma x)$ for $\gamma > 0$. By some simple calculations, we have

$$\|v_0\|_{L^2}^2 = \gamma \|Q\|_{L^2}^2, \quad F(v_0) = \gamma^5 F(Q),$$

and

$$E(v_0) = \frac{\gamma^3}{2} \|\nabla Q\|_{L^2}^2 + \frac{1}{\gamma} \int_{\mathbb{R}^3} W(x) |Q|^2 dx - \frac{\gamma^5}{2} F(Q).$$

These yield

$$4\|v_0\|_{L^2}^2 E(v_0) \leq \|Q\|_{L^2}^2 F(Q) \quad \text{and} \quad \|v_0\|_{L^2}^2 F(v_0) > \|Q\|_{L^2}^2 F(Q) \quad (4.8)$$

for sufficiently large $\gamma > 0$.

Let $u_0(x) = e^{i\lambda|x|^2} v_0(x)$ with $\lambda < 0$ and $u(t)$ be the corresponding solution of (1.3). It easily follows that

$$\|u_0\|_{L^2}^2 = \|v_0\|_{L^2}^2 \quad \text{and} \quad E(u_0) = E(v_0) + 2\lambda^2 \|xv_0\|_{L^2}^2 + 2\lambda \operatorname{Im} \int_{\mathbb{R}^3} \bar{v}_0 x \cdot \nabla v_0 dx, \quad (4.9)$$

$$\operatorname{Im} \int_{\mathbb{R}^3} \bar{u}_0 x \cdot \nabla u_0 dx = \operatorname{Im} \int_{\mathbb{R}^3} \bar{v}_0 x \cdot \nabla v_0 dx + 2\lambda \|xv_0\|_{L^2}^2. \quad (4.10)$$

Combining (4.9) and (4.10), we further infer

$$E(u_0) - \frac{(\operatorname{Im} \int_{\mathbb{R}^3} \bar{u}_0 x \cdot \nabla u_0 dx)^2}{2\|xu_0\|_{L^2}^2} = E(v_0) - \frac{(\operatorname{Im} \int_{\mathbb{R}^3} \bar{v}_0 x \cdot \nabla v_0 dx)^2}{2\|xv_0\|_{L^2}^2},$$

which implies (4.3). In addition, due to (4.8), the estimate (4.2) also holds. Choosing sufficiently small $\lambda \in (-\infty, 0)$, we deduce from formulas (4.9) and (4.10) that

$$4\|u_0\|_{L^2}^2 E(u_0) > \|Q\|_{L^2}^2 F(Q) \quad \text{and} \quad \operatorname{Im} \int_{\mathbb{R}^3} x \bar{u}_0 \nabla u_0 dx < 0.$$

Applying Theorem 4.1, the solution $u(t)$ of (1.3) with initial data u_0 blows up in finite time. On the other hand, we see from (4.9) that $E(u_0) \rightarrow +\infty$ as $\lambda \rightarrow -\infty$. Therefore, the initial energy can be arbitrarily large. This completes the proof. \square

5 Conclusions

In this paper, we study the existence of blow-up solutions for the nonlinear Schrödinger equation arising in trapped dipolar quantum gases with arbitrarily large initial energy. We consider two cases, one is that the system is free, the other is that a harmonic potential is added. In both cases, by constructing an invariant set, we establish a new blow-up criterion, which implies the existence of blow-up solutions with arbitrarily large initial energy.

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Authors' contributions

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