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# Nonexistence and existence of positive radial solutions to a class of quasilinear Schrödinger equations in $\mathbb{R}^N$

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## Abstract

This paper aims to investigate the class of quasilinear Schrödinger equations

$$\begin{aligned} -\Delta u - \left[ \Delta(1 + u^2)^{\frac{\gamma}{2}} \right] \frac{\gamma u}{2(1 + u^2)^{\frac{2-\gamma}{2}}} \\ = \alpha h(|x|)|u|^{p-1}u + \beta H(|x|)|u|^{q-1}u, \quad x \in \mathbb{R}^N, \end{aligned} \quad (0.1)$$

where  $N > 2$ ,  $1 \leq \gamma \leq 2$ ,  $\alpha, \beta \in \mathbb{R}$  and either  $0 < p < 1 < q$  or  $1 < p < q$ . Functions  $h(|x|)$ ,  $H(|x|)$  are continuous and positive in  $\mathbb{R}^N$ . Relying on some special arguments and the Schauder–Tychonoff fixed point theorem, nonexistence criteria, existence of positive ground state solutions and blow-up solutions to Eq. (0.1) with  $0 < p < 1 < q$  or  $1 < p < q$  will be obtained.

**MSC:** 26A33; 34B18

**Keywords:** Quasilinear Schrödinger equation; Ground state solution; Blow-up solution; The Schauder–Tychonoff fixed point theorem

## 1 Preliminaries

This paper is concerned with the following quasilinear Schrödinger equation:

$$-\Delta u - \left[ \Delta(1 + u^2)^{\frac{\gamma}{2}} \right] \frac{\gamma u}{2(1 + u^2)^{\frac{2-\gamma}{2}}} = \alpha h(|x|)|u|^{p-1}u + \beta H(|x|)|u|^{q-1}u, \quad x \in \mathbb{R}^N, \quad (1.1)$$

where  $1 \leq \gamma \leq 2$ ,  $\alpha, \beta \in \mathbb{R}$  and either  $0 < p < 1 < q$  or  $1 < p < q$ .

This class of equations is often referred to as so-called modified nonlinear Schrödinger equations due to the quasilinear term  $\left[ \Delta(1 + u^2)^{\frac{\gamma}{2}} \right] \frac{\gamma u}{2(1 + u^2)^{\frac{2-\gamma}{2}}}$ , whose solutions are related to the standing wave solutions for the quasilinear Schrödinger equation

$$iz_t = -\Delta z + V(x)z - h(|z|^2)z - \left[ \Delta \Psi(|z|^2) \right] \Psi'(|z|^2)z, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where  $V$  is a given potential,  $\Psi$  and  $h$  are real functions.

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The quasilinear Schrödinger equation (1.2) has been derived as models of several physical phenomena corresponding to different types of  $\Psi$ ; see [1, 2]. The super fluid film equation in plasma physics has this structure for  $\Psi(s) = s^{2\alpha}$  [1, 3, 4]. For the case  $\Psi(s) = (1 + s)^{\alpha/2}$ , Eq. (1.2) was used to model the self-channeling of a high-power ultrashort laser in matter [5–8]. Especially, Cheng in [9] solved the existence of positive solutions to the following equation by a dual approach:

$$-\Delta u + Ku - \left[ \Delta(1 + u^2)^{\alpha/2} \right] \frac{\alpha u}{2(1 + u^2)^{(2-\alpha)/2}} = |u|^{q-1}u + |u|^{p-1}u, \tag{1.3}$$

where  $K > 0, \alpha \geq 1, 2 < q + 1 < p + 1 \leq \alpha 2^*$ . Similar work can be found in [10, 11] and the references therein.

Besides, Zhang, Liu, Wu and Cui [12] focused on the existence and nonexistence of entire blow-up solutions for the following quasilinear  $p$ -Laplacian Schrödinger equation with a non-square diffusion term:

$$\begin{cases} -\Delta_p z - \Delta_p(|z|^{2\gamma} |z|^{2\gamma-2} z) = q(x)g(z), \\ z > 0, \quad \text{in } \mathbb{R}^N, \quad z(x) \rightarrow \infty, \quad \text{as } |x| \rightarrow \infty, \end{cases} \tag{1.4}$$

where  $p \geq 2\gamma, \gamma > \frac{1}{2}$ , the nonnegative radial function  $q$  is continuous on  $\mathbb{R}^N$ ,  $g$  is a continuous positive and non-decreasing function on  $[0, \infty)$ . Chen and Chen [13] concentrated on the nonexistence of stable solutions for the quasilinear Schrödinger equation

$$-\Delta u - \left[ \Delta(1 + u^2)^{\frac{1}{2}} \right] \frac{u}{2(1 + u^2)^{\frac{1}{2}}} = h(x)|u|^{q-1}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where  $N \geq 3, q > \frac{5}{2}, h(x)$  is continuous and positive in  $\mathbb{R}^N$ .

Throughout the paper, we consider (1.1) with the following two cases:

- (i)  $0 < p < 1 < q, \alpha, \beta > 0, h(|x|), H(|x|) > 0,$
- (ii)  $1 < p < q, \alpha, \beta > 0, h(|x|), H(|x|) > 0.$

With the aid of a variational argument, the question of the existence and multiplicity of nontrivial solutions to problem (1.1) is largely open. Compared with the work on weak solutions by variational way, we are interested in investigating the radial solutions and asymptotic behavior. In the present paper, the first task is to obtain the nonexistence criteria of positive ground state solutions to (1.1) involving superlinear nonlinearities, which mainly relies on some special techniques. Immediately after that, the sufficient conditions for existence of positive ground state solutions are solved by the Schauder–Tychonoff fixed point theorem. At last, we aim at the sufficient conditions for existence of blow-up solutions involving concave-convex nonlinearities by the Schauder–Tychonoff fixed point theorem. As far as the authors are aware, it seems that there is little work concerning the nonexistence criteria of the positive ground state solutions to problem (1.1) involving superlinear nonlinearities. Furthermore, there is almost no work on the existence of blow-up solutions involving concave-convex nonlinearities.

Motivated by [14–17], we take the changing of variables  $u = g(z)$  or  $z = g^{-1}(u)$ , where  $g(t)$  is given by

$$g'(t) = \left[ 1 + \frac{\gamma^2 g^2(t)}{2(1 + g^2(t))^{2-\gamma}} \right]^{-\frac{1}{2}} = \sqrt{2} (1 + g^2(t))^{\frac{2-\gamma}{2}} (2(1 + g^2(t))^{2-\gamma} + \gamma^2 g^2(t))^{-\frac{1}{2}}, \quad t \geq 0,$$

and  $g(t) = -g(-t)$  on  $(-\infty, 0]$ .

Thus, we can obtain the properties of the function  $g(t)$  as below.

**Lemma 1.1** ([8]) *The function  $g(t)$  satisfies*

- (f<sub>1</sub>)  $g$  is uniquely defined,  $C^\infty$  and invertible;
- (f<sub>2</sub>)  $0 < g'(t) \leq 1$ , for all  $t \in \mathbb{R}$ ;
- (f<sub>3</sub>)  $|g(t)| \leq |t|$ , for all  $t \in \mathbb{R}$ ;
- (f<sub>4</sub>)  $\frac{g(t)}{t} \rightarrow 1$  as  $t \rightarrow 0$ ;
- (f<sub>5</sub>)  $g(t) \leq 2\gamma t g'(t) \leq 2\gamma g(t)$ , for all  $t \in \mathbb{R}^+ = [0, \infty)$ ;
- (f<sub>6</sub>) there exists  $b_0 > 0$  such that

$$|g(t)| \geq \begin{cases} b_0 |t| & \text{if } |t| \leq 1, \\ b_0 |t|^{1/\gamma} & \text{if } |t| \geq 1. \end{cases}$$

After making the change  $u = g(z)$ , (1.1) turns into the following equation:

$$-\Delta z = \alpha h(|x|) |g(z)|^{p-1} g(z) g'(z) + \beta H(|x|) |g(z)|^{q-1} g(z) g'(z), \quad x \in \mathbb{R}^N. \tag{1.6}$$

**Definition 1.1** The function  $z \in C_{loc}^{1,\delta}(\mathbb{R}^N)$  ( $0 < \delta < 1$ ) is said to be a weak solution of (1.6) if

$$\int_{\mathbb{R}^N} \nabla z \nabla \zeta \, dx = \int_{\mathbb{R}^N} \alpha h(|x|) G_1(z) \zeta \, dx + \int_{\mathbb{R}^N} \beta h(|x|) G_2(z) \zeta \, dx, \quad \zeta \in C_0^1(\mathbb{R}^N),$$

where and in the sequel,  $G_1(z) = |g(z)|^{p-1} g(z) g'(z)$ ,  $G_2(z) = |g(z)|^{q-1} g(z) g'(z)$ .

We observe that  $z = z(|x|) = z(r)$  is a positive radial solution of (1.6) if and only if the function  $z(r)$  satisfies the following equation:

$$-(r^{N-1} z'(r))' = \alpha r^{N-1} h(r) |g(z)|^{p-1} g(z) g'(z) + \beta r^{N-1} H(r) |g(z)|^{q-1} g(z) g'(z), \quad r > 0. \tag{1.7}$$

As usual, we focus on the existence and nonexistence of weak solutions to (1.1) via (1.7). Our main conclusions in this work are as below.

**Theorem 1.1** *Let  $1 < p < q$ ,  $\alpha, \beta > 0$ , suppose that functions  $h(t), H(t)$  are positive, continuous in  $\mathbb{R}^N$  and*

$$\begin{aligned} (\mathcal{P}_1) \quad & A_1(r) \rightarrow \infty \text{ or } A_2(r) \rightarrow \infty, \text{ as } r \rightarrow \infty, \text{ where for all } s > 0, A_1(r) = \int_s^r t^{N-1-(N-2)(p+1)} h(t) \, dt, \\ & A_2(r) = \int_s^r t^{N-1-(N-2)(q+1)} H(t) \, dt, \end{aligned}$$

*then problem (1.7) does not possess any positive ground state solutions.*

**Theorem 1.2** *Let  $1 < p < q$ ,  $\alpha, \beta > 0$ , suppose that functions  $h(t), H(t)$  are positive, continuous in  $\mathbb{R}^N$  and*

( $\mathcal{P}_2$ )  $\varphi_1(t) \rightarrow \infty$  or  $\varphi_2(t) \rightarrow \infty$ , as  $t \rightarrow \infty$ , where  $\varphi_1(t) = \int_1^t r^{N-1}h(r) dr \cdot t^{1-N}$ ,  
 $\varphi_2(t) = \int_1^t r^{N-1}H(r) dr \cdot t^{1-N}$ ;  
 ( $\mathcal{P}_3$ )  $(t^N h(t))' \leq \frac{(N-2)(p+1)}{4\gamma} t^N h(t)$ ,  $(t^N H(t))' \leq \frac{(N-2)(q+1)}{4\gamma} t^N H(t)$ , for all  $t > 0$ ,  
 then problem (1.7) has at least one positive ground state solution.

**Theorem 1.3** *Let  $0 < p < 1 < q, \alpha, \beta > 0$ , suppose that  $h(|x|), H(|x|)$  are positive, continuous in  $\mathbb{R}^N$  and satisfy*

$$h(|x|) \leq \frac{L_1}{|x|^{\lambda_1}}, \quad H(|x|) \leq \frac{L_2}{|x|^{\lambda_2}}, \quad |x| \geq 1,$$

where  $L_1, L_2 > 0, 0 < \lambda_1 < 2, \lambda_2 > 2$ , then problem (1.7) has multiple positive blow-up solutions.

The organization of this work is as below. Sufficient conditions for nonexistence of positive ground state solutions to (1.7) will be set up in Sect. 2. Section 3 and Sect. 4 contain the proof of the existence of positive ground state solutions and blow-up solutions.

### 2 Nonexistence criteria of positive ground state solutions

In this section, we aim at deriving some useful lemmas by special techniques and then finishing the proof of Theorem 1.1. Throughout the paper, a function  $z$  is called a ground state solution of problem (1.1) if the weak solution  $z$  tends to zero as  $|x| \rightarrow \infty$ .

Let us denote operator

$$T(z)(r) = -(r^{N-1}z'(r))'.$$

**Lemma 2.1** *Let  $z(r) \in C^2(0, \infty)$  be a positive solution of (1.7), if*

$$T(z)(r) \geq 0 \quad (r > 0), z'(0) = 0, \tag{2.1}$$

then the function  $F(r) = r^{N-2}z(r)$  is increasing for  $r > 0$ . Moreover,  $F'(r) = 0$  iff  $z(r) = cr^{2-N}$ , where  $c$  is a constant.

*Proof* It is easy to get  $T(z)(r) \geq 0$  and

$$\begin{aligned} T(z)(r) &= -(r^{N-1}z'(r))' = -r^{N-2}((N-1)z' + rz'') \\ &= -r^{N-2}((N-2)z + rz')' = -r^{N-2}M'(r), \end{aligned} \tag{2.2}$$

where  $M(r) = (N-2)z + rz'$  and  $M'(r) \leq 0$  ( $r > 0$ ). Integrating (1.7) over  $(0, r)$ , we have  $z'(r) \leq 0$ .

In fact, we have  $M(t) > 0$  for every  $t > 0$ . Otherwise, there exists  $t_0 > 0$  such that  $M(t_0) < 0$ , then

$$rz' \leq M(r) \leq M(t_0), \quad r > t_0,$$

that is,  $z'(r) \leq \frac{M(t_0)}{r}, r > t_0$ .

Integrating the above inequality over  $[t_0, r]$ , one can see

$$z(r) \leq z(t_0) + M(t_0) \ln \frac{r}{t_0} \rightarrow -\infty, \quad \text{as } r \rightarrow \infty,$$

which gives rise to a contradiction with the positive solution  $z(r)$ . Therefore,

$$M(r) = (N - 2)z + rz' = (r^{N-2}z)' r^{3-N} > 0, \quad \text{for all } r > 0.$$

which implies the function  $F(r) = r^{N-2}z(r)$  is increasing for  $r > 0$ . Moreover,  $F'(r) = 0$  iff  $z(r) = cr^{2-N}$ . □

*Proof of Theorem 1.1* By (2.2), one can get

$$T(z)(r) = -r^{N-2}M'(r) = \alpha r^{N-1}h(r)|g(z)|^{p-1}g(z)g'(z) + \beta r^{N-1}H(r)|g(z)|^{q-1}g(z)g'(z).$$

Namely

$$-M'(r) = \alpha rh(r)|g(z)|^{p-1}g(z)g'(z) + \beta rH(r)|g(z)|^{q-1}g(z)g'(z).$$

Since  $z(r)$  is a positive ground state solution and  $g(z)$  satisfies properties

$$(f_6) \quad |g(z)| \geq b_0|z|, \quad |z| \leq 1, \quad (f_4) \quad g'(z) \rightarrow 1 \quad \text{as } z \rightarrow 0,$$

$$(f_5) \quad g(t) \leq 2\gamma tg'(t) \leq 2\gamma g(t),$$

there exist  $b_1, b_2 > 0$  such that

$$-M'(r) \geq b_1 rh(r)z^p(r) + b_2 rH(r)z^q(r), \tag{2.3}$$

integrating (2.3) over  $[s, r]$  yields

$$-M(r) + M(s) \geq b_1 \int_s^r th(t)z^p(t) dt + b_2 \int_s^r tH(t)z^q(t) dt. \tag{2.4}$$

Note that  $r^{N-2}z(r)$  is an increasing function, then

$$\begin{aligned} M(s) &\geq b \int_s^r [th(t)z^p(t) + tH(t)z^q(t)] dt \\ &= b \int_s^r [th(t)(t^{N-2}z(t))^p t^{-(N-2)p} \\ &\quad + tH(t)(t^{N-2}z(t))^q t^{-(N-2)q}] dt \\ &\geq b(s^{N-2}z(s))^p \int_s^r t^{N-1-(N-2)(p+1)}h(t) dt \\ &\quad + b(s^{N-2}z(s))^q \int_s^r t^{N-1-(N-2)(q+1)}H(t) dt \\ &= b \min\{(s^{N-2}z(s))^p, (s^{N-2}z(s))^q\} (A_1(r) + A_2(r)), \end{aligned} \tag{2.5}$$

where  $b = \min\{b_1, b_2\}$ . By  $(\mathcal{P}_1)$ ,  $A_1(r) \rightarrow \infty$  or  $A_2(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , it gives rise to a contradiction. Thus, there is no positive ground state solution to (1.7).

Otherwise,  $A_1(r) < \infty$  and  $A_2(r) < \infty$  as  $r \rightarrow \infty$ . Denote

$$B_1(s) = \int_s^\infty t^{N-1-(N-2)(p+1)} h(t) dt, \quad B_2(s) = \int_s^\infty t^{N-1-(N-2)(q+1)} H(t) dt,$$

which implies that  $B_1(s), B_2(s)$  are bounded for all  $s > 0$ . One can see that

$$B'_1(s) = -s^{N-1-(N-2)(p+1)} h(s), \quad B'_2(s) = -s^{N-1-(N-2)(q+1)} H(s).$$

On the other hand, one can have

$$F'(r) = (r^{N-2} z(r))' = r^{N-3} ((N-2)z + rz') = r^{N-3} M(r)$$

and by (2.5), one can obtain

$$\begin{aligned} F'(s) &= s^{N-3} M(s) \\ &\geq bs^{N-3} \min\{(s^{N-2} z(s))^p, (s^{N-2} z(s))^q\} \\ &\quad \cdot \left( \int_s^\infty t^{N-1-(N-2)(p+1)} h(t) dt + \int_s^\infty t^{N-1-(N-2)(q+1)} H(t) dt \right) \\ &= bs^{N-3} \min\{F^p(s), F^q(s)\} (B_1(s) + B_2(s)), \quad \text{as } r \rightarrow \infty. \end{aligned} \tag{2.6}$$

If  $\min\{F^p(s), F^q(s)\} = F^p(s)$ , one can get

$$\frac{F'(s)}{F^p(s)} \geq bs^{N-3} (B_1(s) + B_2(s)). \tag{2.7}$$

Integrating (2.7) over  $[s, r]$ , one can have

$$\begin{aligned} &\frac{1}{1-p} [F^{1-p}(r) - F^{1-p}(s)] \\ &\geq b \int_s^r t^{N-3} (B_1(t) + B_2(t)) dt \\ &= \frac{b}{N-2} \left[ t^{N-2} (B_1(t) + B_2(t)) \Big|_s^r - \int_s^r t^{N-2} (B'_1(t) + B'_2(t)) dt \right] \\ &= \frac{b}{N-2} [r^{N-2} B_1(r) - s^{N-2} B_1(s) + r^{N-2} B_2(r) - s^{N-2} B_2(s)] \\ &\quad + \frac{b}{N-2} \left[ \int_s^r t^{N-1-(N-2)p} h(t) dt + \int_s^r t^{N-1-(N-2)q} H(t) dt \right], \end{aligned} \tag{2.8}$$

which implies that

$$\begin{aligned} &\frac{1}{p-1} F^{1-p}(s) \\ &\geq \frac{b}{N-2} \left( \int_s^r t^{N-1-(N-2)p} h(t) dt - s^{N-2} B_1(s) \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{b}{N-2} \left( \int_s^r t^{N-1-(N-2)q} H(t) dt - s^{N-2} B_2(s) \right) \\
 & = \frac{b}{N-2} (A_1^*(r) - s^{N-2} B_1(s) + A_2^*(r) - s^{N-2} B_2(s)), \tag{2.9}
 \end{aligned}$$

where  $A_1^*(r) = \int_s^r t^{N-1-(N-2)p} h(t) dt$ ,  $A_2^*(r) = \int_s^r t^{N-1-(N-2)q} H(t) dt$ .

If  $A_1^*(r) \rightarrow \infty$  or  $A_2^*(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , we can get  $F^{1-p}(s) \rightarrow +\infty$ . Since the function  $F(s)$  is increasing, we have  $F(s) \leq 0$ . It yields a contradiction.

As to the other case, if  $\min\{F^p(s), F^q(s)\} = F^p(s)$ , we can apply the same argument. Thus, Eq. (1.7) has no positive ground state solution and the proof is completed.  $\square$

*Remark 2.1* Since

$$A_1(r) = \int_s^r t^{N-1-(N-2)(p+1)} h(t) dt \leq A_1^*(r) = \int_s^r t^{N-1-(N-2)p} h(t) dt, \quad \text{for } s \geq 1,$$

we can get

$$A_1(r) \rightarrow \infty \implies A_1^*(r) \rightarrow \infty, \quad \text{as } r \rightarrow \infty.$$

### 3 Existence of positive ground state solutions

Let us consider (1.7),

$$\begin{cases}
 -(r^{N-1} z'(r))' = \alpha r^{N-1} h(r) |g(z)|^{p-1} g(z) g'(z) + \beta r^{N-1} H(r) |g(z)|^{q-1} g(z) g'(z), \\
 z_0 = z(0) > 0, \quad r > 0, \quad z(r) \rightarrow 0, \quad \text{as } r \rightarrow \infty,
 \end{cases}$$

where  $\alpha, \beta > 0, 1 < p < q$ .

*Proof of Theorem 1.2*

*Step 1.* We claim that, for all  $z_0 > 0$ , there exist  $\delta > 0$  and  $z = z(r)$  such that

$$\begin{aligned}
 -(r^{N-1} z'(r))' & = \alpha r^{N-1} h(r) |g(z)|^{p-1} g(z) g'(z) \\
 & \quad + \beta r^{N-1} H(r) |g(z)|^{q-1} g(z) g'(z), \quad r \in (0, \delta), \tag{3.1}
 \end{aligned}$$

with  $\frac{z_0}{2} \leq z(r) \leq z_0, r \in [0, \delta]; z'(r) < 0, r \in (0, \delta)$ .

By (f<sub>5</sub>), we have  $g(z) > 0$ , thus (3.1) can be rewritten as

$$-(r^{N-1} z'(r))' = \alpha r^{N-1} h(r) g^p(z) g'(z) + \beta r^{N-1} H(r) g^q(z) g'(z), \quad r \in (0, \delta). \tag{3.2}$$

Since the functions  $h(t)$  and  $H(t)$  are positive and continuous, we can get

$$\lim_{r \rightarrow 0} \int_0^r \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} h(s) ds \right) dt = 0, \quad \lim_{r \rightarrow 0} \int_0^r \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} H(s) ds \right) dt = 0,$$

and there exists  $\delta > 0$  such that

$$z_0^p \int_0^\delta \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} h(s) ds \right) dt \leq \frac{z_0}{4\alpha}, \quad z_0^q \int_0^\delta \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} H(s) ds \right) dt \leq \frac{z_0}{4\beta}. \tag{3.3}$$

Let  $U_1$  denote locally convex space of all continuous function on  $[0, \infty)$  with the usual topology and consider the set

$$X = \left\{ z \in U_1 \mid \frac{z_0}{2} \leq z(r) \leq z_0, r \in [0, \delta] \right\},$$

and the operator  $\mathcal{T} : X \rightarrow C[0, \delta]$

$$\begin{aligned} \mathcal{T}z(r) = & z_0 - \alpha \int_0^r \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} h(s)g^p(s)g'(s) ds \right) dt \\ & - \beta \int_0^r \left( \int_0^t \left( \frac{s}{t} \right)^{N-1} H(s)g^q(s)g'(s) ds \right) dt. \end{aligned} \tag{3.4}$$

It is easy to get  $\mathcal{T}X \subset X$ , the operator  $\mathcal{T}$  is continuous and relatively compact. Therefore there exists a  $z \in X$  such that  $\mathcal{T}z = z$  holds by the Schauder–Tychonoff fixed point theorem.

Besides, the solution  $z = z(r)$  can be extended and satisfies  $z'(r) < 0$  as long as  $z(r) > 0$ .

Denote

$$Y = \{ r \geq 0 \mid z(t) > 0, 0 \leq t < r \}.$$

In fact, we have  $Y = [0, \infty)$ . Otherwise, if  $Y \neq [0, \infty)$ , then there exists  $R > 0$  such that

$$z(t) > 0, \quad 0 \leq t < R; \quad z(R) = 0, \quad z'(t) < 0, \quad 0 < t \leq R.$$

Multiplying both sides of (3.2) by  $-rz'(r)$ , we can obtain

$$\frac{1}{2} [r^N (z')^2]' + \frac{N-2}{2} r^{N-1} (z')^2 = -\frac{\alpha}{p+1} r^N h(r) [g^{p+1}(z)]' - \frac{\beta}{q+1} r^N H(r) [g^{q+1}(z)]', \tag{3.5}$$

then integrating (3.5) from  $[0, R]$ , one can see

$$\begin{aligned} & \frac{1}{2} R^N (z'(R))^2 + \int_0^R \left\{ \frac{N-2}{2} r^{N-1} (z')^2 + \frac{\alpha}{p+1} r^N h(r) [g^{p+1}(z)]' \right. \\ & \left. + \frac{\beta}{q+1} r^N H(r) [g^{q+1}(z)]' \right\} dr = 0. \end{aligned} \tag{3.6}$$

On the other hand, by (3.2) we get

$$\begin{aligned} \int_0^R r^{N-1} (z')^2 dr &= [r^{N-1} z' z] \Big|_0^R - \int_0^R z (r^{N-1} z')' dz \\ &= \alpha \int_0^R r^{N-1} h(r) g^p(z) g'(z) z dr + \beta \int_0^R r^{N-1} H(r) g^q(z) g'(z) z dr \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \int_0^R r^N h(r) [g^{p+1}(z)]' dr &= r^N h(r) (g^{p+1}(z)) \Big|_0^R - \int_0^R (r^N h(r))' g^{p+1}(z) dr \\ &= - \int_0^R (r^N h(r))' g^{p+1}(z) dr, \end{aligned} \tag{3.8}$$

and analogously

$$\int_0^R r^N H(r) [g^{q+1}(z)]' dr = - \int_0^R (r^N H(r))' g^{q+1}(z) dr. \tag{3.9}$$

Substituting (3.7), (3.8), (3.9) and (f<sub>5</sub>) into (3.6) gives

$$\begin{aligned} 0 &= \frac{1}{2} R^N (z'(R))^2 + \alpha \int_0^R \left[ \frac{N-2}{2} r^{N-1} h(r) g^p(z) g'(z) z - \frac{1}{p+1} (r^N h(r))' g^{p+1}(z) \right] dr \\ &\quad + \beta \int_0^R \left[ \frac{N-2}{2} r^{N-1} H(r) g^q(z) g'(z) z - \frac{1}{q+1} (r^N H(r))' g^{q+1}(z) \right] dr \\ &\geq \frac{1}{2} R^N (z'(R))^2 + \alpha \int_0^R g^{p+1}(z) \left[ \frac{N-2}{4\gamma} r^{N-1} h(r) - \frac{1}{p+1} (r^N h(r))' \right] dr \\ &\quad + \beta \int_0^R g^{q+1}(z) \left[ \frac{N-2}{4\gamma} r^{N-1} H(r) - \frac{1}{q+1} (r^N H(r))' \right] dr, \end{aligned} \tag{3.10}$$

thus it gives rise to a contradiction with (P<sub>3</sub>), so  $Y = [0, \infty)$ .

*Step 2.* We consider the asymptotic behavior of solution  $z(r)$ , that is,  $z(r) \rightarrow 0$ , as  $r \rightarrow \infty$ .

Since the function  $r^{N-1}(-z')$  is increasing, we have  $r^{N-1}(-z') \geq -z'(1)$ ,  $r \geq 1$ , then integrating from  $[t, R]$ , we get

$$z(t) - z(R) \geq -z'(1) \frac{1}{2-N} (R^{2-N} - t^{2-N}), \quad R > t \geq 1, \tag{3.11}$$

and  $z(t) \geq z(\infty) - z'(1) \frac{1}{N-2} t^{2-N}$ ,  $r \geq 1$ ,  $R \rightarrow \infty$ .

On the other hand, integrating both sides of (3.2) on  $[0, R]$ , we can acquire

$$\begin{aligned} \int_0^R -(r^{N-1} z'(r))' dr &= -R^{N-1} z'(R) \\ &= \alpha \int_0^R r^{N-1} h(r) g^p(z) g'(z) dr + \beta \int_0^R r^{N-1} H(r) g^q(z) g'(z) dr. \end{aligned} \tag{3.12}$$

Let  $0 < z_0 \leq 1$  in  $X$ , since (f<sub>6</sub>), (f<sub>4</sub>) and  $\frac{z_0}{2} \leq z(r) \leq z_0$ , there exist  $c_1, c_2 > 0$  such that

$$\begin{aligned} -R^{N-1} z'(R) &\geq c_1 \int_1^R r^{N-1} h(r) z^p(r) dr + c_2 \int_1^R r^{N-1} H(r) z^q(r) dr \\ &\geq c_1 z^p(R) \int_1^R r^{N-1} h(r) dr + c_2 z^q(R) \int_1^R r^{N-1} H(r) dr \\ &\geq c \min\{z^p(R), z^q(R)\} \left( \int_1^R r^{N-1} h(r) dr + \int_1^R r^{N-1} H(r) dr \right), \end{aligned} \tag{3.13}$$

where  $c = \min\{c_1, c_2\}$ . If  $z^p(R) = \min\{z^p(R), z^q(R)\}$ , then we can have

$$\begin{aligned} \frac{-z'(R)}{z^p(R)} &\geq c \left( \int_1^R r^{N-1} h(r) dr \cdot R^{1-N} + \int_1^R r^{N-1} H(r) dr \cdot R^{1-N} \right) \\ &= c(\varphi_1(R) + \varphi_2(R)), \end{aligned} \tag{3.14}$$

where  $\varphi_i(R)$ ,  $i = 1, 2$  denoted by  $(\mathcal{P}_2)$ . Integrating (3.14) over  $[1, R]$ , one can have

$$\int_1^R \frac{-z'(t)}{z^p(t)} dt = \frac{1}{1-p} [z^{1-p}(R) - z^{1-p}(1)] \geq c \left( \int_1^R \varphi_1(t) dt + \int_1^R \varphi_2(t) dt \right). \tag{3.15}$$

Hence, by hypothesis  $(\mathcal{P}_2)$  we obtain

$$z(R) \leq \left[ z^{1-q}(1) - c(p-1) \left( \int_1^R \varphi_1(t) dt + \int_1^R \varphi_2(t) dt \right) \right]^{\frac{1}{1-p}} \rightarrow 0, \text{ as } R \rightarrow \infty. \tag{3.16}$$

As to the other case, if  $z^q(R) = \min\{z^p(R), z^q(R)\}$ , we can apply the same argument. Thus  $\lim_{r \rightarrow \infty} z(r) = 0$  is proved and the proof of Theorem 1.2 is completed.  $\square$

#### 4 Existence of positive blow-up solutions

In this section, we investigate (1.7) with concave-convex nonlinearities and give a proof of Theorem 1.3. Throughout the paper, a function  $z$  is called a blow-up solution of problem (1.1) if a weak solution  $z$  satisfies  $z \rightarrow \infty$  as  $|x| \rightarrow \infty$ .

We consider (1.7),

$$\begin{cases} -(r^{N-1}z'(r))' = \alpha r^{N-1}h(r)|g(z)|^{p-1}g(z)g'(z) + \beta r^{N-1}H(r)|g(z)|^{q-1}g(z)g'(z), \\ z_0 = z(0) > 0, \quad r > 0, \quad z(r) \rightarrow \infty, \text{ as } r \rightarrow \infty, \end{cases}$$

where  $\alpha, \beta > 0, 0 < p < 1 < q$ .

*Proof of Theorem 1.3* At first, we choose  $z_0 > 0$  such that

$$\begin{cases} 2\alpha(2z_0)^p [\max\{\int_0^1 h(s) ds, \frac{L_1}{N-\lambda_1+kp}\}]^{\frac{1}{2-\lambda_1+kp}} \leq \frac{z_0}{4}, \\ 2\beta(2z_0)^q [\max\{\int_0^1 H(s) ds, \frac{L_2}{N-\lambda_2+kq}\}]^{\frac{1}{2-\lambda_2+kq}} \leq \frac{z_0}{4}, \end{cases} \tag{4.1}$$

and

$$\alpha(2z_0)^p \int_0^1 h(s) ds \leq \frac{z_0}{4}, \quad \beta(2z_0)^q \int_0^1 H(s) ds \leq \frac{z_0}{4}, \tag{4.2}$$

where  $k = \frac{2-\lambda_1}{1-p} = \frac{\lambda_2-2}{q-1} \geq 0, 0 < \lambda_1 < 2, \lambda_2 > 2$ .

It is well known that (1.7) is equivalent to the following integral form:

$$\begin{aligned} z(r) = z_0 + \alpha \int_0^r t^{1-N} \int_0^t s^{N-1} h(s) g^p(z) g'(z) ds dt \\ + \beta \int_0^r t^{1-N} \int_0^t s^{N-1} H(s) g^q(z) g'(z) ds dt. \end{aligned} \tag{4.3}$$

Let  $U_2$  denote locally convex space of all continuous function on  $[0, \infty)$  with the usual topology and consider the set

$$W = \{z \in U_2 | z_0 \leq z(r) \leq A(r), r \geq 0\},$$

where

$$A(r) = \begin{cases} 2z_0, & 0 \leq r \leq 1, \\ 2z_0r^k, & r \geq 1, \end{cases} \tag{4.4}$$

and the operator  $\mathcal{T} : W \rightarrow C\bar{R}_+, \bar{R}_+ = [0, +\infty)$  is given by  $\mathcal{T}z = \tilde{z}$ ,

$$\begin{aligned} \tilde{z}(r) &= z_0 + \alpha \int_0^r t^{1-N} \int_0^t s^{N-1} h(s) g^p(z) g'(z) ds dt \\ &\quad + \beta \int_0^r t^{1-N} \int_0^t s^{N-1} H(s) g^q(z) g'(z) ds dt. \end{aligned} \tag{4.5}$$

Obviously,  $W$  is a nonempty closed convex set of  $C\bar{R}_+$ . In order to apply the Schauder–Tychonoff fixed point theorem, we are going to verify in three steps.

*Step 1.*  $\mathcal{T}$  maps  $W$  into itself. For  $0 \leq r \leq 1$ , by (4.2) we have

$$\begin{aligned} \tilde{z}(r) &\leq z_0 + \alpha \int_0^r t^{1-N} \int_0^t s^{N-1} h(s) z^p(s) ds dt + \beta \int_0^r t^{1-N} \int_0^t s^{N-1} H(s) z^q(s) ds dt \\ &\leq z_0 + \alpha (2z_0)^p \int_0^r \int_0^t h(s) ds dt + \beta (2z_0)^q \int_0^r \int_0^t H(s) ds dt \\ &\leq z_0 + \alpha (2z_0)^p \int_0^1 h(s) ds + \beta (2z_0)^q \int_0^1 H(s) ds \leq z_0 + \frac{z_0}{4} + \frac{z_0}{4} \leq 2z_0, \end{aligned} \tag{4.6}$$

and for  $r \geq 1$ , by (4.1),(4.2), we have

$$\begin{aligned} \tilde{z}(r) &= z_0 + \alpha \left( \int_0^1 + \int_1^r \right) t^{1-N} \int_0^t s^{N-1} h(s) g^p(z) g'(z) ds dt \\ &\quad + \beta \left( \int_0^1 + \int_1^r \right) t^{1-N} \int_0^t s^{N-1} H(s) g^q(z) g'(z) ds dt \\ &\leq z_0 + \frac{z_0}{4} + \frac{z_0}{4} + \alpha \int_1^r t^{1-N} \int_0^t s^{N-1} h(s) z^p(s) ds dt \\ &\quad + \beta \int_1^r t^{1-N} \int_0^t s^{N-1} H(s) z^q(s) ds dt \\ &= \frac{3z_0}{2} + \alpha \int_1^r t^{1-N} \left( \int_0^1 + \int_1^t \right) s^{N-1} h(s) z^p(s) ds dt \\ &\quad + \beta \int_1^r t^{1-N} \left( \int_0^1 + \int_1^t \right) s^{N-1} H(s) z^q(s) ds dt \\ &\leq \frac{3z_0}{2} + \alpha \int_1^r t^{1-N} \left[ (2z_0)^p \int_0^1 h(s) ds + (2z_0)^p L_1 \int_1^t s^{N-1-\lambda_1} s^{kp} ds \right] dt \\ &\quad + \beta \int_1^r t^{1-N} \left[ (2z_0)^q \int_0^1 H(s) ds + (2z_0)^q L_2 \int_1^t s^{N-1-\lambda_2} s^{kq} ds \right] dt \\ &\leq \frac{3z_0}{2} + 2\alpha (2z_0)^p \left[ \max \left\{ \int_0^1 h(s) ds, \frac{L_1}{N - \lambda_1 + kp} \right\} \right] \int_1^r t^{1-N} t^{N-\lambda_1+kp} dt \\ &\quad + 2\beta (2z_0)^q \left[ \max \left\{ \int_0^1 H(s) ds, \frac{L_2}{N - \lambda_2 + kq} \right\} \right] \int_1^r t^{1-N} t^{N-\lambda_2+kq} dt \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{3z_0}{2} + 2\alpha(2z_0)^p \left[ \max \left\{ \int_0^1 h(s) ds, \frac{L-1}{N-\lambda_1+kp} \right\} \right] \cdot \frac{1}{2-\lambda_1+kp} \cdot r^{2-\lambda_1+kp} \\
 &\quad + 2\beta(2z_0)^q \left[ \max \left\{ \int_0^1 H(s) ds, \frac{L_2}{N-\lambda_2+kq} \right\} \right] \cdot \frac{1}{2-\lambda_2+kq} \cdot r^{2-\lambda_2+kq} \\
 &\leq \frac{3z_0}{2} + \frac{z_0}{4}r^k + \frac{z_0}{4}r^k \leq 2z_0r^k,
 \end{aligned} \tag{4.7}$$

where  $2 - \lambda_1 + kp = k$ ,  $2 - \lambda_2 + kq = k$ , that is,  $k = \frac{2-\lambda_1}{1-p} = \frac{\lambda_2-2}{q-1}$  with  $0 < \lambda_1 < 2$ ,  $\lambda_2 > 2$ , thus we prove that  $\mathcal{T}W \subset W$ .

*Step 2.*  $\mathcal{T}$  is continuous. Let  $\{z_n\}$  be a sequence in  $W$  which converges to  $z \in W$  uniformly on each compact subinterval of  $\bar{R}_+$ .

Let

$$\begin{aligned}
 \varphi_m(r) &= r^{1-N} \int_0^r s^{N-1} h(s) g^p(z_m) g'(z_m) ds + r^{1-N} \int_0^r s^{N-1} H(s) g^q(z_m) g'(z_m) ds, \\
 \varphi(r) &= r^{1-N} \int_0^r s^{N-1} h(s) g^p(z) g'(z) ds + r^{1-N} \int_0^r s^{N-1} H(s) g^q(z) g'(z) ds.
 \end{aligned}$$

Then one can see

$$\begin{aligned}
 |\varphi_m(r) - \varphi(r)| &\leq \int_0^r h(s) |g^p(z_m) g'(z_m) - g^p(z) g'(z)| ds \\
 &\quad + \int_0^r H(s) |g^q(z_m) g'(z_m) - g^q(z) g'(z)| ds
 \end{aligned} \tag{4.8}$$

and

$$|\tilde{z}_m(r) - \tilde{z}(r)| \leq \int_0^r |\varphi_m(s) - \varphi(s)| ds, \tag{4.9}$$

from (4.8), (4.9), we see that  $\{\varphi_m\}$  converges to  $\varphi$  uniformly and  $\{\tilde{z}_m\}$  converges to  $\tilde{z}$  uniformly on each compact subinterval of  $\bar{R}_+$ . Hence, the mapping  $\mathcal{T}$  is continuous.

*Step 3.*  $\mathcal{T}(W)$  is relatively compact. For  $R > 0$  an arbitrary constant, one can have

$$\begin{aligned}
 \tilde{z}(r) &= \alpha \int_0^r \left(\frac{s}{r}\right)^{N-1} h(s) g^p(z) g'(z) ds + \beta \int_0^r \left(\frac{s}{r}\right)^{N-1} H(s) g^q(z) g'(z) ds \\
 &\leq \alpha \int_0^r h(s) z^p(s) ds + \beta \int_0^r H(s) z^q(s) ds \\
 &\leq \alpha \int_0^R h(s) A^p(s) ds + \beta \int_0^R H(s) A^q(s) ds,
 \end{aligned} \tag{4.10}$$

it implies the local boundedness of the set  $\{\tilde{z}(r) | z \in W\}$ . Thus, the relatively compactness of  $\mathcal{T}(W)$  can be shown by the Ascoli–Arzelà theorem.

Therefore, the Schauder–Tychonoff fixed point theorem guarantees a  $z \in W$  satisfying  $\mathcal{T}z = z$ , namely,  $z(r)$  satisfies (1.7). Thus,  $z(|x|)$  gives a solution of (1.1). Besides, multiple occurrences  $z_0$  fulfill (4.1) and (4.2), so multiple positive radial solutions of (1.1) can be constructed.

*Step 4.* We consider the asymptotic behavior of solution  $z(r)$ , that is,  $z(r) \rightarrow \infty$ , as  $r \rightarrow \infty$ .

Given  $z_0 \geq 1, r \geq 1, \gamma - 1 < p < 1 < q$  and  $z_0 \leq z(r) \leq A(r)$ , one can get

$$g^p(z)g'(z) \geq \frac{1}{2\gamma}g^p(z)\frac{g(z)}{z} = \frac{1}{2\gamma}\frac{g^{p+1}(z)}{z} \geq b_1z^{\frac{p+1}{\gamma}-1}(r) \geq b_1(z_0)^{\frac{p+1}{\gamma}-1} = b_2 > 0, \tag{4.11}$$

and  $g^q(z)g'(z) \geq b_3 > 0$ . Then

$$\begin{aligned} & r^{1-N} \int_0^r s^{N-1}h(s)g^p(z)g'(z) ds + r^{1-N} \int_0^r s^{N-1}H(s)g^q(z)g'(z) ds \\ & \geq b_2r^{1-N} \int_0^r s^{N-1}h(s) ds + b_3r^{1-N} \int_0^r s^{N-1}H(s) ds. \end{aligned} \tag{4.12}$$

By the hypothesis  $(\mathcal{P}_2)$ , we can verify that  $\lim_{r \rightarrow \infty} z(r) = \infty$  and the proof of Theorem 1.3 is completed. □

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**Authors' contributions**

JL formulated and proved the theorems on the nonexistence and existence of the solutions, YW checked all the computations. All authors read and approved the final manuscript.

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