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Coupled elliptic systems depending on the gradient with nonlocal BCs in exterior domains

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Abstract

We study the existence and multiplicity of positive radial solutions for a coupled elliptic system in exterior domains where the nonlinearities depend on the gradients and the boundary conditions are nonlocal. We use a new cone to establish the existence of solutions by means of fixed point index theory.

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1 Introduction

In this paper, we study the existence and multiplicity of positive radial solutions for the coupled elliptic system

$$
\begin{cases}
-\Delta u = h_1(|x|)f_1(u, v, |\nabla u|, |\nabla v|), & |x| \in [r_0, +\infty), \\
-\Delta v = h_2(|x|)f_2(u, v, |\nabla u|, |\nabla v|), & |x| \in [r_0, +\infty), \\
\lim_{|x|\to\infty} u(x) = \alpha_1[u], \quad c_1 u + \tilde{d}_1 \frac{\partial u}{\partial r} = \beta_1[u] & \text{for } |x| = r_0, \\
\lim_{|x|\to\infty} v(x) = \alpha_2[v], \quad c_2 v + \tilde{d}_2 \frac{\partial v}{\partial r} = \beta_2[v] & \text{for } |x| = r_0,
\end{cases}
\tag{1.1}
$$

where $\alpha_i[\cdot]$, and $\beta_i[\cdot]$ are bounded linear functionals, h_i and f_i are nonnegative functions, $c_i \geq 0, \tilde{d}_i \leq 0, r_0 > 0$, and $\frac{\partial}{\partial r}$ denotes (as in [23]) differentiation in the radial direction $r = |x|$. The functions f_i are continuous, and every singularity is captured by the term $h_i \in L^1$, which may have pointwise singularities.

Many papers study the existence of radial solutions of elliptic equations in the exterior part of a ball. A variety of methods have been used, for instance, when the boundary conditions (BCs) are *homogeneous*; a priori estimates were utilized by Castro et al. [7], sub and super solutions were used by Djedali and Orpel [16] and Sankar et al. [39]; variational methods were used by Orpel [37], and topological methods where employed by Abebe and coauthors [1], do Ó et al. [17], Hai and Shivaji [29], Han and Wang [30], Lee [35], Orpel [38], and Stanczy [40].

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In particular, recently, Hai and Shivaji [29] proved the existence and multiplicity of positive radial solutions for the superlinear elliptic system

$$
\begin{cases}
-\Delta u = \lambda h_1(|x|)f_1(v), & |x| \in [r_0, +\infty), \\
-\Delta v = \lambda h_2(|x|)f_2(u), & |x| \in [r_0, +\infty), \\
\lim_{|x|\to\infty} u(x) = 0, & d_1 \frac{\partial u}{\partial n} + \tilde{c}_1(u)u = 0 \quad \text{for } |x| = r_0, \\
\lim_{|x|\to\infty} v(x) = 0, & d_2 \frac{\partial v}{\partial n} + \tilde{c}_2(v)v = 0 \quad \text{for } |x| = r_0,
\end{cases}
$$

using a fixed point result of Krasnoselskii type applied to suitable completely continuous integral operators on $C[0,1] \times C[0,1]$. These results seem to be the first ones proving the multiplicity of positive solutions for this kind of systems.

On the other hand, in the context of nonhomogeneous BCs, elliptic problems were studied by Aftalion and Busca [2], Butler et al. [6], Cianciaruso and al. [10], Dhanya et al. [15], do Ó et al. [18–21], Goodrich [26, 27], Ko and et al. [31], and Lee et al. [34].

The existence of positive radial solutions of elliptic equations with nonlinearities depending on the gradient subject to Neumann, Dirichlet, or Robin boundary conditions has been investigated by a number of authors; see, for example, Averna et al. [5], Cianciaruso et al. [8, 9, 12], De Figueiredo et al. [13, 14], Faria et al. [22], and Montreanu et al. [36].

Our system (1.1) is quite general:

(1) the nonlinearities f_i depend on the functions u and v and their gradients; no monotonicity hypotheses are supposed.

(2) the boundary conditions are nonlocal and represent feedback mechanisms. They have been deeply studied for ordinary differential equations, for example, in [11, 24, 25, 41, 42]).

To search solutions of the elliptic PDE

$$
-\Delta w = g(|x|)f(w, |\nabla w|)
$$

with some boundary conditions, a topological approach is associating, by using standard transformations, an integral operator of the form

$$
Sw(t) = \int_0^1 G(t,s)g(r(s))\tilde{f}(w(s), |w'(s)|) \, ds.
$$

It is straightforward, in the local problems, to find the Green's function G by integration and by using the BCs. However, let us remark that, in the nonlocal problems, this is a long and technical calculation, often resulting in a sum of terms of different signs.

Here, as in [41], we treat the nonlocal problem as the perturbation of the simpler local problem. In such a way, we handle the positivity properties of the simpler Green's function of the local problem.

Often, the associated integral operator is studied in the cone of nonnegative functions in the space $C^1[0,1]$ or in a weighed space of differential functions as in [3]. In our case and in particular when seeking for multiple solutions, it is suitable to work in a smaller cone: we will introduce a new cone in which we will use Harnack-type inequalities.

Moreover, since we are interested in positive solutions, the functionals α_i and β_i must satisfy some positivity conditions; we will not suppose this in the whole space, but we choose to include the requirement in the definition of the cone.

We show that, under suitable conditions on the nonlinear terms, the fixed point index is 0 on certain open bounded subsets of the cone and 1 on the others; the choice of these subsets allows us to have more freedom on the conditions of the growth of the nonlinearities. These conditions relate the upper and lower bounds of the nonlinearities f_i on stripes and some constants, depending on the kernel of the integral operator and on the nonlocal BCs that are easily estimable as we show in an example.

2 The associate integral operator

Consider in $\mathbb{R}^n$, $n \geq 3$, the equation

$$-\triangle w = h(|x|)f(w, |\nabla w|), \quad |x| \in [r_0, +\infty). \tag{2.1}$$

Since we are interested in radial solutions $w = w(r)$, $r = |x|$, following [6], we rewrite (2.1) as

$$-w''(r) - \frac{n-1}{r}w'(r) = h(r)f(w(r), |w'(r)|), \quad r \in [r_0, +\infty). \tag{2.2}$$

By using the transformation

$$r(t) := r_0 t^{\frac{1}{2-n}}, \quad t \in (0, 1],$$

equation (2.2) becomes

$$w''(r(t)) + g(t)f\left(w(r(t)), \frac{|w'(r(t))|}{|r'(t)|}\right) = 0, \quad t \in (0, 1],$$

with

$$g(t) = \frac{r_0^2}{(n-2)^2}t^{\frac{2n-2}{2-n}}h(r(t)).$$

Consider in $\mathbb{R}^n$ the system of boundary value problems

$$\begin{cases} -\Delta u = h_1(|x|)f_1(u, v, |\nabla u|, |\nabla v|), & |x| \in [r_0, +\infty), \\ -\Delta v = h_2(|x|)f_2(u, v, |\nabla u|, |\nabla v|), & |x| \in [r_0, +\infty), \\ \lim_{|x|\to\infty} u(x) = \alpha_1[u], \quad c_1 u + \tilde{d}_1 \frac{\partial u}{\partial r} = \beta_1[u] & \text{for } |x| = r_0, \\ \lim_{|x|\to\infty} v(x) = \alpha_2[v], \quad c_2 v + \tilde{d}_2 \frac{\partial v}{\partial r} = \beta_2[v] & \text{for } |x| = r_0. \end{cases} \tag{2.3}$$

Set $u(t) = u(r(t))$ and $v(t) = v(r(t))$. Thus with system (2.3) we associate the system of ODEs

$$\begin{cases} u''(t) + g_1(t)f_1(u(t), v(t), \frac{|u'(t)|}{|r'(t)|}, \frac{|v'(t)|}{|r'(t)|}) = 0, & t \in (0, 1), \\ v''(t) + g_2(t)f_2(u(t), v(t), \frac{|u'(t)|}{|r'(t)|}, \frac{|v'(t)|}{|r'(t)|}) = 0, & t \in (0, 1), \\ u(0) = \alpha_1[u], \quad c_1 u(1) + d_1 u'(1) = \beta_1[u], \\ v(0) = \alpha_2[v], \quad c_2 v(1) + d_2 v'(1) = \beta_2[v], \end{cases} \tag{2.4}$$

where $d_i = \frac{r_0}{2-n}\tilde{d}_i$ and $g_i(t) = \frac{r_0^2}{(n-2)^2}t^{\frac{2n-2}{2-n}}h_i(r(t))$.

We study the existence of positive solutions of system (2.4) by means of the associated system of perturbed Hammerstein integral equations

$$
\begin{cases}
u(t) = \gamma_1(t)\alpha_1[u] + \delta_1(t)\beta_1[u] + \int_0^1 k_1(t,s)g_1(s)f_1(u(s),v(s),\frac{|u'(s)|}{|r'(s)|},\frac{|v'(s)|}{|r'(s)|})\,ds, \\
v(t) = \gamma_2(t)\alpha_2[v] + \delta_2(t)\beta_2[v] + \int_0^1 k_2(t,s)g_2(s)f_2(u(s),v(s),\frac{|u'(s)|}{|r'(s)|},\frac{|v'(s)|}{|r'(s)|})\,ds,
\end{cases}
$$

where γ_i is the solution of the BVP $w''(t) = 0$, $w(0) = 1$, $c_i w(1) + d_i w'(1) = 0$, that is,

$$
\gamma_i(t) = 1 - \frac{c_i t}{d_i + c_i};
$$

δ_i is the solution of the BVP $w''(t) = 0$, $w(0) = 0$, $c_i w(1) + d_i w'(1) = 1$, that is,

$$
\delta_i(t) = \frac{t}{d_i + c_i};
$$

and k_i is the Green's function associated with the homogeneous problem in which $\alpha_i[w] = \beta_i[w] = 0$, that is,

$$
k_i(t,s) := \begin{cases}
s(1 - \frac{c_i t}{d_i+c_i}), & s \le t, \\
t(1 - \frac{c_i s}{d_i+c_i}), & s > t.
\end{cases}
$$

In the following proposition, we resume the properties of the functions γ_i, δ_i and k_i, which will be further useful.

Proposition 2.1 *We have, for $i = 1, 2$:*

- *The functions γ_i, δ_i are in $C^1[0,1]$; moreover, for $t \in [a_i, b_i] \subset (0,1)$, with $a_i + b_i < 1$,*

$$
\|\gamma_i\|_\infty = 1 \quad and \quad \gamma_i(t) \ge 1 - t \ge 1 - b_i = (1 - b_i)\|\gamma_i\|_\infty > a_i\|\gamma_i\|_\infty;
$$

$$
\|\delta_i\|_\infty = \frac{1}{d_i + c_i} \quad and \quad \delta_i(t) = t\|\delta_i\|_\infty \ge a_i\|\delta_i\|_\infty.
$$

- *The kernels k_i are nonnegative and continuous in $[0,1] \times [0,1]$. Moreover, for $t \in [a_i, b_i]$, we have*

$$
k_i(t,s) \le \phi_i(s) \quad for\ (t,s) \in [0,1] \times [0,1], \quad and
$$

$$
k_i(t,s) \ge a_i\phi_i(s) \quad for\ (t,s) \in [a_i, b_i] \times [0,1],
$$

with $\phi_i(s) := s(1 - \frac{c_i}{d_i+c_i}s)$.

Let $\omega(t) = t(1 - t)$. Our setting will be the Banach space (see [3])

$$
C_\omega^1[0,1] = \left\{ w \in C[0,1] \cap C^1(0,1) : \sup_{t \in (0,1)} \omega(t)\left|w'(t)\right| < +\infty \right\}
$$

endowed with the norm

$$
\|w\| := \max\left\{ \|w\|_\infty, \left\|w'\right\|_\omega \right\},
$$

where $\|w\|_\infty := \max_{t \in [0,1]} |w(t)|$ and $\|w'\|_\omega := \sup_{t \in (0,1)} \omega(t)|w'(t)|$.

For $i = 1, 2$ and fixed $[a_i, b_i] \subset (0, 1)$ such that $a_i + b_i < 1$, we consider the cones

$$K_i := \Big\{ w \in C^1_\omega[0, 1] : w \geq 0, \min_{t \in [a_i, b_i]} w(t) \geq a_i \|w\|_\infty,$$

$$\|w'\|_\omega \leq 4w(1/2), \alpha_i[w] \geq 0, \beta_i[w] \geq 0 \Big\}$$

in $C^1_\omega[0, 1]$ and the cone

$$K := K_1 \times K_2$$

in $C^1_\omega[0, 1] \times C^1_\omega[0, 1]$.

Note that the functions in K_i are strictly positive on the subinterval $[a_i, b_i]$ and that for $w \in K_i$, we have the inequalities $\|w'\|_\omega \leq \|w\| \leq 4\|w\|_\infty$.

Set

$$F_i(u, v)(t) := \int_0^1 k_i(t, s) g_i(s) f_i\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds$$

and

$$T(u, v)(t) := \begin{pmatrix} \gamma_1(t)\alpha_1[u] + \delta_1(t)\beta_1[u] + F_1(u, v)(t) \\ \gamma_2(t)\alpha_2[v] + \delta_2(t)\beta_2[v] + F_2(u, v)(t) \end{pmatrix} := \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix}. \tag{2.5}$$

We will further assume that, for $i = 1, 2$,

- $f_i : [0, +\infty)^4 \to [0, +\infty)$ is continuous, and there exist $\psi_i : [0, +\infty)^4 \to [0, +\infty)$ such that $f_i(u, v, \frac{|u'|}{|r'|}, \frac{|v'|}{|r'|}) \leq \psi_i(u, v, \frac{\omega|u'|}{|r'|}, \frac{\omega|v'|}{|r'|})$ for all $(u, v) \in K$;
- $h_i : [r_0, +\infty) \to [0, +\infty)$ is continuous, and $h_i(r) \leq \frac{1}{r^{n+\mu_i}}$ as $r \to +\infty$ for some $\mu_i > 0$.
- $0 \leq \alpha_i[\gamma_i] < 1$ and $0 \leq \beta_i[\delta_i] < 1$;
- $\alpha_i[k_i] := \alpha_i[k_i(\cdot, s)]$ and $\beta_i[k_i] := \beta_i[k_i(\cdot, s)]$ are nonnegative;
- $D_i = (1 - \alpha_i[\gamma_i])(1 - \beta_i[\delta_i]) - \alpha_i[\delta_i]\beta_i[\gamma_i] > 0$.

Note that the hypotheses on the nonlinearities f_i are used in [3] and are satisfied, for example, if the functions f_i are continuous and, with respect to the last two variables, decreasing or bounded.

We now prove that T leaves the cone K invariant and is completely continuous.

Theorem 2.2 *The operator T maps K into K and is completely continuous.*

Proof To prove that T leaves the cone K invariant, it suffices to prove that $T_i K \subset K_i$.

For $(u_1, u_2) \in K$, we have

$$\|T_i(u_1, u_2)\|_\infty \leq \|\gamma_i\|_\infty \alpha_i[u_i] + \|\delta_i\|_\infty \beta_i[u_i]$$

$$+ \int_0^1 \phi_i(s) g_i(s) f_i\left(u_1(s), u_2(s), \frac{|u'_1(s)|}{|r'(s)|}, \frac{|u'_2(s)|}{|r'(s)|} \right) ds$$

$$\leq \|\gamma_i\|_\infty \alpha_i[u_i] + \|\delta_i\|_\infty \beta_i[u_i]$$

$$+ \int_0^1 \phi_i(s) g_i(s) \psi_i\left(u_1(s), u_2(s), \frac{\omega(s)|u'_1(s)|}{|r'(s)|}, \frac{\omega(s)|u'_2(s)|}{|r'(s)|} \right) ds$$

$$< \infty. \tag{2.6}$$

On the other hand, we have

$$\min_{t\in[a_i,b_i]} T_i(u_1,u_2)(t) \geq a_i\bigg(\|\gamma_i\|_\infty \alpha_i[u_i] + \|\delta_i\|_\infty \beta_i[u_i]$$

$$+ \int_0^1 \phi_i(s)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds\bigg)$$

$$\geq a_i\big\|T_i(u_1,u_2)\big\|_\infty.$$

Now we prove that, for every $(u_1,u_2)\in K$,

$$\big\|\big(F_i(u_1,u_2)\big)'\big\|_\omega \leq 4F_i(u_1,u_2)(1/2).$$

We have

$$\omega(t)\big|\big(F_i(u_1,u_2)\big)'(t)\big|$$

$$= \bigg| -t(1-t)\int_0^t \frac{c_i s}{d_i+c_i}g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds$$

$$+ t(1-t)\int_t^1 \bigg(1-\frac{c_i s}{d_i+c_i}\bigg)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds\bigg|$$

$$\leq \int_0^t t(1-t)\frac{c_i s}{d_1+c_i}g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds$$

$$+ \int_t^1 t(1-t)\bigg(1-\frac{c_i s}{d_i+c_i}\bigg)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds.$$

Since $\frac{c_i t}{d_i+c_i}\leq t$, we have that $(1-t)\leq(1-\frac{c_i t}{d_i+c_i})$, and consequently

$$\omega(t)\big|\big(F_i(u_1,u_2)\big)'(t)\big| \leq \int_0^t s\bigg(1-\frac{c_i t}{d_i+c_i}\bigg)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds$$

$$+ \int_t^1 t\bigg(1-\frac{c_i s}{d_i+c_i}\bigg)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds$$

$$= \int_0^1 k_i(t,s)g_i(s)f_i\bigg(u_1(s),u_2(s),\frac{|u_1'(s)|}{|r'(s)|},\frac{|u_2'(s)|}{|r'(s)|}\bigg)\,ds$$

$$= F_i(u_1,u_2)(t) \leq \big\|F_i(u_1,u_2)\big\|_\infty.$$

Let $\tau_i\in[0,1]$ be such that

$$F_i(u_1,u_2)(\tau_i) = \big\|F_i(u_1,u_2)\big\|_\infty.$$

For any $t\in[0,1]$, we can easily compute that

$$\frac{k_i(t,s)}{k_i(\tau_i,s)} = \begin{cases} t/\tau_i, & \tau_i,t\leq s, \\ (1-\frac{c_i t}{d_i+c_i})(1-\frac{c_i\tau_i}{d_i+c_i})^{-1}, & \tau_i,t> s, \\ t(1-\frac{c_i s}{d_i+c_i})s^{-1}(1-\frac{c_i\tau_i}{d_i+c_i})^{-1}, & t\leq s\leq\tau_i, \\ s(1-\frac{c_i t}{d_i+c_i})\tau_i^{-1}(1-\frac{c_i s}{d_i+c_i})^{-1}, & \tau_i\leq s\leq t, \end{cases}$$

and that $\frac{k_i(t,s)}{k_i(\tau_i,s)} \geq t(1-t)$ for $t, s \in [0,1]$. Then, for all $t \in [0,1]$, we have

$$
\begin{aligned}
F_i(u_1, u_2)(t) &= \int_0^1 \frac{k_i(t,s)}{k_i(\tau_i,s)} k_i(\tau_i,s) g_i(s) f_i\left(u_1(s), u_2(s), \frac{|u_1'(s)|}{|r'(s)|}, \frac{|u_2'(s)|}{|r'(s)|} \right) ds \\
&\geq t(1-t) \int_0^1 k_i(\tau_i,s) g_i(s) f_i\left(u_1(s), u_2(s), \frac{|u_1'(s)|}{|r'(s)|}, \frac{|u_2'(s)|}{|r'(s)|} \right) ds \\
&= t(1-t) \big\| F_i(u_1, u_2) \big\|_\infty.
\end{aligned}
$$

For $t = \frac{1}{2}$, we obtain

$$
\big\| F_i(u_1, u_2) \big\|_\infty \leq 4 F_i(u_1, u_2)\left(\frac{1}{2} \right).
$$

Therefore we conclude that

$$
\big\| \big(F_i(u_1, u_2)\big)' \big\|_\omega \leq \big\| F_i(u_1, u_2) \big\|_\infty \leq 4 F_i(u_1, u_2)\left(\frac{1}{2} \right).
$$

Since $\gamma_i, \delta_i \in K_i$, we have

$$
\begin{aligned}
\big| \omega(t) \big(T_i(u_1, u_2) \big)'(t) \big| &\leq \omega(t) \big| \gamma_i'(t) \big| \alpha_i[u_i] + \omega(t) \big| \delta_i'(t) \big| \beta_i[u_i] \\
&\quad + \omega(t) \big| \big(F_i(u_i, u_2) \big)'(t) \big| \\
&\leq \big\| \gamma_i' \big\|_\omega \alpha_i[u_i] + \big\| \delta_i' \big\|_\omega \beta_i[u_i] + \big\| F_i'(u_1, u_2) \big\|_\omega \\
&\leq 4\gamma_i\left(\frac{1}{2} \right) \alpha_i[u_i] + 4\delta_i\left(\frac{1}{2} \right) \beta_i[u_i] + 4 F_i(u_i, u_2)\left(\frac{1}{2} \right) \\
&= 4 T_i(u_1, u_2)\left(\frac{1}{2} \right).
\end{aligned}
$$

Taking the supremum on $[0,1]$, we obtain

$$
\big\| \big(T_i(u_1, u_2) \big)' \big\|_\omega \leq 4 T_i(u_1, u_2)\left(\frac{1}{2} \right).
$$

Since α_i and β_i are linear functionals, it follows that $\alpha_i[T_i(u_1, u_2)]$ and $\beta_i[T_i(u_1, u_2)]$ are nonnegative.

Summarizing, we have $TK \subset K$.

To prove the complete continuity of T, let us note that the continuity of f, k_i, α_i, and β_i give the continuity of each T_i and thus the continuity of T.

Let U be a bounded subset of K; from (2.6) it follows that $T(U)$ is bounded in K. Now we prove that $T(U)$ is relatively compact in K. It is a standard argument based on the uniform continuity of the kernels k_i on $[0,1] \times [0,1]$ and on the Ascoli–Arzelà theorem that $T_i(U)$ is relatively compact in $C[0,1]$.

Now let $(u_n, v_n)_{n \in \mathbb{N}}$ be a sequence in U. Then $T_i(u_n, v_n) \subset K_i$.

There exists $(u_{n_k}, v_{n_k})_{k \in \mathbb{N}}$ such that $(T_1(u_{n_k}, v_{n_k}))_{k \in \mathbb{N}}$ converges in $C[0,1]$.

Since $T_2(U)$ is relatively compact, there exists $(u_{n_{k_p}}, v_{n_{k_p}})_{p \in \mathbb{N}} := (u_{n_p}, v_{n_p})_{p \in \mathbb{N}} \subset (u_{n_k}, v_{n_k})_{k \in \mathbb{N}}$ such that $(T_i(u_{n_p}, v_{n_p}))_{p \in \mathbb{N}} \to w_i \in C[0,1]$ for $i = 1, 2$. Since

$$
\big\| \big(T_i(u_{n_p}, v_{n_p}) \big)' - \big(T_i(u_{n_m}, v_{n_m}) \big)' \big\|_\omega \leq 4 \big\| T_i(u_{n_p}, v_{n_p}) - T_i(u_{n_m}, v_{n_m}) \big\|_\infty,
$$

that is, $((T_i(u_{n_p}, v_{n_p}))')_{p\in\mathbb{N}}$ is a Cauchy sequence in $\|\cdot\|_\omega$ for $i = 1, 2$. Then $(T_i(u_{n_p}, v_{n_p}))_{p\in\mathbb{N}}$ is a Cauchy sequence in $C^1_\omega[0, 1]$, and so it converges to $w_i \in C^1_\omega[0, 1]$. The closedness of K implies that $(w_1, w_2) \in K$, and therefore $T(U)$ is relatively compact in K. $\qquad\square$

To use the fixed point index, we utilize the following sets in K for $\rho_1, \rho_2 > 0$:

$$K_{\rho_1,\rho_2} := \left\{(u, v) \in K : \|u\|_\infty < \rho_1 \text{ and } \|v\|_\infty < \rho_2\right\},$$

$$V_{\rho_1,\rho_2} := \left\{(u, v) \in K : \min_{t\in[a_1,b_1]} u(t) < \rho_1 \text{ and } \min_{t\in[a_2,b_2]} v(t) < \rho_2\right\}.$$

Since $\|w'\|_\omega \leq 4\|w\|_\infty$ in K, we have $\|w\| \leq 4\|w\|_\infty$, and therefore K_{ρ_1,ρ_2} and V_{ρ_1,ρ_2} are open and bounded with respect to K. It is straightforward to verify that these sets satisfy the following properties:

(P_1) $K_{\rho_1,\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\rho_1/a_1,\rho_2/a_2}$.

(P_2) $(w_1, w_2) \in \partial K_{\rho_1,\rho_2}$ if and only if $(w_1, w_2) \in K$ and for some $i \in \{1, 2\}$ $\|w_i\|_\infty = \rho_i$ and $a_i\rho_i \leq w_i(t) \leq \rho_i$ for $t \in [a_i, b_i]$.

(P_3) $(w_1, w_2) \in \partial V_{\rho_1,\rho_2}$ if and only if $(w_1, w_2) \in K$ and for some $i \in \{1, 2\}$ $\min_{t\in[a_i,b_i]} w_i(t) = \rho_i$ and $\rho_i \leq w_i(t) \leq \rho_i/a_i$ for $t \in [a_i, b_i]$.

The following theorem follows from classical results about fixed point index (more details can be seen, for example, in [4, 28]).

Theorem 2.3 *Let K be a cone in an ordered Banach space X. Let Ω be an open bounded subset with $0 \in \Omega \cap K$ and $\overline{\Omega \cap K} \neq K$. Let Ω^1 be open in X with $\overline{\Omega^1} \subset \Omega \cap K$. Let $F : \overline{\Omega \cap K} \to K$ be a compact map. Suppose that*

(1) $Fx \neq \mu x$ for all $x \in \partial(\Omega \cap K)$ and $\mu \geq 1$.

(2) There exists $h \in K \setminus \{0\}$ such that $x \neq Fx + \lambda h$ for all $x \in \partial(\Omega^1 \cap K)$ and $\lambda > 0$.

Then F has at least one fixed point $x \in (\Omega \cap K) \setminus \overline{(\Omega^1 \cap K)}$.

Denoting by $i_K(F, U)$ the fixed point index of F in some $U \subset X$, we have

$$i_K(F, \Omega \cap K) = 1 \quad and \quad i_K(F, \Omega^1 \cap K) = 0.$$

The same result holds if

$$i_K(F, \Omega \cap K) = 0 \quad and \quad i_K(F, \Omega^1 \cap K) = 1.$$

3 A system of elliptic PDE

We define the following sets:

$$\Omega^{\rho_1,\rho_2} = [0, \rho_1] \times [0, \rho_2] \times [0, +\infty) \times [0, +\infty),$$

$$A_1^{s_1,s_2} = \left[s_1, \frac{s_1}{a_1}\right] \times \left[0, \frac{s_2}{a_2}\right] \times [0, +\infty) \times [0, +\infty),$$

$$A_2^{s_1,s_2} = \left[0, \frac{s_1}{a_1}\right] \times \left[s_2, \frac{s_2}{a_2}\right] \times [0, +\infty) \times [0, +\infty),$$

and the numbers

$$
\begin{aligned}
C_i := \Bigg[\frac{1}{D_i} \Bigg(&\big[\big(1 - \beta_i[\delta_i] \big) + \|\delta_i\|_\infty \beta_i[\gamma_i] \big] \int_0^1 \alpha_i[k_i] g_i(s)\, ds \\
&+ \big[\alpha_i[\delta_i] + \|\delta_i\|_\infty \big(1 - \alpha_i[\gamma_i] \big) \big] \int_0^1 \beta_i[k_i] g_i(s)\, ds \Bigg) + \sup_{t \in [0,1]} \int_0^1 k_i(t,s) g_i(s)\, ds \Bigg]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
M_i = \Bigg[\frac{1}{D_i} \Bigg(&\big[a_i \big(1 - \beta_i[\delta_i] \big) + a_i \|\delta_i\|_\infty \beta_i[\gamma_i] \big] \int_{a_i}^{b_i} \alpha_i[k_i] g_i(s)\, ds \\
&+ \big[a_i \alpha_i[\delta_i] + a_i \|\delta_i\|_\infty \big(1 - \alpha_i[\gamma_i] \big) \big] \int_{a_i}^{b_i} \beta_1[k_i] g_i(s)\, ds \Bigg) \\
&+ \inf_{t \in [a_i,b_i]} \int_{a_i}^{b_i} k_i(t,s) g_i(s)\, ds \Bigg]^{-1}.
\end{aligned}
$$

Theorem 3.1 *Suppose that there exist* $\rho_1, \rho_2, s_1, s_2 \in (0, +\infty)$, *with* $\rho_i < s_i$, $i = 1, 2$, *such that*

$$
\sup_{\Omega^{\rho_1,\rho_2}} f_i(w_1, w_2, z_1, z_2) < C_i \rho_i \tag{3.1}
$$

and

$$
\inf_{A_i^{s_1,s_2}} f_i(w_1, w_2, z_1, z_2) > M_i s_i. \tag{3.2}
$$

Then system (2.3) *has at least one positive radial solution.*

Proof Note that the choice of the numbers ρ_i and s_i ensures the compatibility of conditions (3.1) and (3.2).

We want to show that $i_K(T, K_{\rho_1,\rho_2}) = 1$ and $i_K(T, V_{s_1,s_2}) = 0$, so that from Theorem 2.3 it follows that the completely continuous operator T has a fixed point in $V_{s_1,s_2} \setminus \overline{K}_{\rho_1,\rho_2}$. Then system (2.3) admits a positive radial solution.

First, we claim that $\lambda(u,v) \neq T(u,v)$ for all $(u,v) \in \partial K_{\rho_1,\rho_2}$ and $\lambda \geq 1$, which implies that the index of T is 1 on K_{ρ_1,ρ_2}. Suppose this is not true. Let $\lambda \geq 1$, and let $(u,v) \in \partial K_{\rho_1,\rho_2}$ be such that

$$
\lambda(u,v) = T(u,v).
$$

In view of (P_2), without loss in generality, let us suppose that $\|u\|_\infty = \rho_1$. Then

$$
\begin{aligned}
\lambda u(t) = {}&\gamma_1(t)\alpha_1[u] + \delta_1(t)\beta_1[u] \\
&+ \int_0^1 k_1(t,s) g_1(s) f_1 \Bigg(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \Bigg) ds. \tag{3.3}
\end{aligned}
$$

Applying α_1 to both terms, we have

$$
\lambda \alpha_1[u] = \alpha_1[\gamma_1]\alpha_1[u] + \alpha_1[\delta_1]\beta_1[u] + \int_0^1 \alpha_1[k_1] g_1(s) f_1 \Bigg(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \Bigg) ds,
$$

wich implies

$$\left(\lambda - \alpha_1[\gamma_1]\right)\alpha_1[u] - \alpha_1[\delta_1]\beta_1[u] = \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds.$$

In a similar way, applying β_1, we obtain

$$\left(\lambda - \beta_1[\delta_1]\right)\beta_1[u] - \beta_1[\gamma_1]\alpha_1[u] = \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds.$$

Denoting

$$N_1^\lambda := \begin{pmatrix} \lambda - \alpha_1[\gamma_1] & -\alpha_1[\delta_1] \\ -\beta_1[\gamma_1] & \lambda - \beta_1[\delta_1] \end{pmatrix}, \qquad N_1^1 := N_1, \quad \text{and} \quad D_1 := \det N_1,$$

we can write the previous conditions as

$$N_1^\lambda \begin{pmatrix} \alpha_1[u] \\ \beta_1[u] \end{pmatrix} = \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \end{pmatrix}.$$

Therefore we get that

$$\begin{pmatrix} \alpha_1[u] \\ \beta_1[u] \end{pmatrix} = \left(N_1^\lambda\right)^{-1} \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \end{pmatrix}$$

$$\leq (N_1)^{-1} \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \end{pmatrix},$$

so that formula (3.3) becomes, for $t \in [0, 1]$,

$$\begin{aligned}
u(t) \leq \frac{1}{D_1}\bigg[&\gamma_1(t)\left(1 - \beta_1[\delta_1]\right) \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \\
&+ \gamma_1(t)\alpha_1[\delta_1] \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \\
&+ \delta_1(t)\beta_1[\gamma_1] \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \\
&+ \delta_1(t)\left(1 - \alpha_1[\gamma_1]\right) \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \bigg] \\
&+ \int_0^1 k_1(t,s)g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \\
= \frac{1}{D_1}\bigg[&\left[\gamma_1(t)\left(1 - \beta_1[\delta_1]\right) + \delta_1(t)\beta_1[\gamma_1]\right] \\
&\times \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \\
&+ \left[\gamma_1(t)\alpha_1[\delta_1] + \delta_1(t)\left(1 - \alpha_1[\gamma_1]\right)\right] \\
&\times \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds \bigg]
\end{aligned}$$

$$+ \int_0^1 k_1(t,s)g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds$$

$$\leq \sup_{\Omega^{\rho_1,\rho_2}} f_1(w_1, w_2, z_1, z_2) \left[\frac{1}{D_1} \left[\left(\gamma_1(t)\left(1 - \beta_1[\delta_1]\right) + \delta_1(t)\beta_1[\gamma_1] \right) \int_0^1 \alpha_1[k_1]g_1(s)\,ds \right.\right.$$

$$+ \left.\left(\gamma_1(t)\alpha_1[\delta_1] + \delta_1(t)\left(1 - \alpha_1[\gamma_1]\right) \right) \int_0^1 \beta_1[k_1]g_1(s)\,ds \right]$$

$$+ \left. \int_0^1 k_1(t,s)g_1(s)\,ds \right]. \tag{3.4}$$

Taking the supremum on $[0,1]$ in the last inequality, it follows that

$$\rho_1 = \|u\|_\infty$$

$$\leq \sup_{\Omega^{\rho_1,\rho_2}} f_1(w_1, w_2, z_1, z_2) \left[\frac{1}{D_1} \left[\left[\left(1 - \beta_1[\delta_1]\right) + \|\delta_1\|_\infty \beta_1[\gamma_1] \right] \int_0^1 \alpha_1[k_1]g_1(s)\,ds \right.\right.$$

$$+ \left. \left[\alpha_1[\delta_1] + \|\delta_1\|_\infty \left(1 - \alpha_1[\gamma_1]\right) \right] \int_0^1 \beta_1[k_1]g_1(s)\,ds \right]$$

$$+ \left. \sup_{t\in[0,1]} \int_0^1 k_1(t,s)g_1(s)\,ds \right]$$

$$= \frac{1}{C_1} \sup_{\Omega^{\rho_1,\rho_2}} f_1(w_1, w_2, z_1, z_2) < \rho_1,$$

which is a contradiction.

Now we show that that the index of T is 0 on V_{s_1,s_2}.

Consider $l(t) = 1$ for $t \in [0,1]$ and note that $(l,l) \in K$. Now we claim that

$$(u,v) \neq T(u,v) + \lambda(l,l) \quad \text{for } (u,v) \in \partial V_{s_1,s_2} \text{ and } \lambda \geq 0.$$

Assume, by contradiction, that there exist $(u,v) \in \partial V_{s_1,s_2}$ and $\lambda \geq 0$ such that $(u,v) = T(u,v) + \lambda(l,l)$.

Without loss of generality, we can assume that $\min_{t\in[a_1,b_1]} u(t) = s_1$ and $s_1 \leq u(t) \leq s_1/a_1$ for $t \in [a_1,b_1]$. Then, for $t \in [a_1,b_1]$, we obtain

$$u(t) = \gamma_1(t)\alpha_1[u] + \delta_1(t)\beta_1[u] + \int_0^1 k_1(t,s)g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds + \lambda. \tag{3.5}$$

Applying α_1 and β_1 to both sides of (3.5) gives

$$\alpha_1[u] = \alpha_1[\gamma_1]\alpha_1[u] + \alpha_1[\delta_1]\beta_1[u]$$

$$+ \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds + \lambda\alpha_1[1],$$

$$\beta_1[u] = \beta_1[\gamma_1]\alpha_1[u] + \beta_1[\delta_1]\beta_1[u]$$

$$+ \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \right) ds + \lambda\beta_1[1].$$

Thus we have

$$\big(1 - \alpha_1[\gamma_1]\big)\alpha_1[u] - \alpha_1[\delta_1]\beta_1[u]$$

$$= \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda\alpha_1[1],$$

$$-\beta_1[\gamma_1]\alpha_1[u] + \big(1 - \beta_1[\delta_1]\big)\beta_1[u]$$

$$= \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda\beta_1[1].$$

Therefore

$$N_1 \begin{pmatrix} \alpha_1[u] \\ \beta_1[u] \end{pmatrix} = \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds + \lambda\alpha_1[1] \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds + \lambda\beta_1[1] \end{pmatrix}.$$

Applying the matrix $(N_1)^{-1}$ to both sides of the last equality, we obtain

$$\begin{pmatrix} \alpha_1[u] \\ \beta_1[u] \end{pmatrix} = (N_1)^{-1} \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds + \lambda\alpha_1[1] \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds + \lambda\beta_1[1] \end{pmatrix}$$

$$\geq (N_1)^{-1} \begin{pmatrix} \int_0^1 \alpha_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \\ \int_0^1 \beta_1[k_1]g_1(s)f_1(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}) ds \end{pmatrix}.$$

Thus, as in the previous step, we have

$$u(t) \geq \frac{1}{D_1}\bigg[\big[\gamma_1(t)\big(1 - \beta_1[\delta_1]\big) + \delta_1(t)\beta_1[\gamma_1]\big]$$

$$\times \int_0^1 \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds$$

$$+ \big[\gamma_1(t)\alpha_1[\delta_1] + \delta_1(t)\big(1 - \alpha_1[\gamma_1]\big)\big]$$

$$\times \int_0^1 \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds\bigg]$$

$$+ \int_0^1 k_1(t, s)g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda.$$

Then, for $t \in [a_1, b_1]$, we obtain

$$u(t) \geq \frac{1}{D_1}\bigg[\big[\gamma_1(t)\big(1 - \beta_1[\delta_1]\big) + \delta_1(t)\beta_1[\gamma_1]\big]$$

$$\times \int_{a_1}^{b_1} \alpha_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds$$

$$+ \big[\gamma_1(t)\alpha_1[\delta_1] + \delta_1(t)\big(1 - \alpha_1[\gamma_1]\big)\big]$$

$$\times \int_{a_1}^{b_1} \beta_1[k_1]g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds\bigg]$$

$$+ \int_{a_1}^{b_1} k_1(t, s)g_1(s)f_1\left(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|}\right) ds + \lambda$$

$$\geq \inf_{A_1^{s_1,s_2}} f_1(w_1, w_2, z_1, z_2) \left[\frac{1}{D_1} \left[\left[\gamma_1(t)\left(1 - \beta_1[\delta_1]\right) + \delta_1(t)\beta_1[\gamma_1] \right] \int_{a_1}^{b_1} \alpha_1[k_1]g_1(s)\,ds \right. \right.$$

$$+ \left. \left[\gamma_1(t)\alpha_1[\delta_1] + \delta_1(t)\left(1 - \alpha_1[\gamma_1]\right) \right] \int_{a_1}^{b_1} \beta_1[k_1]g_1(s)\,ds \right]$$

$$+ \left. \int_{a_1}^{b_1} k_1(t,s)g_1(s)\,ds \right] + \lambda.$$

Taking the minimum over $[a_1, b_1]$ gives

$$s_1 \geq \inf_{A_1^{s_1,s_2}} f_1(w_1, w_2, z_1, z_2) \left[\frac{1}{D_1} \left[\left[a_1\left(1 - \beta_1[\delta_1]\right) + a_1\|\delta_1\|_\infty \beta_1[\gamma_1] \right] \int_{a_1}^{b_1} \alpha_1[k_1]g_1(s)\,ds \right. \right.$$

$$+ \left. \left[a_1\alpha_1[\delta_1] + a_1\|\delta_1\|_\infty \left(1 - \alpha_1[\gamma_1]\right) \right] \int_{a_1}^{b_1} \beta_1[k_1]g_1(s)\,ds \right]$$

$$+ \left. \inf_{t\in[a_1,b_1]} \int_{a_1}^{b_1} k_1(t,s)g_1(s)\,ds \right] + \lambda$$

$$> M_1 s_1 \frac{1}{M_1} + \lambda,$$

a contradiction. □

By means of Theorem 3.1 and the fixed point index properties in Theorem 2.3 we can state results on the existence of *multiple* positive solutions for system (2.3). Here we enunciate a result on the existence of two positive solutions (see [32, 33] for the conditions that ensure three or more positive results).

Theorem 3.2 *Suppose that there exist* $\rho_i, s_i, \theta_i \in (0, \infty)$ *with* $\rho_i < s_i$ *and* $\frac{s_i}{a_i} < \theta_i$, $i = 1, 2$, *such that*

$$\sup_{\Omega^{\rho_1,\rho_2}} f_i(w_1, w_2, z_1, z_2) < C_i\rho_i,$$

$$\inf_{A_i^{s_1,s_2}} f_i(w_1, w_2, z_1, z_2) > M_i s_i,$$

$$\sup_{\Omega^{\theta_1,\theta_2}} f_i(w_1, w_2, z_1, z_2) < C_i\theta_i.$$

Then system (2.3) *has at least two positive radial solutions.*

Example 3.3   Note that Theorems 3.1 and 3.2 can be applied when the nonlinearities f_i are of the type

$$f_i\left(u, v, |\nabla u|, |\nabla v|\right) = \left(\delta_i u^{\alpha_i} + \gamma_i v^{\beta_i}\right)\zeta_i\left(u, v, |\nabla u|, |\nabla v|\right)$$

with continuous functions ζ_i bounded by a strictly positive constant, $\alpha_i, \beta_i > 1$, and suitable $\delta_i, \gamma_i \geq 0$.

Setting

$$q(w) = w^3 \chi_{[0,40]}(w) + \left|65{,}600 - w^2\right|\chi_{(40,360)}(w) + \frac{100}{101}\left|65{,}000 - w\right|\chi_{[360,+\infty)}(w),$$

we can consider in $\mathbb{R}^3$ the system of BVPs

$$
\begin{cases}
-\Delta u = \frac{1}{30|x|^4}(2 - \sin(|\nabla u|^2 + |\nabla v|^2)q(u) & \text{in } \Omega, \\
-\Delta v = \frac{1}{\pi|x|^4}\arctan(1 + |\nabla u|^2 + |\nabla v|^2)q(v) & \text{in } \Omega, \\
\lim_{|x|\to\infty} u(x) = u(\frac{1}{4}), \qquad 2u - 4\frac{\partial u}{\partial r} = u(\frac{1}{2}) & \text{for } |x| = 1, \\
\lim_{|x|\to\infty} v(x) = v(\frac{1}{4}), \qquad 3v - 2\frac{\partial v}{\partial r} = v(\frac{1}{2}) & \text{for } |x| = 1.
\end{cases}
\tag{3.6}
$$

Let $[a_1, b_1] = [a_2, b_2] = [\frac{1}{4}, \frac{1}{2}]$. By direct computation we obtain

$$
D_i = \frac{c_i - 1}{4(c_i + d_i)};
$$

$$
\sup_{t\in[0,1]} \int_0^1 k_i(t,s)g_i(s)\,ds = \frac{(c_i + 2d_i)^2}{8(c_i + d_i)^2}; \qquad \inf_{t\in[\frac{1}{4},\frac{1}{2}]} \int_{\frac{1}{4}}^{\frac{1}{2}} k_i(t,s)g_i(s)\,ds = \frac{5c_i + 8d_i}{128(c_i + d_i)};
$$

$$
\int_0^1 \alpha_i[k_i]g_i(s)\,ds = \frac{(3c_i + 7d_i)}{32(c_i + d_i)}; \qquad \int_0^1 \beta_i[k_i]g_i(s)\,ds = \frac{(c_i + 3d_i)}{8(c_i + d_i)};
$$

$$
\int_{a_i}^{b_i} \alpha_i[k_i]g_i(s)\,ds = \frac{5c_i + 8d_i}{128(c_i + d_i)}; \qquad \int_{a_i}^{b_i} \beta_i[k_i]g_i(s)\,ds = \frac{3}{32}\left(1 - \frac{c_i}{2(c_i + d_i)}\right).
$$

Since in our example the mixed perturbed conditions state that $c_1 = 2$, $d_1 = 4$, $c_2 = 3$, and $d_2 = 2$, we easily compute C_i and M_i:

$$
C_1 = \frac{9}{47}, \qquad C_2 = \frac{25}{48}, \qquad M_1 = \frac{96}{41}, \qquad M_2 = \frac{16}{3}.
$$

Choosing $\rho_1 = \rho_2 = 1$, $s_1 = 9$, $s_2 = 5$, $\theta_1 = 40{,}000$, and $\theta_2 = 65{,}000$, we have

$$
\sup_{\Omega^{\rho_1,\rho_2}} f_1 \le \frac{1}{10}\sup_{u\in[0,\rho_1]} q(u) = \frac{1}{10}q(\rho_1) = \frac{1}{10} < \frac{9}{47} = C_1\rho_1,
$$

$$
\inf_{A_1^{s_1,s_2}} f_1 \ge \frac{1}{30}\inf_{u\in[s_1,4s_1]} q(u) = \frac{1}{30}q(s_1) = \frac{243}{10} > \frac{192}{41} = M_1 s_1,
$$

$$
\sup_{\Omega^{\theta_1,\theta_2}} f_1 \le \frac{1}{10}\sup_{u\in[0,\theta_1]} q(u) = \frac{1}{10}q(40) = 6400 < \frac{36{,}000}{47} = C_1\theta_1,
$$

$$
\sup_{\Omega^{\rho_1,\rho_2}} f_2 \le \frac{1}{2}\sup_{v\in[0,\rho_2]} q(v) = \frac{1}{2}q(\rho_2) = \frac{1}{2} < \frac{25}{48} = C_2\rho_2,
$$

$$
\inf_{A_2^{s_1,s_2}} f_2 \ge \frac{1}{4}\inf_{v\in[s_2,4s_2]} q(v) = \frac{1}{4}q(s_2) = \frac{125}{4} > \frac{80}{3} = M_2 s_2,
$$

$$
\sup_{\Omega^{\theta_1,\theta_2}} f_2 \le \frac{1}{2}\sup_{v\in[0,\theta_2]} q(v) = \frac{1}{2}q(40) = 3200 < \frac{203{,}125}{6} = C_2\theta_2,
$$

where the suprema and infima are computed on

$$
\Omega^{\rho_1,\rho_2} = \Omega^{1,1} = [0,1] \times [0,1] \times [0,+\infty) \times [0,+\infty);
$$

$$
A_1^{s_1,s_2} = A_1^{9,5} = [9,36] \times [0,20] \times [0,+\infty) \times [0,+\infty);
$$

$$
A_2^{s_1,s_2} = A_2^{9,5} = [0,36] \times [5,20] \times [0,+\infty) \times [0,+\infty);
$$

$$\Omega^{\theta_1,\theta_2} = \Omega^{40,000,60,000} = [0, 40,000] \times [0, 60,000] \times [0, +\infty) \times [0, +\infty).$$

Then the hypotheses of Theorem 3.2 are satisfied, and hence system (3.6) has at least two positive solutions.

4 Nonexistence results

We now show a nonexistence result for the system of elliptic equations (2.3) when the the functions f_i have an enough "small" or "large" growth.

Theorem 4.1 *Assume that one of following conditions holds*:

$$f_i(w_1, w_2, z_1, z_2) < C_i w_i, \quad w_i > 0 \text{ for } i = 1, 2, \tag{4.1}$$

$$f_i(w_1, w_2, z_1, z_2) > M_i w_i, \quad w_i > 0 \text{ for } i = 1, 2. \tag{4.2}$$

Then the only possible positive solution of system (2.3) *is the zero one*.

Proof Suppose that (4.1) holds and assume that there exists a solution $(\bar{u}, \bar{v})$ of (2.3), $(\bar{u}, \bar{v}) \neq (0, 0)$; then $(u, v) := (\bar{u} \circ r, \bar{v} \circ r)$ is a fixed point of T. Let, for example, $\|(u, v)\| = \|u\| \leq 4\|u\|_\infty \neq 0$.

Then, for $t \in [0, 1]$, taking into account (3.4), we have

$$\begin{aligned}
u(t) &< C_1 \bigg(\frac{1}{D_1} \bigg[\big[(1 - \beta_1[\delta_1]) + \|\delta_1\|_\infty \beta_1[\gamma_1] \big] \int_0^1 \alpha_1[k_1] g_1(s) u(s) \, ds \\
&\quad + \big[\alpha_1[\delta_1] + \|\delta_1\|_\infty (1 - \alpha_1[\gamma_1]) \big] \int_0^1 \beta_1[k_1] g_1(s) u(s) \, ds \bigg] + \int_0^1 k_1(t, s) g_1(s) u(s) \, ds \bigg) \\
&\leq C_1 \|u\|_\infty \bigg(\frac{1}{D_1} \bigg[\big[(1 - \beta_1[\delta_1]) + \|\delta_1\|_\infty \beta_1[\gamma_1] \big] \int_0^1 \alpha_1[k_1] g_1(s) \, ds \\
&\quad + \big[\alpha_1[\delta_1] + \|\delta_1\|_\infty (1 - \alpha_1[\gamma_1]) \big] \int_0^1 \beta_1[k_1] g_1(s) \, ds \bigg] + \int_0^1 k_1(t, s) g_1(s) \, ds \bigg).
\end{aligned}$$

For $u > 0$, taking the supremum for $t \in [0, 1]$, we have $\|u\|_\infty < \|u\|_\infty$, a contradiction.

Suppose that (4.2) holds and assume that there exists $(u, v) \in K$ such that $(u, v) = T(u, v)$ and $(u, v) \neq (0, 0)$. Let, for example, $\|u\|_\infty \neq 0$; then $\sigma := \min_{t \in [a_1, b_1]} u(t) > 0$ since $u \in K_1$. Thus, as in the proof of Theorem 3.1, we have, for $t \in [a_1, b_1]$,

$$\begin{aligned}
u(t) &\geq \frac{1}{D_1} \bigg[\big[a_1 (1 - \beta_1[\delta_1]) + a_1 \|\delta_1\|_\infty \beta_1[\gamma_1] \big] \\
&\quad \times \int_{a_1}^{b_1} \alpha_1[k_1] g_1(s) f_1 \bigg(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \bigg) \, ds \\
&\quad + \big[a_1 \alpha_1[\delta_1] + a_1 \|\delta_1\|_\infty (1 - \alpha_1[\gamma_1]) \big] \\
&\quad \times \int_{a_1}^{b_1} \beta_1[k_1] g_1(s) f_1 \bigg(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \bigg) \, ds \bigg] \\
&\quad + \int_{a_1}^{b_1} k_1(t, s) g_1(s) f_1 \bigg(u(s), v(s), \frac{|u'(s)|}{|r'(s)|}, \frac{|v'(s)|}{|r'(s)|} \bigg) \, ds
\end{aligned}$$

$$> M_1 \bigg(\frac{1}{D_1} \bigg[\big[a_1 \big(1 - \beta_1[\delta_1] \big) + a_1 \|\delta_1\|_\infty \beta_1[\gamma_1] \big] \int_{a_1}^{b_1} \alpha_1[k_1] g_1(s) u(s) \, ds$$

$$+ \big[a_1 \alpha_1[\delta_1] + a_1 \|\delta_1\|_\infty \big(1 - \alpha_1[\gamma_1] \big) \big] \int_{a_1}^{b_1} \beta_1[k_1] g_1(s) u(s) \, ds \bigg]$$

$$+ \int_{a_1}^{b_1} k_1(t,s) g_1(s) u(s) \, ds \bigg)$$

$$\geq M_1 \sigma \bigg(\frac{1}{D_1} \bigg[\big[a_1 \big(1 - \beta_1[\delta_1] \big) + a_1 \|\delta_1\|_\infty \beta_1[\gamma_1] \big] \int_{a_1}^{b_1} \alpha_1[k_1] g_1(s) \, ds$$

$$+ \big[a_1 \alpha_1[\delta_1] + a_1 \|\delta_1\|_\infty \big(1 - \alpha_1[\gamma_1] \big) \big] \int_{a_1}^{b_1} \beta_1[k_1] g_1(s) \, ds \bigg] + \int_{a_1}^{b_1} k_1(t,s) g_1(s) \, ds \bigg).$$

For $u > 0$, taking the infimum for $t \in [a_1, b_1]$, we obtain $\sigma > \sigma$, a contradiction. $\qquad \square$

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