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A general stability result for second order stochastic quasilinear evolution equations with memory

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Abstract

The goal of this paper is to discuss an initial boundary value problem for the stochastic quasilinear viscoelastic evolution equation with memory driven by additive noise. We prove the existence of global solution and asymptotic stability of the solution using some properties of the convex functions. Moreover, our result is established without imposing restrictive assumptions on the behavior of the relaxation function at infinity.

MSC: 60H15; 35L05; 35L70

Keywords: Quasilinear stochastic viscoelastic wave equations; Explosive solutions; Energy inequality

1 Introduction

The quasilinear viscoelastic wave equation of the following form:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + h(u_t) = f(u), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in D, \end{cases} \tag{1.1}$$

describes a viscoelastic material, with $u(x,t)$ giving the position of material particle x at time t, where D is a bounded domain in $\mathbb{R}^d$ with a smooth boundary ∂D, $\rho > 0$, g is the relaxation function, f denotes the body force, and h is the damping term. The properties of the solution to (1.1) have been studied by many authors (see [1–7]). For instance, in [1], Cavalcanti et al. considered (1.1) for $h(u_t) = -\gamma\Delta u_t$ and $f(u) = 0$, where $0 < \rho \le 2/(d-2)$ if $d \ge 3$ or $\rho > 0$ if $d = 1, 2$. They proved a global existence result when the constant $\gamma \ge 0$ and an exponential decay result for the case $\gamma > 0$. Messaoudi et al. [4] studied (1.1) for $h(u_t) = -\Delta u_{tt}$ and $f(u) = 0$, they proved an explicit and general decay rate result with some properties of the convex functions. Liu [5] considered (1.1) for $h(u_t) = 0$ and $f(u) = b|u|^{p-2}u$, where $b > 0, p > 2$. The author proved that, for a certain class of relaxation functions and certain initial data in the stable set, the decay rate of the solution energy is similar to that of the relaxation function. Conversely, he also obtained for certain initial data in the unstable set that there are solutions that blow up in finite time. In [6], Song investigated (1.1) for $h(u_t) = |u_t|^{q-2}u_t$ and $f(u) = |u|^{p-2}u$, where $q > 2$, and ρ, p satisfy

$$\begin{cases} 2 < p < \infty, \qquad 2 < \rho < \infty, \quad \text{if } d = 1, 2, \\ 2 < p < 2(d-1)/(d-2), \qquad 2 < \rho \leq d/(d-2), \quad \text{if } d \geq 3. \end{cases}$$

He proved the global nonexistence of the positive initial energy solutions of the quasilinear viscoelastic wave equation. Cavalcanti et al. [7] also studied (1.1) with $h(u_t) = a(x)u_t$ and $f(u) = b|u|^{p-2}u$, where $a(x)$ can be null on a part of the boundary, they obtained an exponential rate of decay of solutions.

In fact, the driving force may be affected by the random environment. In view of this, we consider the following stochastic quasilinear viscoelastic wave equations:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u \, ds = \sigma(x,t)\partial_t W(t,x), \quad \text{in } D \times (0,\infty), \\ u(x,t) = 0, \quad \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } D, \end{cases} \tag{1.2}$$

where g is a positive function satisfying some conditions to be specified later, σ is local Lipschitz continuous, $W(t,x)$ is an infinite dimensional Wiener random field, and the initial data $u_0(x)$ and $u_1(x)$ are $\mathcal{F}_0$-measurable given functions.

To motivate our work, let us firstly recall some results regarding $\rho = 0$ and $g \equiv 0$, then (1.2) can be rewritten as the following stochastic wave equation:

$$u_{tt} - \Delta u + h(u_t) = f(u) + \sigma(x,t)\partial_t W(t,x), \quad x \in D, t \in (0,T). \tag{1.3}$$

In [8, 9], Chow considered the large-time asymptotic properties of solutions to a class of semi-linear stochastic wave equations with linear damping in a bounded domain. Under appropriate conditions, the author obtained the exponential stability of an equilibrium solution in mean-square and the almost sure sense by energy inequality. Using Lyapunov function techniques, Brzeźniak et al. [10] proved global existence and stability of solutions for the stochastic nonlinear beam equations. In [11], Brzeźniak and Zhu studied a type of stochastic nonlinear beam equation with locally Lipschitz coefficients. Using a suitable Lyapunov function and applying the Khasminskii test they showed the nonexplosion of the mild solutions. In addition, under some additional assumptions they proved the exponential stability of the solution. Kim [12] and Barbu et al. [13] investigated initial boundary value stochastic wave equations with nonlinear damping and dissipative damping, respectively. There are also many results on the stochastic wave equations, see the references in [10, 14–20].

When $\rho = 0$ and $g \neq 0$, (1.2) can be rewritten as the following stochastic viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u \, ds + h(u_t) = f(u) + \varepsilon\sigma(x,t)\partial_t W(t,x), \quad x \in D, t \in (0,T). \tag{1.4}$$

For the current equation (1.4), the memory part makes it difficult to estimate the energy by using these methods which are used in stochastic wave equation. Hence, Wei and Jiang

[21] studied (1.4) with $\sigma \equiv 1$ and $q = 2$ in another way. They showed the existence and uniqueness of solution for (1.4) and obtained the decay estimate of the energy function of the solution under some appropriate assumption on g. In [22], Liang and Gao extended the existence and uniqueness results of [21] with $\sigma = \sigma(u, \nabla u, x, t)$. In the case of $\sigma = \sigma(x, t)$, they proved that the solution either blows up in finite time with positive probability or is explosive in L^2 using the energy inequality. Furthermore, Liang and Gao [23] considered (1.4) driven by Lévy noise, they proved the global existence and uniqueness of the mild solution with the appropriate energy function and obtained the exponential stability of the solutions. Liang and Guo [24] studied (1.4) driven by multiplicative noise, the authors proved the global existence and asymptotic stability of the mild solution by the Lyapunov function.

Furthermore, Kim et al. [25] considered (1.2) with $\rho \neq 0$ and $g \neq 0$ driven by an additive noise, i.e., $\sigma = \sigma(x, t)$. By an appropriate energy inequality, they proved that finite time blow-up is possible for equation (1.2) if $p > \{q, \rho + 2\}$ and the initial energy is sufficiently negative.

We note that in the above literature, Messaoudi et al. [4], Liang and Gao [23], Chen et al. [24], and Kim et al. [25] did not discuss the optimality of the decay rate of (1.2) under the influence of random environment. We prove the stability of solutions to (1.2) by modifying the convex functions. The result of this paper provides an explicit energy decay formula that allows a larger class of functions g from which the energy decay rates are not necessarily of exponential or polynomial types.

This paper is organized as follows. In Sect. 2, we present some assumptions and definitions needed for our work. Section 3 shows the statement and proof of our main result.

2 Preliminaries

Firstly, let us introduce some notations used throughout this paper. We set $H = L^2(D)$ with the inner product and norm denoted by $(\cdot, \cdot)$ and $\|\cdot\|_2$, respectively. Denote by $\|\nabla \cdot\|_2$ the Dirichlet norm in $V = H_0^1(D)$. We consider the following hypotheses.

(A1) ρ, p, q satisfy

$$\begin{cases} 0 < \rho \leq \frac{2}{d-2}, & \text{if } d \geq 3, \\ \rho > 0, & \text{if } d = 1, 2. \end{cases} \tag{2.1}$$

(A2) $g \in C^1[0, \infty)$ is a nonnegative and nonincreasing function satisfying

$$1 - \int_0^\infty g(s)\,ds = l > 0. \tag{2.2}$$

(A3) There exists a positive function $H \in C^1(\mathbb{R}+)$, with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex C^2 function on $(0, r]$ for some $r < 1$ such that

$$g'(t) \leq -H\bigl(g(t)\bigr), \quad \forall t \geq 0. \tag{2.3}$$

Definition 2.1 Assume that $(u_0, u_1) \in H_0^1(D) \times L^2(D)$ and $\mathbf{E} \int_0^T \|\sigma(t)\|^2\,dt < \infty$, u is said to be the solution of (1.2) on the interval $[0, T]$, if (u, u_t) is $H_0^1(D) \times L^2(D)$-valued progressively measurable, $(u, u_t) \in L^2(\Omega; C([0, T]; H_0^1(D) \times L^2(D)))$, $u_t \in L^q((0, T) \times D)$, and such that (1.2) holds in the sense of distributions over $(0, T) \times D$ for almost all ω.

Theorem 2.1 *Assume that* $(u_0,u_1)\in H_0^1(D)\times L^2(D)$, $\mathbf{E}\int_0^T\|\sigma(t)\|^2\,dt<\infty$, *and condition* (2.1) *holds.* u_t *is a solution of* (1.2) *with initial data* $(u_0,u_1)\in H_0^1(D)\times L^2(D)$, *according to Definition* 2.1 *on the interval* $[0,T]$, *for any* $T>0$, *we have*

$$\mathbf{E}\sup_{0\le t\le T}e\bigl(u(t)\bigr)<\infty.$$

Similar to Theorem 4.1 of [25], we can explicitly drive the proof of the above theorem. Now, we introduce the "modified" energy associated with problem (1.2):

$$\mathcal{E}(t)=\frac{1}{\rho+2}\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2}+\frac{1}{2}l\bigl\|\nabla u(t)\bigr\|_2^2+\frac{1}{2}\bigl\|\nabla u_t(t)\bigr\|_2^2+\frac{1}{2}(g\circ\nabla u)(t), \tag{2.4}$$

where, for any $w\in L^2(D)$,

$$(g\circ w)(t)=\int_0^t g(t-s)\bigl\|w(t)-w(s)\bigr\|_2^2\,ds.$$

Let $(\Omega,P,\mathcal{F})$ be a complete probability space for which $\{\mathcal{F}_t,t\ge 0\}$ of sub-σ-fields of $\mathcal{F}$ is given. A point of D will be denoted by D and $\mathbf{E}(\cdot)$ stands for expectation with respect to probability measure P. When $\mathcal{O}$ is a topological space, $\mathcal{B}$ denotes the Borel σ-algebra over $\mathcal{O}$. Suppose that $\{W(t,x):t\ge 0\}$ is a H-value Q-Wiener process on the probability space with the variance operator Q satisfying $\operatorname{Tr}Q<\infty$. Moreover, we can assume that Q has the following form: $Qe_i=\lambda_i e_i$, $i=1,2,\dots$, where λ_i are eigenvalues of Q satisfying $\sum_{i=1}^\infty\lambda_i<\infty$ and $\{e_i\}$ are the corresponding eigenfunctions with $c_0:=\sup_{i\ge 1}\|e_i\|_\infty<\infty$ ($\|\cdot\|_\infty$ denotes the super-norm). To simplify the computations, we assume that the covariance operator Q and $-\Delta$ with homogeneous Dirichlet boundary condition have a common set of eigenfunctions, i.e., $\{e_i\}_{i=1}^\infty$ satisfy

$$\begin{cases}-\Delta e_i=\mu_i e_i, & x\in D,\\ e_i=0, & x\in\partial D,\end{cases} \tag{2.5}$$

and form an orthonormal base of H. In this case, $W(t,x)=\sum_{i=1}^\infty\sqrt{\lambda_i}B_i(t)e_i$, where $\{B_i(t)\}$ is a sequence of independent copies of standard Brownian motions in one dimension. In addition, $\{W(t,x):t\ge 0\}$ is an H-valued Q-Wiener process. For more details about the infinite-dimensional Wiener process and the stochastic integral, see in [26, 27].

3 Stability properties of solutions

In this section, we state and prove our main stability result. Throughout this section, we suppose that $\sigma(x,t,w)=\sigma(x,t)$ such that

$$\int_0^\infty\int_D\sigma^2(x,t)\,dx\,dt<\infty. \tag{3.1}$$

As is well known, equation (1.2) is equivalent to the following Itô system:

$$\begin{cases}du=v\,dt,\\ d(\frac{1}{\rho+1}|v|^\rho v)=(\Delta u+\Delta u_{tt}-\int_0^t g(t-s)\Delta u(s)\,ds)\,dt+\sigma(x,t)\,dW(t,x).\end{cases} \tag{3.2}$$

In order to prove our stability result, we need the following lemmas.

Lemma 3.1 *Let $u_0(x)$ and $u_1(x)$ be $\mathcal{F}_0$-measurable with $u_0(x) \in H_0^1(D)$ and $u_1(x) \in L^2(D)$. Assume (2.1) holds. Let (u,v) be a solution of system (3.2). Then we have*

$$\begin{aligned}\frac{d}{dt}\mathbf{E}\mathcal{E}(t) &= \frac{1}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx - \frac{1}{2}\mathbf{E}\bigl(-g' \circ \nabla u\bigr)(t) - \frac{1}{2}g(t)\mathbf{E}\bigl\|\nabla u(t)\bigr\|_2^2 \\ &\leq \frac{1}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx + \frac{1}{2}\mathbf{E}\bigl(g' \circ \nabla u\bigr)(t).\end{aligned} \tag{3.3}$$

Proof Applying Itô's formula to $\frac{2}{\rho+2}\|v\|_{\rho+2}^{\rho+2}$, we get

$$\begin{aligned}\frac{2}{\rho+2}\|v\|_{\rho+2}^{\rho+2} &= \frac{2}{\rho+2}\|v_0\|_{\rho+2}^{\rho+2} - 2\int_0^t (\nabla u, \nabla v)\,ds + 2\int_D\biggl(\int_0^t g(t-s)\nabla u(s)\,ds, \nabla v(t)\biggr)\,dx \\ &\quad - 2\int_0^t (\nabla u_{tt}, \nabla v)\,ds + 2\int_0^t \bigl(v, \sigma(x,s)\,dW_s\bigr) + \int_0^t \bigl\|\sigma(x,s)\bigr\|_{L_2^0}^2\,ds \\ &= 2\mathbf{E}\mathcal{E}(0) - \bigl\|\nabla u(t)\bigr\|_2^2 - \bigl\|\nabla u_t(t)\bigr\|_2^2 + 2\int_D\biggl(\int_0^t g(t-s)\nabla u(s)\,ds, \nabla v(t)\biggr)\,dx \\ &\quad + 2\int_0^t \bigl(v, \sigma(x,s)\,dW_s\bigr) + \int_0^t \bigl\|\sigma(x,s)\bigr\|_{L_2^0}^2\,ds.\end{aligned} \tag{3.4}$$

For the third term on the right-hand side of (3.4), we obtain

$$\begin{aligned}&2\biggl(\int_0^t g(t-s)\nabla u(s)\,ds, v(t)\biggr) \\ &\quad = 2\int_0^t g(t-s)\int_D \nabla v(t)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,dx\,ds + 2\int_0^t g(t-s)\int_D \nabla v(t)\nabla u(t)\,dx\,ds \\ &\quad = -\int_0^t g(t-s)\frac{d}{dt}\int_D \bigl|\nabla u(s) - \nabla u(t)\bigr|^2\,dx\,ds + \int_0^t g(s)\frac{d}{dt}\int_D \bigl|\nabla u(t)\bigr|^2\,dx\,ds \\ &\quad = \frac{d}{dt}\biggl(\int_0^t g(s)\,ds\bigl\|\nabla u(t)\bigr\|_2^2 - (g \circ \nabla u)(t)\biggr) + \bigl(g' \circ \nabla u\bigr)(t) - g(t)\bigl\|\nabla u(t)\bigr\|_2^2.\end{aligned} \tag{3.5}$$

Inserting (3.5) into (3.4) and taking the expectation for (3.4), we get (3.3). □

Let

$$S(t) = \frac{1}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_0^t\int_D \lambda_i e_i^2(x)\sigma^2(x,s)\,dx\,ds.$$

From (3.1), we have

$$\begin{aligned}S(\infty) &= \frac{1}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_0^{\infty}\int_D \lambda_i e_i^2(x)\sigma^2(x,s)\,dx\,ds \\ &\leq \frac{1}{2}c_0^2\,\mathrm{Tr}\,R\mathbf{E}\int_0^{\infty}\int_D \sigma^2(x,s)\,dx\,ds \\ &= E_1 < \infty.\end{aligned} \tag{3.6}$$

Then we integrate (3.3) on $(0, T)$, which yields

$$\mathbf{E}\mathcal{E}(t) \le \mathbf{E}\mathcal{E}(0) + E_1. \tag{3.7}$$

Lemma 3.2 *Let u be a solution of (1.2). The functional*

$$\Psi(t) := \frac{1}{\rho+1}\int_D |u_t|^\rho u_t u\,dx + \int_D \nabla u \cdot \nabla u_t\,dx$$

satisfies, along the solution of (1.2), the estimate

$$\begin{aligned}\mathbf{E}\Psi'(t) \le{}& -\frac{l}{2}\mathbf{E}\int_D |\nabla u|^2\,dx + \mathbf{E}\int_D |\nabla u_t|^2\,dx + \frac{1}{\rho+1}\mathbf{E}\int_D |u_t|^{\rho+2}\,dx \\ &+ \frac{1-l}{2l}\mathbf{E}(g\circ\nabla u)(t).\end{aligned} \tag{3.8}$$

Proof Direct differentiation of Ψ, using (1.2), yields

$$\begin{aligned}\Psi'(t) ={}& -\int_D |\nabla u|^2\,dx + \int_D \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx + \int_D |\nabla u_t|^2\,dx \\ &+ \frac{1}{\rho+1}\int_D |u_t|^{\rho+2}\,dx + \int_D \bigl(\sigma(x,s)\,dW_s, u\bigr)\,dx.\end{aligned}$$

We take the expectation of the above formula to get the following result:

$$\begin{aligned}\mathbf{E}\Psi'(t) ={}& -\mathbf{E}\int_D |\nabla u|^2\,dx + \mathbf{E}\int_D \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &+ \mathbf{E}\int_D |\nabla u_t|^2\,dx + \frac{1}{\rho+1}\mathbf{E}\int_D |u_t|^{\rho+2}\,dx\end{aligned} \tag{3.9}$$

for any general solution. With simple density parameters, this estimate is still applicable for weak solutions. Then we estimate that the second item on the right-hand side of (3.9) is as follows:

$$\begin{aligned}&\int_D \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &\quad\le \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D\left(\int_0^t g(t-s)\bigl(\bigl|\nabla u(t)-\nabla u(s)\bigr| + \bigl|\nabla u(t)\bigr|\bigr)\,ds\right)^2 dx.\end{aligned}$$

By using

$$\int_0^t g(s)\,ds < \int_0^\infty g(s)\,ds = 1-l \tag{3.10}$$

and

$$(a+b)^2 \le (1+\eta)a^2 + \left(1+\frac{1}{\eta}\right)b^2, \quad \forall \eta > 0,$$

we arrive at

$$\int_D \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx$$

$$\begin{aligned}
&\le \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}(1+\eta)(1-l)^2\int_D |\nabla u|^2\,dx \\
&\quad + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_D\left(\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|\,ds\right)^2 dx \\
&\le \frac{1}{2}\left[1+(1+\eta)(1-l)^2\right]\int_D |\nabla u|^2\,dx + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t).
\end{aligned}$$

By taking $\eta = \frac{l}{1-l}$, we obtain

$$\int_D \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx \le \frac{2-l}{2}\int_D |\nabla u|^2\,dx + \frac{1-l}{2l}(g\circ\nabla u)(t). \tag{3.11}$$

Inserting (3.11) in (3.9), we get (3.8). □

Lemma 3.3 *Let u be a solution of* (1.2). *The functional*

$$\chi(t) := \int_D\left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \tag{3.12}$$

satisfies the solution of equation (1.2) *and, for any $\delta_1,\delta_2>0$, the estimate*

$$\begin{aligned}
\mathbf{E}\chi'(t) &\le \big(1+2(1-l)^2\big)\delta_1\mathbf{E}\int_D |\nabla u|^2\,dx - \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\mathbf{E}\int_D |u_t|^{\rho+2}\,dx \\
&\quad + (1-l)\left(2\delta_1+\frac{1}{2\delta_1}\right)\mathbf{E}(g\circ\nabla u)(t) + \frac{g(0)}{4\delta_2}\left(1+\frac{C_p}{\rho+1}\right)\mathbf{E}\big(-g'\circ\nabla u\big)(t) \\
&\quad + \left[\delta_2 + c\frac{\delta_2}{\rho+1}\big(2\big(\mathcal{E}(0)+E_1\big)\big)^\rho - \int_0^t g(s)\,ds\right]\mathbf{E}\int_D |\nabla u_t|^2\,dx.
\end{aligned} \tag{3.13}$$

Proof Differentiating (3.12) with respect to t and making use of (1.2), we arrive at

$$\begin{aligned}
\chi'(t) &= \int_D \nabla u(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad - \int_D\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\cdot\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx \\
&\quad - \left(\int_0^t g(s)\,ds\right)\int_D |\nabla u_t|^2\,dx - \int_D \nabla u_t\cdot\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\quad - \frac{1}{\rho+1}\int_D |u_t|^\rho u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad - \int_D \sigma(x,s)\,dW_s\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad - \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\int_D |u_t|^{\rho+2}\,dx.
\end{aligned}$$

We take the expectation of the above formula to get the following result:

$$\mathbf{E}\chi'(t) = \mathbf{E}\int_D \nabla u(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx$$

$$
\begin{aligned}
&-\mathbf{E}\int_D\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\cdot\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx\\
&-\mathbf{E}\left(\int_0^t g(s)\,ds\right)\int_D|\nabla u_t|^2\,dx-\mathbf{E}\int_D\nabla u_t\cdot\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\\
&-\frac{1}{\rho+1}\mathbf{E}\int_D|u_t|^\rho u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx\\
&-\frac{1}{\rho+1}\mathbf{E}\left(\int_0^t g(s)\,ds\right)\int_D|u_t|^{\rho+2}\,dx.
\end{aligned}
\tag{3.14}
$$

Now we repeat the Cauchy–Schwarz inequality, Hölder's inequality and Young's inequality, to estimate each term on the right-hand side of equation (3.14).

The first item on the right can be estimated as follows:

$$
\begin{aligned}
&\int_D\nabla u(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\\
&\quad\le\delta_1\int_D|\nabla u|^2\,dx+\frac{1-l}{4\delta_1}(g\circ\nabla u)(t),\quad \delta_1>0.
\end{aligned}
\tag{3.15}
$$

As for the second item, we can get the following result from the previously obtained formula (3.10) and $(a+b)^2\le 2(a^2+b^2)$:

$$
\begin{aligned}
&\int_D\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\cdot\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx\\
&\quad\le\delta_1\int_D\left|\int_0^t g(t-s)\nabla u(s)\,ds\right|^2dx+\frac{1}{4\delta_1}\int_D\left|\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right|^2dx\\
&\quad\le\delta_1\int_D\left(\int_0^t g(t-s)\big(\big|\nabla u(t)-\nabla u(s)\big|+\big|\nabla u(t)\big|\big)\,ds\right)^2dx\\
&\qquad+\frac{1}{4\delta_1}\left(\int_0^t g(t-s)\,ds\right)\int_D\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2\,ds\,dx\\
&\quad\le 2\delta_1\int_D\left(\int_0^t g(t-s)\big|\nabla u(t)-\nabla u(s)\big|\,ds\right)^2dx\\
&\qquad+2\delta_1\left(\int_0^t g(s)\,ds\right)^2\int_D|\nabla u|^2\,dx+\frac{1}{4\delta_1}\left(\int_0^t g(s)\,ds\right)(g\circ\nabla u)(t)\\
&\quad\le\left(2\delta_1+\frac{1}{4\delta_1}\right)(1-l)(g\circ\nabla u)(t)+2\delta_1(1-l)^2\int_D|\nabla u|^2\,dx.
\end{aligned}
\tag{3.16}
$$

For the fourth term on the right-hand side of (3.14), it is easy to draw, $\forall\delta_2>0$,

$$
\begin{aligned}
&\int_D\nabla u_t(t)\cdot\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx\\
&\quad\le\delta_2\int_D|\nabla u_t|^2\,dx+\frac{g(0)}{4\delta_2}\int_D\int_0^t -g'(t-s)\big|\nabla u(t)-\nabla u(s)\big|^2\,ds\,dx.
\end{aligned}
\tag{3.17}
$$

For the fifth item, we can similarly get the following results:

$$
\frac{1}{\rho+1}\int_D|u_t|^\rho u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx
$$

$$\leq \frac{1}{\rho+1}\bigg[\delta_2 \int_D |u_t|^{2(\rho+1)}\,dx$$
$$+ \frac{g(0)}{4\delta_2} C_p \int_D \int_0^t -g'(t-s)\big|\nabla u(t) - \nabla u(s)\big|^2\,ds\,dx\bigg], \tag{3.18}$$

where C_p is the Poincaré constant and $\delta_2 > 0$. By using the Sobolev embedding

$$H_0^1(D) \hookrightarrow L^{2(\rho+1)}(D) \quad \text{for } 0 < \rho \leq 2/(n-2) \text{ if } n \geq 3 \text{ and } \rho > 0 \text{ if } n = 1, 2, \tag{3.19}$$

and by (3.7), $\forall t \geq 0$, we get

$$\int_D |u_t|^{2(\rho+1)}\,dx \leq c\big(2\big(\mathcal{E}(0) + E_1\big)\big)^\rho \int_D |\nabla u_t|^2\,dx. \tag{3.20}$$

Then (3.18) has the following form:

$$\frac{1}{\rho+1}\int_D |u_t|^\rho u_t \int_0^t g'(t-s)\big(u(t) - u(s)\big)\,ds\,dx$$
$$\leq c\delta_2\big(2\big(\mathcal{E}(0) + E_1\big)\big)^\rho \int_D |\nabla u_t|^2\,dx + \frac{g(0)C_p}{4\delta_2(\rho+1)}\big(-g' \circ \nabla u\big)(t). \tag{3.21}$$

Combining (3.14)–(3.17) and (3.21), we get (3.13). The proof is completed. □

Theorem 3.4 *Let $u_0(x)$ and $u_1(x)$ be $\mathcal{F}_0$-measurable with $u_0(x) \in L^2(\Omega; H_0^1(D))$ and $u_1(x) \in L^2(\Omega; L^2(D))$. Assume that* (A1)–(A3) *hold. Then there exist positive constants k_1, k_2, k_3, and ε_0 such that the solution of* (1.2) *satisfies*

$$\mathbf{E}\mathcal{E}(t) \leq k_3 H_1^{-1}(k_1 t + k_2) \quad \forall t \geq 0, \tag{3.22}$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)}\,ds \quad \text{and} \quad H_0(t) = H\big(J(t)\big)$$

provided that J is a positive C^1 function, with $J(0) = 0$, for which H_0 is a strictly increasing and convex C^2 function on $(0, r]$ and

$$\int_t^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))}\,ds < +\infty, \tag{3.23}$$

$H(t) = ct^p$, for $1 < p < \frac{3}{2}$. Moreover, if $\int_0^t H_1(t)\,dt < +\infty$ for some choice of J, then we have the improved estimate:

$$\mathbf{E}\mathcal{E}(t) \leq k_3 G^{-1}(k_1 t + k_2) \quad \text{where } G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)}\,ds. \tag{3.24}$$

Remark 3.1

1. By using the property of H, we can show that the H_1 function is strictly decremented and raised on $(0, 1]$, with $\lim_{t\to 0} H_1(t) = +\infty$. So Theorem 3.4 ensures

$$\lim_{t\to 0} \mathcal{E}(t) = 0.$$

2. Our result is obtained under the very general assumption of the relaxation function g, which allows the processing of the larger class function g, which guarantees uniform stability of (1.2) and has a decay rate explicit formula energy.
3. The usual exponential and polynomial decay rate estimates have proven that g is satisfied (2.2) and $g' \leq -kg^p$, $1 \leq p < 3/2$, it is a special case of our results. For these special cases, we will prove that this is a "simple" proof.
4. Our results allow the relaxation function to not necessarily exhibit exponential decay or polynomial decay. For example, if

$$g(t) = ae^{-t^q}$$

for $0 < q < 1$ and a is chosen so that g satisfies (2.2), then $g'(t) = -H(g(t))$ where, for $t \in (0, r]$, $r < a$,

$$H(t) = \frac{qt}{[\ln(a/t)]^{\frac{1}{q}-1}},$$

which satisfies hypothesis (A3). Also, by taking $J(t) = t^\alpha$, (3.23) is satisfied with any $\alpha > 1$. For this reason, we can use Theorem 3.4 and perform some calculations to infer that the energy is attenuated by the same g, i.e.,

$$\mathbf{E}\mathcal{E}(t) \leq ce^{-kt^q}.$$

5. With (A2) and (A3), we can easily infer $\lim_{t\to\infty} g(t) = 0$. This means that $\lim_{t\to+\infty}(-g'(t))$ cannot be equal to a positive number, so it is natural to assume $\lim_{t\to+\infty}(-g'(t)) = 0$. Therefore, there is $t_1 > 0$ big enough so that $g(t_1) > 0$ and

$$\max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \geq t_1. \tag{3.25}$$

As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$, then $g(t) > 0$ for any $t \in [0, t_1]$ and

$$0 < g(t_1) \leq g(t) \leq g(0), \quad \forall t \in [0, t_1].$$

Hence, since H is a positive continuous function, then

$$a \leq H\big(g(t)\big) \leq b, \quad \forall t \in [0, t_1]$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$g'(t) \leq -H\big(g(t)\big) \leq -a = -\frac{a}{g(0)} g(0) \leq -\frac{a}{g(0)} g(t),$$

which gives, for some positive constant d, we have

$$g'(t) \leq -dg(t), \quad \forall t \in [0, t_1]. \tag{3.26}$$

Now let us prove Theorem 3.4.

Proof We consider the functional

$$L(t) = \mathbf{E}\big(M\mathcal{E}(t) + \varepsilon\Psi(t) + \chi(t)\big),$$

where M and ε are to be specified later. Let $g_0 = \int_0^{t_1} g(s)\,ds$, where $t_1 > 0$ was introduced in (3.25). Using (3.3), (3.8), (3.13), and $\int_0^t g(s)\,ds \geq g_0$, $t \geq t_1$, we obtain

$$\begin{aligned} L'(t) \leq \mathbf{E}\Bigg(&\left[\frac{M}{2} - \frac{g(0)}{4\delta_2}\left(1 + \frac{C_p}{\rho+1}\right)\right]\big(g' \circ \nabla u\big)(t) + \frac{\varepsilon - g_0}{\rho+1}\int_D |u_t|^{\rho+2}\,dx \\ &- \left[\frac{l}{2}\varepsilon - \big(1 + 2(1-l)^2\big)\delta_1\right]\int_D |\nabla u|^2\,dx \\ &- \big[g_0 - \varepsilon - \delta_2 - c\delta_2\big(2\big(\mathcal{E}(0) + E_1\big)\big)^\rho\big]\int_D |\nabla u_t|^2\,dx \\ &+ (1-l)\left(\frac{\varepsilon}{2l} + 2\delta_1 + \frac{1}{2\delta_1}\right)(g \circ \nabla u)(t) \\ &+ \frac{M}{2}\sum_{i=1}^\infty \int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\Bigg). \end{aligned} \tag{3.27}$$

At this point, we choose our constant carefully. First, we choose $\varepsilon < g_0$, then δ_1 and δ_2 are small enough so that

$$\frac{l}{2}\varepsilon - \big(1 + 2(1-l)^2\big)\delta_1 > 0, \qquad g_0 - \varepsilon - \delta_2 - c\delta_2\big(2\big(\mathcal{E}(0) + E_1\big)\big)^\rho > 0.$$

Finally, we take M sufficiently large so that

$$\frac{M}{2} - \frac{g(0)}{4\delta_2}\left(1 + \frac{C_p}{\rho+1}\right) \geq 0.$$

Therefore, (3.27) reduces to

$$L'(t) \leq -k\mathbf{E}\mathcal{E}(t) + c\mathbf{E}(g \circ \nabla u)(t) + \frac{M}{2}\sum_{i=1}^\infty \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx, \quad \forall t \geq t_1. \tag{3.28}$$

On the other hand, we can choose M even larger (if needed) so that

$$L \sim \mathbf{E}\mathcal{E}. \tag{3.29}$$

Now,we use (3.3) and (3.26), for any $t \geq t_1$,

$$\begin{aligned} &\mathbf{E}\int_0^{t_1} g(s)\int_D \big|\nabla u(t) - \nabla u(t-s)\big|^2\,dx\,ds \\ &\quad \leq -\frac{1}{d}\mathbf{E}\int_0^{t_1} g'(s)\int_D \big|\nabla u(t) - \nabla u(t-s)\big|^2\,dx\,ds \\ &\quad \leq -c\left(\mathbf{E}\mathcal{E}'(t) - \sum_{i=1}^\infty \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\right). \end{aligned} \tag{3.30}$$

Next, we take $F(t) = L(t) + c\mathbf{E}\mathcal{E}(t) - (M + 2c)S(t)$, here M was mentioned in (3.27), $S(t) = \frac{1}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_0^t\int_D \lambda_i e_i^2(x)\sigma^2(x,s)\,dx\,ds$, we assume that $\sigma(x,t)$ satisfies

(A4) $S(t) \le \frac{1}{M^\gamma}, \gamma > 1$.

This means that $F(t)$ is equivalent to $\mathbf{E}\mathcal{E}(t)$, therefore, using (3.28)–(3.30), for all $t \ge t_1$, we get

$$F'(t) \le -m\mathbf{E}\mathcal{E}(t) + c\mathbf{E}\int_{t_1}^t g(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds. \tag{3.31}$$

In the case of $p = 1$, we use estimate (3.31) to get the following result:

$$\begin{aligned} F'(t) &\le -m\mathbf{E}\mathcal{E}(t) + c\mathbf{E}(g \circ \nabla u)(t) \\ &\le -m\mathbf{E}\mathcal{E}(t) - c\mathbf{E}\mathcal{E}'(t), \quad \forall t \ge t_1, \end{aligned}$$

which gives

$$(F + c\mathbf{E}\mathcal{E})'(t) \le -m\mathbf{E}\mathcal{E}(t), \quad \forall t \ge t_1.$$

Hence, using the fact that $F + c\mathbf{E}\mathcal{E} \sim \mathbf{E}\mathcal{E}$, we easily obtain

$$\mathbf{E}\mathcal{E}(t) \le c'e^{-ct} = cG^{-1}(t).$$

In the case of $1 < p < \frac{3}{2}$, one can easily show that $\int_0^{+\infty} g^{1-\delta_0}(s)\,ds < +\infty$ for any $\delta_0 < 2 - p$. Using (3.3) and (3.7), and choosing t_1 even larger if needed, we deduce that, for all $t \ge t_1$,

$$\begin{aligned} \eta(t) &:= \int_{t_1}^t g^{1-\delta_0}(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \\ &\le 2\int_{t_1}^t g^{1-\delta_0}(s)\int_D \left(\left|\nabla u(t)\right|^2 + \left|\nabla u(t-s)\right|^2\right) dx\,ds \\ &\le c\big(\mathcal{E}(0) + E_1\big)\int_{t_1}^t g^{1-\delta_0}(s)\,ds < 1. \end{aligned} \tag{3.32}$$

Then, Jensen's inequality, (3.3), hypotheses (A2) and (A3), and (3.32) altogether lead to

$$\begin{aligned} &\mathbf{E}\int_{t_1}^t g(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \\ &\quad= \mathbf{E}\int_{t_1}^t g^{\delta_0}(s)g^{1-\delta_0}(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \\ &\quad= \mathbf{E}\int_{t_1}^t g^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})}(s)g^{1-\delta_0}(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \\ &\quad\le \mathbf{E}\left(\eta(t)\left[\frac{1}{\eta(t)}\int_{t_1}^t g(s)^{(p-1+\delta_0)}g^{1-\delta_0}(s)\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds\right]^{\frac{\delta_0}{p-1+\delta_0}}\right) \\ &\quad\le \mathbf{E}\left[\int_{t_1}^t g(s)^p\int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds\right]^{\frac{\delta_0}{p-1+\delta_0}} \end{aligned}$$

$$
\begin{aligned}
&\le c\mathbf{E}\biggl[\int_{t_1}^{t} -g'(s)\int_D \bigl|\nabla u(t)-\nabla u(t-s)\bigr|^2\,dx\,ds\biggr]^{\frac{\delta_0}{p-1+\delta_0}}\\
&\le c\Biggl[-\mathbf{E}\mathcal{E}'(t)+\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\Biggr]^{\frac{\delta_0}{p-1+\delta_0}}.
\end{aligned}
$$

Then, in particular, for $\delta_0 = \frac{1}{2}$ we conclude that (3.31) becomes

$$
\begin{aligned}
F'(t) &\le -m\mathbf{E}\mathcal{E}(t) + c\Biggl[-\mathbf{E}\mathcal{E}'(t)+\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\Biggr]^{\frac{1}{2p-1}}\\
&\quad+\frac{M}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx.
\end{aligned}
$$

Now, we multiply both sides by $(\mathbf{E}\mathcal{E}(t))^{\alpha}$, using $\alpha = 2p-2$ and (3.3) to get

$$
\begin{aligned}
&\bigl(F(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha}\bigr)'\\
&\quad= F'(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha} + \alpha F(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha-1}\bigl(\mathbf{E}\mathcal{E}(t)\bigr)'\\
&\quad\le -m\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha} + c\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha}\Biggl[-\mathbf{E}\mathcal{E}'(t)+\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\Biggr]^{\frac{1}{1+\alpha}}\\
&\qquad+\frac{M}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha} + \alpha F(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha-1}\bigl(\mathbf{E}\mathcal{E}(t)\bigr)'.
\end{aligned}
$$

Then, applying Young's inequality, $q = 1+\alpha$, and $q' = \frac{1+\alpha}{\alpha}$, we get

$$
\begin{aligned}
&\bigl(F(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha}\bigr)'\\
&\quad\le -m\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha} + \varepsilon\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha} - C_\varepsilon\mathbf{E}\mathcal{E}'(t) + C_\varepsilon\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\\
&\qquad+\alpha F(t)\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha-1}\mathbf{E}\mathcal{E}'(t) + \frac{M}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha}\\
&\quad\le -m\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha} + \varepsilon\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha} - C_\varepsilon\mathbf{E}\mathcal{E}'(t) + C_\varepsilon\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\\
&\qquad+\frac{\alpha}{1+\alpha}\bigl(\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha}\bigr)' + \frac{M}{2}\sum_{i=1}^{\infty}\mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{\alpha}.
\end{aligned}
$$

Let $F_0(t) = F(t)(\mathbf{E}\mathcal{E}(t))^{\alpha} - \frac{\alpha}{1+\alpha}(\mathbf{E}\mathcal{E}(t))^{1+\alpha} + C_\varepsilon\mathbf{E}\mathcal{E}(t) - 2C_\varepsilon S(t) - M(\mathbf{E}\mathcal{E}(0)+E_1)^{\alpha}S(t)$. So choose $\varepsilon < m$ so that

$$
F_0'(t) \le -m'\bigl(\mathbf{E}\mathcal{E}(t)\bigr)^{1+\alpha}.
$$

It can be known from (A4) that it is obviously equivalent to $\mathbf{E}\mathcal{E}(t)$. Therefore we have, for some $a_0 > 0$,

$$F_0'(t) \le -a_0 F_0^{1+\alpha}(t),$$

from which we easily infer that

$$\mathbf{E}\mathcal{E}(t) \le \frac{c}{(c't + c'')^{\frac{1}{2p-2}}}. \tag{3.33}$$

By recalling that $p < \frac{3}{2}$ and using (3.33), we find that $\int_0^{+\infty} \mathcal{E}(s)\,ds < +\infty$. Therefore, noting that

$$\int_0^t \int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \le c \int_0^t \mathcal{E}(s)\,ds,$$

estimate (3.31) gives

$$\begin{aligned} F'(t) &\le -m\mathbf{E}\mathcal{E}(t) + c\mathbf{E}\bigl(g^{p\cdot\frac{1}{p}} \circ \nabla u\bigr)(t) \le -m\mathbf{E}\mathcal{E}(t) + c\bigl[\mathbf{E}\bigl(g^p \circ \nabla u\bigr)(t)\bigr]^{\frac{1}{p}} \\ &\le -m\mathbf{E}\mathcal{E}(t) + c\bigl[\mathbf{E}\bigl(-g' \circ \nabla u\bigr)(t)\bigr]^{\frac{1}{p}} \le -m\mathbf{E}\mathcal{E}(t) + c\bigl[-\mathbf{E}\mathcal{E}'(t)\bigr]^{\frac{1}{p}}. \end{aligned}$$

Hence, repeating the above steps, with $\alpha = p - 1$, we obtain

$$\mathbf{E}\mathcal{E}(t) \le \frac{c}{(c't + c'')^{\frac{1}{p-1}}} = cG^{-1}\bigl(c't + c''\bigr).$$

Thus the proof of Theorem 3.4 is completed. □

Remark 3.2 In particular, when $p = 1$, we also need not assume (A4). That is, taking $F(t) = L(t) + c\mathbf{E}\mathcal{E}(t)$, which is clearly equivalent to $\mathbf{E}\mathcal{E}(t)$, and using (3.28)–(3.30), for all $t \ge t_1$, we have

$$\begin{aligned} F'(t) \le &- m\mathbf{E}\mathcal{E}(t) + c\mathbf{E}\int_{t_1}^t g(s) \int_D \left|\nabla u(t) - \nabla u(t-s)\right|^2 dx\,ds \\ &+ \left(\frac{M}{2} + c\right) \sum_{i=1}^{\infty} \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx. \end{aligned} \tag{3.34}$$

Estimate (3.31) yields

$$\begin{aligned} F'(t) &\le -m\mathbf{E}\mathcal{E}(t) + c\mathbf{E}(g \circ \nabla u)(t) + \left(\frac{M}{2} + c\right) \sum_{i=1}^{\infty} \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx \\ &\le -m\mathbf{E}\mathcal{E}(t) - c\mathbf{E}\mathcal{E}'(t) + \left(\frac{M}{2} + c\right) \sum_{i=1}^{\infty} \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx, \quad \forall t \ge t_1, \end{aligned}$$

which gives

$$(F + c\mathbf{E}\mathcal{E})'(t) \le -m\mathbf{E}\mathcal{E}(t) + \left(\frac{M}{2} + c\right) \sum_{i=1}^{\infty} \mathbf{E}\int_D \lambda_i e_i^2(x)\sigma^2(x,t)\,dx, \quad \forall t \ge t_1.$$

Hence, using the fact that $F + c\mathbf{E}\mathcal{E} \sim \mathbf{E}\mathcal{E}$, it is easy to obtain

$$\mathbf{E}\mathcal{E}(t) \le c' e^{-ct} + (M + 2c)E_1.$$

This means that it is progressively stable and degenerates to $(M + 2c)E_1$.

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