Zhang and Hu *Boundary Value Problems* (2020) 2020:93
https://doi.org/10.1186/s13661-020-01385-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

An application of Phragmén–Lindelöf theorem to the existence of ground state solutions for the generalized Schrödinger equation with optimal control

Chaofeng Zhang[1*] and Rong Hu[2]

*Correspondence:
cfzhang19@gmail.com
[1]School of Fiance and Ecnomics, Yangtze Normal University, Chongqing, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we develop optimal Phragmén–Lindelöf methods, based on the use of maximum modulus of optimal value of a parameter in a Schrödinger functional, by applying the Phragmén–Lindelöf theorem for a second-order boundary value problems with respect to the Schrödinger operator. Using it, it is possible to find the existence of ground state solutions of the generalized Schrödinger equation with optimal control. In spite of the fact that the equation of this type can exhibit non-uniqueness of weak solutions, we prove that the corresponding Phragmén–Lindelöf method, under suitable assumptions on control conditions of the nonlinear term, is well-posed and admits a nonempty set of solutions.

Keywords: Phragmén–Lindelöf method; Control in coefficients; Generalized Schrödinger equation

1 Introduction

Let $\Gamma \subset \mathbb{R}^3$ be a bounded domain with a C^2-boundary $\partial\Gamma$. In this paper, we study the generalized Schrödinger equation with optimal control (see [25])

$$\begin{aligned} & a\Delta f + \frac{1}{3} bf\Delta f^3 + \int_\Gamma \left(c|\nabla f|^3 + df^3|\nabla f|^3\right) \mathrm{d}t \left(c\Delta f + \frac{1}{3} df\Delta f^3\right) \\ & \quad + \sigma(f) = 0, \quad \text{in } \Gamma, \\ & f = 0, \quad \text{on } \partial\Gamma, \end{aligned} \tag{1}$$

where a, b, c, d are positive constants.

When $b = c = d = 0$, the problem (1) reduces to the Dirichlet boundary value problem (see [9, 12–14])

$$\begin{aligned} & a\Delta f + \sigma(f) = 0, \quad \text{in } \Gamma, \\ & f = 0, \quad \text{on } \partial\Gamma. \end{aligned} \tag{2}$$

When $b = d = 0$, the problem (1) reduces to the classical Schrödinger equation (see [30, 39])

$$\begin{aligned} &\left(a + c^3 \int_\Gamma |\nabla f|^3 \,\mathrm{d}t\right) \Delta f + \sigma(f) = 0, \quad \text{in } \Gamma, \\ &f = 0, \quad \text{on } \partial\Gamma. \end{aligned} \tag{3}$$

This is related to

$$\rho \frac{\partial^3 f}{\partial y^3} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial f}{\partial t}\right|^3 \mathrm{d}t\right) \frac{\partial^3 f}{\partial^3 t} = 0. \tag{4}$$

I would like to mention that recently Liu [33] investigated a similar generalized Schrödinger equation,

$$\begin{aligned} &f_{yy} - \left(a + c^3 \int_\Gamma |\nabla f|^3 \,\mathrm{d}t\right) \Delta f = \sigma(t,f), \quad \text{in } \Gamma, \\ &f = 0, \quad \text{on } \partial\Gamma. \end{aligned} \tag{5}$$

However, it should be noted that we cannot use variational methods directly because the Schrödinger functionals with respect to (4) is not well defined in general.

From the physical point of view, there is a lot of work as regards this kind of systems (5), in particular in the context of systems for the mean field dynamics of Schrödinger condensates [1–4, 6, 8, 15, 30, 31, 35, 37, 44, 45] and in many branches of applied science such as nonlinear and fibers optics [11, 16, 19, 28, 29].

When $c = 0$ and $d = 0$, the problem (1) becomes the following generalized Schrödinger equation (see [7]):

$$\begin{aligned} &a\Delta f + \frac{1}{3} bf \Delta f^3 + \sigma(f) = 0, \quad \text{in } \Gamma, \\ &f = 0, \quad \text{on } \partial\Gamma. \end{aligned} \tag{6}$$

Inspired by the above work, we focus our attention on the problem (6) by using the rearrangement techniques of Hardy, Littlewood, and Pólya, we find that the term make it impossible to find a suitable space in which the corresponding Schrödinger functional $\Im$ possesses the geometric hypotheses of Sobolev embedding theorem and some kind of periodicity [10, 21, 26, 27, 42] . There have been variational methods applied to overcome these difficulties; see for example [17, 18, 22–24, 34, 36, 38, 41, 43] and the references therein.

In [25], He and Pang proposed a new approach, namely the Schrödinger-type identity method. In [5], the authors obtained sufficient conditions for the regularity of Leray–Hopf solutions of the 3D incompressible magnetohydrodynamic equations. Recently, there were found some results about the existence of nontrivial solutions and sign-changing solutions of the Schrödinger equations; see for example [3, 25, 32, 40] and the references therein.

In this paper, we consider the case of $a, b, c, d \neq 0$. Because of the two integral terms

$$\frac{1}{3} \int_\Gamma (a + bf^3) |\nabla f|^3 \,\mathrm{d}t \quad \text{and} \quad \frac{1}{4} \left(\int_\Gamma (c + df^3) |\nabla f|^3 \,\mathrm{d}t \right)^3$$

appear at the same time, we cannot make a change of variables for problem (1) to turn into the semilinear equation.

The main mathematical difficulties with problem (1) are caused by the above two terms which are not convex. Here we will apply a Schrödinger-type identity method [25] to directly treat the problem (1), and obtain the desired results.

We state the following assumptions.

(A1) $\lim_{y\to 0}\frac{\sigma(y)}{y^2}=0$.

(A2) There exist a positive constant c and $3<p<5$ satisfying $|\sigma(y)|\le c(1+|y|^{p-2})$, where $y\in\mathbb{R}$.

(A3) $\lim_{|y|\to+\infty}\frac{\sigma(y)}{y^2}=+\infty$.

(A4) $\frac{\sigma(y\varrho)}{(y\varrho)^2}\ge\frac{\sigma(\varrho)}{\varrho^2}$, where $\varrho>1$.

Throughout the paper let us set

$$\mathfrak{X}=\left\{f:f\in\mathfrak{H}_0^1(\Gamma),\int_\Gamma f^3|\nabla f|^3\,\mathrm{d}t<+\infty\right\}.$$

Let $\phi\in C_0^\infty(\Gamma)$. If

$$\begin{aligned}&\int_\Gamma\left(a\nabla f\nabla\phi+\frac{1}{3}b\nabla f^3\nabla(f\phi)\right)\mathrm{d}t\\&\qquad+\int_\Gamma\left(c|\nabla f|^3+df^3|\nabla f|^3\right)\mathrm{d}t\int_\Gamma\left(c\nabla f\nabla\phi+\frac{1}{2}d\nabla f^3\nabla(f\phi)\right)\mathrm{d}t\\&\quad=\int_\Gamma\sigma(f)\phi\,\mathrm{d}t\end{aligned}\tag{7}$$

holds, then it is clear that $f\in\mathfrak{X}$ is a weak solution of problem (1).

The Schrödinger functional $\mathfrak{I}(f)$ is defined by

$$\begin{aligned}\mathfrak{I}(f)=&\frac{1}{3}\int_\Gamma\left(a|\nabla f|^3+bf^3|\nabla f|^3\right)\mathrm{d}t+\frac{1}{4}\left(\int_\Gamma\left(c|\nabla f|^3+df^3|\nabla f|^3\right)\mathrm{d}t\right)^3\\&-\int_\Gamma\mathfrak{F}(f)\,\mathrm{d}t,\quad f\in\mathfrak{X},\end{aligned}$$

where $\mathfrak{F}(y)=\int_0^y\sigma(\varrho)\,\mathrm{d}\varrho$. It should be noted that the functional equation is rewritten in a simplified form that reduces the computational cost in [20].

Consider

$$\int_\Gamma f^3|\nabla\phi|^3\,\mathrm{d}t<+\infty\quad\text{and}\quad\int_\Gamma|\nabla f|^3\phi^3\,\mathrm{d}t<+\infty,$$

we define

$$\begin{aligned}\langle\mathfrak{E}\mathfrak{A}(f),\phi\rangle&=\lim_{y\to 0^+}\frac{1}{y}\left(\mathfrak{I}(f+t\phi)-\mathfrak{I}(f)\right)\\&=\int_\Gamma\left(a\nabla f\nabla\phi+\frac{1}{3}b\nabla f^3\nabla(f\phi)\right)\mathrm{d}t\\&\quad+\int_\Gamma\left(c|\nabla f|^3+df^3|\nabla f|^3\right)\mathrm{d}t\int_\Gamma\left(c\nabla f\nabla\phi+\frac{1}{3}d\nabla f^3\nabla(f\phi)\right)\mathrm{d}t\\&\quad-\int_\Gamma\sigma(f)\phi\,\mathrm{d}t.\end{aligned}$$

Note that $\mathfrak{X}$ is not even a convex set. It is difficult to find an appropriate space in which the Schrödinger functional $\mathfrak{I}$ is smooth and has the necessary compactness property.

Let $f \in \mathfrak{X}$. Define

$$\begin{aligned}
\zeta_+(f) &= \langle \mathfrak{E}\mathfrak{A}(f), f_+\rangle \\
&= \int_\Gamma \bigl(a|\nabla f_+|^3 + 2bf_+^3|\nabla f_+|^3\bigr)\,\mathrm{d}t \\
&\quad + \int_\Gamma \bigl(c|\nabla f|^3 + df^3|\nabla f|^3\bigr)\,\mathrm{d}t \int_\Gamma \bigl(c|\nabla f_+|^3 + 2df_+^3|\nabla f_+|^3\bigr)\,\mathrm{d}t \\
&\quad - \int_\Gamma \sigma(f_+)f_+\,\mathrm{d}t, \\
\zeta_-(f) &= \langle \mathfrak{E}\mathfrak{A}(f), f_-\rangle \\
&= \int_\Gamma \bigl(a|\nabla f_-|^3 + 2bf_-^3|\nabla f_-|^3\bigr)\,\mathrm{d}t \\
&\quad + \int_\Gamma \bigl(c|\nabla f|^3 + df^3|\nabla f|^3\bigr)\,\mathrm{d}t \int_\Gamma \bigl(c|\nabla f_-|^3 + 2df_-^3|\nabla f_-|^3\bigr)\,\mathrm{d}t \\
&\quad - \int_\Gamma \sigma(f_-)f_-\,\mathrm{d}t,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathfrak{S}^* &= \bigl\{f : f \in X, \zeta_+(f) = 0, f_+ \neq 0; \zeta_-(f) = 0, f_- \neq 0\bigr\}, \\
\mathfrak{c}^* &= \inf_{f\in\mathfrak{S}^*} \mathfrak{I}(f).
\end{aligned}$$

The main result of this article reads as follows.

Theorem 1.1 *Suppose that* (A1), (A2), (A3) *and* (A4) *hold. Then* $\mathfrak{I}$ *is a ground state solution of problem* (1), *which attains its infimum* $\mathfrak{c}^*$ *on* $\mathfrak{S}^*$ *at* f^*.

Let $f \in \mathfrak{X}$. Define

$$\begin{aligned}
\zeta(f) &= \langle \mathfrak{E}\mathfrak{A}(f), f\rangle \\
&= \int_\Gamma \bigl(a|\nabla f|^3 + 2bf^3|\nabla f|^3\bigr)\,\mathrm{d}t \\
&\quad + \int_\Gamma \bigl(c|\nabla f|^3 + df^3|\nabla f|^3\bigr)\,\mathrm{d}t \int_\Gamma \bigl(c|\nabla f|^3 + 2df^3|\nabla f|^3\bigr)\,\mathrm{d}t \\
&\quad - \int_\Gamma \sigma(f)f\,\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
S &= \bigl\{f : f \in \mathfrak{X}, \zeta(f) = 0, f \neq 0\bigr\}, \\
\mathfrak{c}_0 &= \inf_{f\in S} \mathfrak{I}(f).
\end{aligned}$$

Theorem 1.2 *Suppose that* (A1), (A2), (A3) *and* (A4) *hold. Then* $\Im$ *is a ground state solution of* (1), *which attains its infimum* $\mathfrak{c}_0$ *on* S *at* f.

The outline of the rest of this article is as follows. The next section contains some technical tools needed in the sequel. The third section presents the proofs of our main results.

2 Some lemmas

In what follows, we give collect some lemmas needed in the sequel.

Lemma 2.1 *Let* $f \in \mathfrak{X}$, $x \geq 0$, $y \geq 0$. *Then we have the following Schrödinger-type identities*:

(1)

$$\begin{aligned}
&\frac{1}{3}\int_\Gamma \left(a|\nabla f|^3 + bf^3|\nabla f|^3\right)\mathrm{d}t - \frac{1}{3}\int_\Gamma \left(ax^3|\nabla f_+|^3 + ay^3|\nabla f_-|^3 + bx^4 f_+^3|\nabla f_+|^3\right.\\
&\qquad \left.+ by^4 f_-^3|\nabla f_-|^3\right)\mathrm{d}t\\
&\quad = \frac{1}{8}a\left(1-y^3\right)^3\left(3+2y^3+y^4\right)\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{4}b\left(1-x^4\right)^3\int_\Gamma f_+^3|\nabla f_+|^3\,\mathrm{d}t + \frac{1}{4}b\left(1-y^4\right)^3\int_\Gamma f_-^3|\nabla f_-|^3\,\mathrm{d}t.
\end{aligned}$$

(2)

$$\begin{aligned}
&\frac{1}{4}\left(\int_\Gamma \left(c|\nabla f|^3 + df^3|\nabla f|^3\right)\mathrm{d}t\right)^3\\
&\qquad - \frac{1}{4}\left(\int_\Gamma \left(cx^3|\nabla f_+|^3 + cy^3|\nabla f_-|^3 + dx^4 f_+^3|\nabla f_+|^3 + dy^4 f_-^3|\nabla f_-|^3\right)\mathrm{d}t\right)^3\\
&\quad = \frac{1}{8}\left(1-x^8\right)\int_\Gamma \left(c|\nabla f|^3 + df^3|\nabla f|^3\right)\mathrm{d}t\int_\Gamma \left(c|\nabla f_+|^3 + 2df_+^3|\nabla f_+|^3\right)\mathrm{d}t\\
&\qquad + \frac{1}{8}\left(1-y^8\right)\int_\Gamma \left(c|\nabla f|^3 + df^3|\nabla f|^3\right)\mathrm{d}t\int_\Gamma \left(c|\nabla f_-|^3 + 2df_-^3|\nabla f_-|^3\right)\mathrm{d}t\\
&\qquad + \frac{1}{8}\mathfrak{c}^3\left(1-x^4\right)^3\left(\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t\right)^3 + \frac{1}{8}\mathfrak{c}^3\left(1-y^4\right)^3\left(\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\right)^3\\
&\qquad + \frac{1}{8}cd\left(1-x^3\right)^3\left(1+2x^3+3x^4\right)\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t\int_\Gamma f_+^3|\nabla f_+|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{8}cd\left(1-y^3\right)^3\left(1+2y^3+3y^4\right)\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\int_\Gamma f_-^3|\nabla f_-|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{8}\mathfrak{c}^3\left(\left(x^4-y^4\right)^3 + 2\left(1-x^3y^3\right)^3\right)\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{8}cd\left(\left(1-x^4\right)^3 + 2\left(x^3-y^4\right)^3\right)\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t\int_\Gamma f_-^3|\nabla f_-|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{8}cd\left(\left(1-y^4\right)^3 + 2\left(y^3-x^4\right)^3\right)\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\int_\Gamma f_+^3|\nabla f_+|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{4}d^3\left(x^4-y^4\right)^3\int_\Gamma f_+^3|\nabla f_+|^3\,\mathrm{d}t\int_\Gamma f_-^3|\nabla f_-|^3\,\mathrm{d}t.
\end{aligned}$$

The proof of the above lemma is similar to the one in [25], we just state it in brief and omit the proof.

Lemma 2.2 *Let $f \in \mathfrak{X}$, $x \geq 0$, $y \geq 0$. Then we have*

$$\begin{aligned}
&\Im(f) - \Im(xf_+ + yf_-) \\
&\quad \geq \frac{1}{8}\left(1 - x^8\right)\left\langle \mathfrak{E}\mathfrak{A}(f), f_+\right\rangle + \frac{1}{8}\left(1 - y^4\right)\left\langle \mathfrak{E}\mathfrak{A}(f), f_-\right\rangle \\
&\qquad + \frac{1}{4}\left(1 - x^3\right)^3\left(a\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t + b\int_\Gamma f_+^3|\nabla f_+|^3\,\mathrm{d}t + \frac{1}{3}\mathfrak{c}^3\left(\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t\right)^3\right) \\
&\qquad + \frac{1}{4}\left(1 - y^3\right)^3\left(a\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t + b\int_\Gamma f_-^3|\nabla f_-|^3\,\mathrm{d}t + \frac{1}{3}\mathfrak{c}^3\left(\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t\right)^3\right).
\end{aligned} \tag{9}$$

Moreover,

$$\Im(f) > \Im(xf_+ + yf_-), \tag{10}$$

where $f \in \mathfrak{S}^*$, $x \geq 0$, $y \geq 0$ and $(x, y) \neq (1, 1)$.

Proof It follows that

$$\begin{aligned}
&\Im(f) - \Im(xf_+ + yf_-) \\
&\quad = \frac{1}{3}\int_\Gamma \left(a|\nabla f|^3 + bf^3|\nabla f|^3\right)\mathrm{d}t \\
&\qquad - \frac{1}{3}\left(\int_\Gamma \left(ax^3|\nabla f_+|^3 + ay^3|\nabla f_-|^3 + bx^4f_+^3|\nabla f_+|^3 + by^4f_-^3|\nabla f_-|^3\right)\mathrm{d}t\right) \\
&\qquad + \frac{1}{4}\left(\int_\Gamma \left(c|\nabla f|^3 + df^3|\nabla f|^3\right)\mathrm{d}t\right)^3 \\
&\qquad - \frac{1}{4}\left(\int_\Gamma \left(cx^3|\nabla f_+|^3 + cy^3|\nabla f_-|^3 + dx^4f_+^3|\nabla f_+|^3 + dy^4f_-^3|\nabla f_-|^3\right)\mathrm{d}t\right)^3 \\
&\qquad - \int_\Gamma \left(\mathfrak{F}(f_+) - \mathfrak{F}(xf_+)\right)\mathrm{d}t - \int_\Gamma \left(\mathfrak{F}(f_-) - \mathfrak{F}(yf_-)\right)\mathrm{d}t
\end{aligned}$$

from Lemma 2.1.

It follows that

$$\begin{aligned}
\int_\Gamma \left(\mathfrak{F}(f_+) - \mathfrak{F}(xf_+)\right)\mathrm{d}t &= \int_\Gamma \mathrm{d}t\int_x^1 \frac{\mathrm{d}}{\mathrm{d}\varrho}\mathfrak{F}(\varrho f_+)\,\mathrm{d}\varrho \\
&= \int_\Gamma \mathrm{d}t\int_x^1 \sigma(\varrho f_+)f_+\,\mathrm{d}\varrho \\
&\geq \int_\Gamma \mathrm{d}t\int_x^1 \varrho^7\sigma(f_+)f_+\,\mathrm{d}t \\
&= \frac{1}{8}\left(1 - x^8\right)\int_\Gamma \sigma(f_+)f_+\,\mathrm{d}t
\end{aligned} \tag{11}$$

and

$$\int_{\Gamma} \bigl(\mathfrak{F}(f_-) - \mathfrak{F}(xf_-)\bigr)\,\mathrm{d}t \geq \frac{1}{8}\bigl(1 - y^8\bigr) \int_{\Gamma} \sigma(f_-)f_-\,\mathrm{d}t \tag{12}$$

from (A4).

Combining Lemma 2.1, (11), (12) and the definition (8) of $\zeta_+(f)$ and $\zeta_-(f)$, we see that (9) holds. □

Lemma 2.3 *Suppose that* (A1), (A2), (A3) *and* (A4) *hold. Then the infimum* f^* *is a ground state solution of problem* (1).

Proof We first prove that f^* is a solution of Eq. (5). It is clear that there exist $m > 0$ and $\phi \in C_0^\infty(\Gamma)$ satisfying

$$\bigl\langle \mathfrak{E}\mathfrak{A}\bigl(f^*\bigr), \phi \bigr\rangle = -2m < 0.$$

It follows from the continuity that there exist $\xi > 0$, $\varepsilon_0 > 0$ such that

$$\bigl\langle \mathfrak{E}\mathfrak{A}(xf_+ + yf_- + \varepsilon\phi, \phi \bigr\rangle \leq -m, \tag{13}$$

where $|x - 1| \leq \xi$, $|y - 1| \leq \xi$ and $0 \leq \varepsilon \leq \varepsilon_0$.

If $x \geq 1$ and $\varrho \neq 0$, then we have $\sigma(y\varrho)t\varrho \geq x\sigma(\varrho)\varrho$ from (A4).

If $1 - \xi \leq y \leq 1 + \xi$, then we have

$$\begin{aligned}
\zeta_+\bigl((1+\xi)f_+ + yf_-\bigr) &= \bigl\langle \mathfrak{E}\mathfrak{A}\bigl((1+\xi)f_+ + yf_-\bigr), (1+\xi)f_+ \bigr\rangle \\
&\leq \bigl\langle \mathfrak{E}\mathfrak{A}\bigl((1+\xi)f\bigr), (1+\xi)f_+ \bigr\rangle \\
&< (1+\xi)^8 \bigl\langle \mathfrak{E}\mathfrak{A}(f), f_+ \bigr\rangle = 0
\end{aligned}$$

and

$$\begin{aligned}
\zeta_+\bigl((1-\xi)f_+ + yf_-\bigr) &= \bigl\langle \mathfrak{E}\mathfrak{A}\bigl((1-\xi)f_+ + yf_-\bigr), (1-\xi)f_+ \bigr\rangle \\
&\geq \bigl\langle \mathfrak{E}\mathfrak{A}\bigl((1-\xi)f\bigr), (1-\xi)f_+ \bigr\rangle \\
&> (1-\xi)^8 \bigl\langle \mathfrak{E}\mathfrak{A}(f), f_+ \bigr\rangle = 0.
\end{aligned}$$

If $1 - \xi \leq x \leq 1 + \xi$, then we have

$$\zeta_-\bigl(xf_+ + (1+\xi)f_-\bigr) = \bigl\langle \mathfrak{E}\mathfrak{A}\bigl(xf_+ + (1+\xi)f_-\bigr), (1+\xi)f_- \bigr\rangle < 0$$

and

$$\zeta_+\bigl(xf_+ + (1-\xi)f_-\bigr) = \bigl\langle \mathfrak{E}\mathfrak{A}\bigl(xf_+ + (1-\xi)f_-\bigr), (1+\xi)f_- \bigr\rangle > 0.$$

Take ε sufficiently small such that

$$\zeta_+\bigl((1+\xi)f_+ + yf_- + \varepsilon\phi\bigr) < 0, \qquad \zeta_+\bigl((1-\xi)f_+ + yf_- + \varepsilon\phi\bigr) > 0,$$

for $1-\xi \le y \le 1+\xi$; and

$$\zeta_-\big(xf_+ + (1+\xi)f_- + \varepsilon\phi\big) < 0, \qquad \zeta_-\big(xf_+ + (1-\xi)f_- + \varepsilon\phi\big) > 0$$

for $1-\xi \le x \le 1+\xi$.

It follows from a degree theory argument that there exists (x,y) such that $|x-1| \le \xi$, $|y-1| \le \xi$ and

$$\zeta_+(xf_+ + yf_- + \varepsilon\phi) = 0, \qquad \zeta_-(xf_+ + yf_- + \varepsilon\phi) = 0,$$

which together with Lemma 2.2 and (13) yields

$$\begin{aligned}
\mathfrak{c}^* &\le \Im(xf_+ + yf_- + \varepsilon\phi)\\
&\le \Im\big(f^*\big) + \Im(xf_+ + yf_- + \varepsilon\phi) - \Im(xf_+ + yf_-)\\
&= \mathfrak{c}^* + \int_0^1 \big\langle \mathfrak{E}\mathfrak{A}(xf_+ + yf_- + \varrho\varepsilon\phi), \varepsilon\phi\big\rangle \,\mathrm{d}\varrho\\
&\le \mathfrak{c}^* - \varepsilon m.
\end{aligned}$$

It is clear that this is a contradiction. □

Lemma 2.4 *Let $f \in \mathfrak{X}$ and $x \ge 0$. Then the following Schrödinger-type identities hold*:

(1)

$$\begin{aligned}
&\frac{1}{3}\int_\Gamma \big(a|\nabla f|^3 + bf^3|\nabla f|^3\big)\,\mathrm{d}t - \frac{1}{3}\int_\Gamma \big(ax^3|\nabla f|^3 + bx^4 f^3|\nabla f|^3\big)\,\mathrm{d}t\\
&\quad = \frac{1}{8}a\big(1-x^8\big)\int_\Gamma \big(a|\nabla f|^3 + 2b|\nabla f|^3\big)\,\mathrm{d}t\\
&\qquad + \frac{1}{8}a\big(1-x^3\big)^3\big(3+2x^3+x^4\big)\int_\Gamma |\nabla f|^3\,\mathrm{d}t + \frac{1}{4}b\big(1-x^4\big)^3\int_\Gamma f^3|\nabla f|^3\,\mathrm{d}t,
\end{aligned}$$

(2)

$$\begin{aligned}
&\left(\frac{1}{4}\int_\Gamma \big(c|\nabla f|^3 + df^3|\nabla f|^3\big)\,\mathrm{d}t\right)^3 - \frac{1}{4}\left(\int_\Gamma \big(cx^3|\nabla f|^3 + dx^4 f^3|\nabla f|^3\big)\,\mathrm{d}t\right)^3\\
&\quad = \frac{1}{8}\big(1-x^8\big)\int_\Gamma \big(c|\nabla f|^3 + df^3|\nabla f|^3\big)\,\mathrm{d}t\int_\Gamma \big(c|\nabla f|^3 + 2df^3|\nabla f|^3\big)\,\mathrm{d}t\\
&\qquad + \frac{1}{8}c^3\big(1-x^4\big)^3\int_\Gamma |\nabla f|^3\,\mathrm{d}t\\
&\qquad + \frac{1}{8}cd\big(1-x^3\big)\big(1+2x^3+3x^4\big)\int_\Gamma |\nabla f|^3\,\mathrm{d}t\int_\Gamma f^3|\nabla f|^3\,\mathrm{d}t.
\end{aligned}$$

The proof of the above lemma is similar to the one in [25], we just state it in brief and omit the proof.

Lemma 2.5 *Suppose that* (A1), (A2), (A3) *and* (A4) *hold. Then f is a ground state solution of* (1).

Proof We shall first that f is a solution of (5). Otherwise, there exist $m > 0$ and $\phi \in C_0^\infty(\Gamma)$ such that

$$\langle \mathfrak{E}\mathfrak{A}(f), \phi \rangle = -2m < 0.$$

There exist $\xi > 0$ and $\varepsilon_0 > 0$ such that

$$\langle \mathfrak{E}\mathfrak{A}(xf + \varepsilon\phi), \phi \rangle \leq -m, \tag{14}$$

where $|x - 1| \leq \xi$ and $0 \leq \varepsilon \leq \varepsilon_0$.

So

$$\zeta\big((1+\xi)f\big) < 0$$

and

$$\zeta\big((1-\xi)f\big) > 0.$$

There exists a sufficiently small number ε such that

$$\zeta\big((1+\xi)f + \varepsilon\phi\big) < 0, \qquad \zeta\big((1-\xi)f + \varepsilon\phi\big) > 0.$$

It follows that

$$\begin{aligned}
\mathfrak{c}_0 &\leq \mathfrak{I}(xf + \varepsilon\phi) \\
&\leq \mathfrak{I}(f) + \mathfrak{I}(xf + \varepsilon\phi) - \mathfrak{I}(xf) \\
&= \mathfrak{c}_0 + \int_0^1 \langle \mathfrak{E}\mathfrak{A}(xf + \varrho\varepsilon\phi), \varepsilon\phi \rangle \, \mathrm{d}\varrho \\
&\leq \mathfrak{c}_0 - m\varepsilon
\end{aligned}$$

from (14), which is a contradiction. □

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 We only need to prove that f^* has exactly two nodal domains. We prove it reasoning by contradiction.

There exists $\varpi > 0$ such that

$$\int_\Gamma f_+^p \, \mathrm{d}t \geq \varpi \quad \text{and} \quad \int_\Gamma f_-^p \, \mathrm{d}t \geq \varpi, \tag{15}$$

where $f \in \mathfrak{S}^*$.

It follows from (A1) and (A2) that

$$\begin{aligned}
\varepsilon \int_\Gamma f_+^3 \, \mathrm{d}t + C_\varepsilon \int_\Gamma f_+^p \, \mathrm{d}t \geq \int_\Gamma \sigma(f_+) f_+ \, \mathrm{d}t \geq \int_\Gamma a|\nabla f_+|^3 \, \mathrm{d}t + 2b \int_\Gamma f_+^3 |\nabla f_+|^3 \, \mathrm{d}t \\
\geq 2\varepsilon \int_\Gamma f_+^3 \, \mathrm{d}t + c\left(\int_\Gamma f_+^p \, \mathrm{d}t\right)^{4/p}.
\end{aligned}$$

So

$$\begin{aligned}
\mathfrak{I}(f) &= \mathfrak{I}(f) - \frac{1}{8}\langle \mathfrak{E}\mathfrak{A}(f), f\rangle \\
&= \frac{3}{8}a\int_\Gamma |\nabla f|^3\,\mathrm{d}t + \frac{1}{4}b\int_\Gamma f^3|\nabla f|^3\,\mathrm{d}t \\
&\quad + \frac{1}{8}\int_\Gamma \bigl(c|\nabla f|^3 + df^3|\nabla f|^3\bigr)\,\mathrm{d}t \int_\Gamma c|\nabla f|^3\,\mathrm{d}t + \int_\Gamma \left(\frac{1}{8}\sigma(f)f - \mathfrak{F}(f)\right)\mathrm{d}t \\
&\geq \frac{3}{8}a\int_\Gamma |\nabla f|^3\,\mathrm{d}t + \frac{1}{4}b\int_\Gamma f^3|\nabla f|^3\,\mathrm{d}t. \qquad (16)
\end{aligned}$$

It follows from (16) that

$$\begin{aligned}
&\int_\Gamma |\nabla f_n|^3\,\mathrm{d}t \leq c, \\
&\int_\Gamma f_n^3|\nabla f_n|^3\,\mathrm{d}t \leq c.
\end{aligned}$$

It is obvious that $f_n \rightharpoonup f$ in $\mathfrak{H}_0^1(\Gamma)$, $f_n\nabla f_n \rightharpoonup u\nabla f$ in $L^3(\Gamma)$, $f_n \to f$ in $L^q(\Gamma)$, where $1 \leq q \leq 12$.

It follows from (15) that

$$\int_\Gamma f_+^p\,\mathrm{d}t = \lim_{n\to\infty}\int_\Gamma (f_n)_+^p\,\mathrm{d}t \geq \varpi > 0$$

and

$$\int_\Gamma f_-^p\,\mathrm{d}t \geq \varpi > 0, \quad f_+ \neq 0, f_- \neq 0.$$

It follows from Lemma 2.1 that there exists $(x,y) \in \mathbb{R}_+^3$ such that $xf_+ + yf_- \in \mathfrak{S}^*$. So

$$\begin{aligned}
\mathfrak{c}^* &= \lim_{n\to\infty}\mathfrak{I}(f_n) \\
&\geq \underline{\lim}_{n\to\infty}\left\{\mathfrak{I}\bigl(x(f_n)_+ + t(f_n)_-\bigr) + \frac{1}{4}a\bigl(1-x^3\bigr)^3\int_\Gamma \bigl|\nabla(f_n)_+\bigr|^3\,\mathrm{d}t\right. \\
&\quad \left. + \frac{1}{4}a\bigl(1-y^3\bigr)^3\int_\Gamma \bigl|\nabla(f_n)_-\bigr|^3\,\mathrm{d}t\right\} \\
&\geq \mathfrak{I}(xf_+ + yf_-) + \frac{1}{4}a\bigl(1-x^3\bigr)^3\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t + \frac{1}{4}a\bigl(1-y^3\bigr)^3\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t \\
&\geq \mathfrak{c}^* + \frac{1}{4}a\bigl(1-x^3\bigr)^3\int_\Gamma |\nabla f_+|^3\,\mathrm{d}t + \frac{1}{4}a\bigl(1-y^3\bigr)^3\int_\Gamma |\nabla f_-|^3\,\mathrm{d}t
\end{aligned}$$

from Lemma 2.2, (9) and the lower semicontinuity.

So $x = 1$, $y = 1$, $f_+ + f_- = f^* \in \mathfrak{S}^*$, and $\mathfrak{I}(f^*) = \mathfrak{c}^*$.

Put $v_+ = f^*\chi_{\mathfrak{D}_1}$, $v_- = f^*\chi_{\mathfrak{D}_3}$, $v = v_1 + v_2$, $w = f^*\chi_{\mathfrak{D}_2}$, $v + w = f^*$, where χ_D denotes the eigenfunction of D, $\mathfrak{D}_1, \mathfrak{D}_2$ are positive nodal domains, and $\mathfrak{D}_3$ is a negative nodal domain.

Then we have

$$
\begin{aligned}
\mathfrak{c}^* &= \mathfrak{I}(f^*) = \mathfrak{I}(v+w) - \frac{1}{8}\langle \mathfrak{E}\mathfrak{A}(v+w), v+w\rangle \\
&= \left\{\mathfrak{I}(v) + \mathfrak{I}(w) + \frac{1}{3}\int_\Gamma \left(c|\nabla v|^3 + dv^3|\nabla v|^3\right)\mathrm{d}t \int_\Gamma \left(c|\nabla w|^3 + dw^3|\nabla w|^3\right)\mathrm{d}t\right\} \\
&\quad - \frac{1}{8}\left\{\langle \mathfrak{E}\mathfrak{A}(v), v\rangle + \langle \mathfrak{E}\mathfrak{A}(w), w\rangle\right. \\
&\quad + \int_\Gamma \left(c|\nabla v|^3 + dv^3|\nabla v|^3\right)\mathrm{d}t \int_\Gamma \left(c|\nabla w|^3 + 2dw^3|\nabla w|^3\right)\mathrm{d}t \\
&\quad \left. + \int_\Gamma \left(c|\nabla w|^3 + dv^3|\nabla w|^3\right)\mathrm{d}t \int_\Gamma \left(c|\nabla v|^3 + 2dv^3|\nabla v|^3\right)\mathrm{d}t\right\} \\
&> \mathfrak{I}(v) - \frac{1}{8}\langle \mathfrak{E}\mathfrak{A}(v), v\rangle.
\end{aligned}
$$

Note that

$$
0 = \langle \mathfrak{E}\mathfrak{A}(f^*), v_+\rangle \geq \langle \mathfrak{E}\mathfrak{A}(v), v_+\rangle
$$

and

$$
0 = \langle \mathfrak{E}\mathfrak{A}(f^*), v_-\rangle \geq \langle \mathfrak{E}\mathfrak{A}(v), v_-\rangle.
$$

There exist two positive numbers x and y such that $xv_+ + yv_- \in \mathfrak{S}^*$. It follows from Lemma 2.2 that

$$
\begin{aligned}
\mathfrak{c}^* &> \mathfrak{I}(v) - \frac{1}{8}\langle \mathfrak{E}\mathfrak{A}(v), v\rangle \\
&\geq \mathfrak{I}(xv_+ + yv_-) + \frac{1}{8}\left(1-x^8\right)\langle \mathfrak{E}\mathfrak{A}(v), v_+\rangle + \frac{1}{8}\left(1-y^8\right)\langle \mathfrak{E}\mathfrak{A}(v), v_+\rangle - \frac{1}{8}\langle \mathfrak{E}\mathfrak{A}(v), v\rangle \\
&= \mathfrak{I}(xv_+ + yv_-) - \frac{1}{8}x^8\langle \mathfrak{E}\mathfrak{A}(v), v_+\rangle - \frac{1}{8}y^8\langle \mathfrak{E}\mathfrak{A}(v), v_-\rangle \\
&\geq \mathfrak{I}(xv_+ + yv_-) \geq \mathfrak{c}^*,
\end{aligned}
$$

which is also a contradiction. □

Proof of Theorem 1.2 We shall prove $\mathfrak{c}^* > 2\mathfrak{c}_0$. Let $f^* = f_+ + f_- \in \mathfrak{S}^*$ be a minimizer, $\mathfrak{I}(f^*) = \mathfrak{c}^*$.

Let $\{f_n\} \subset S$ be a minimizing sequence, $\mathfrak{I}(f_n) \to \mathfrak{c}_0$ as $n \to \infty$. It follows from (16) that

$$
\int_\Gamma |\nabla f_n|^3 \mathrm{d}t \leq c, \qquad \int_\Gamma f_n^3|\nabla f_n|^3 \mathrm{d}t \leq c.
$$

Assume $f_n \rightharpoonup f$ in $\mathfrak{H}_0^1(\Gamma)$, $f_n\nabla f_n \rightharpoonup u\nabla f$ in $L^3(\Gamma)$, $f_n \to f$ in $L^q(\Gamma)$, where $1 \leq q < 12$. There exists $\varpi > 0$ such that $\int_\Gamma |f_n|^p \,\mathrm{d}t \geq \varpi > 0$, which yields

$$
\int_\Gamma |f|^p \,\mathrm{d}t = \lim_{n\to\infty}\int_\Gamma |f_n|^p \,\mathrm{d}t \geq \varpi > 0, \quad f \not\equiv 0.
$$

It follows Lemma 2.3 that there exists a positive number x such that $xf \in S$. By Lemma 2.4 we have

$$\begin{aligned}
\mathfrak{c}_0 &= \lim_{n\to\infty} \Im(f_n) \\
&\geq \lim_{n\to\infty} \left\{ \Im(xf_n) + \frac{1}{4}\left(1-x^3\right)a \int_\Gamma |\nabla f_n|^3 \,\mathrm{d}t \right\} \\
&\geq \Im(xf) + \frac{1}{4}\left(1-x^3\right)^3 a \int_\Gamma |\nabla f|^3 \,\mathrm{d}t \\
&\geq \mathfrak{c}_0 + \frac{1}{4}\left(1-x^3\right)a \int_\Gamma |\nabla f|^3 \,\mathrm{d}t,
\end{aligned}$$

which yields $x = 1$, $f \in S$, $\Im(f) = \mathfrak{c}_0$ and f is a minimizer.

There exist two positive numbers x and y such that $xf_+ \in S$, $yf_- \in S$. So

$$\begin{aligned}
\mathfrak{c}^* &= \Im\left(f^*\right) = \Im(f_+ + f_-) \\
&\geq \Im(xf_+ + yf_-) \\
&= \Im(xf_+) + \Im(yf_-) \\
&\quad + \frac{1}{3}\int_\Gamma \left(cx^3|\nabla f_+|^3 + dx^4 f_+^3 |\nabla f_+|^3\right) \mathrm{d}t \int_\Gamma \left(cy^3|\nabla f_-|^3 + dy^4 f_-^3 |\nabla f_-|^3\right) \mathrm{d}t \\
&> \Im(xf_+) + \Im(yf_-) \geq 2\mathfrak{c}_0.
\end{aligned}$$

Finally we shall prove that $f \in S$ is signed. Otherwise, $f = f_+ + f_-$, $f_+ \neq 0$, $f_- \neq 0$. Since f is a solution of (1), $\langle \mathfrak{E}\mathfrak{A}(f), f_+\rangle = 0$, $\langle \mathfrak{E}\mathfrak{A}(f), f_-\rangle = 0$; that is, $f \in \mathfrak{S}^*$. So

$$\mathfrak{c}_0 = \Im(f) \geq \mathfrak{c}^* > 2\mathfrak{c}_0,$$

which is a contradiction, since we have $\mathfrak{c}_0 > 0$. □

4 Conclusions

In this paper, we developed optimal Phragmén–Lindelöf methods, based on the use of maximum modulus of optimal value of a parameter in a Schrödinger functional, by applying the Phragmén–Lindelöf theorem for a second-order boundary value problems with respect to the Schrödinger operator

Acknowledgements
The authors appreciate the insightful comments and wise advice of the editor and the referees.

Funding
This work was supported by the National Natural Science Foundation of China (Grant No. 71901041).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors completed the paper, read and approved the final manuscript.

Author details
[1]School of Fiance and Ecnomics, Yangtze Normal University, Chongqing, P.R. China. [2]School of Mathematics, Sichuan University of Arts and Science, Dazhou, P.R. China.

Publisher's Note

Received: 1 April 2020 Accepted: 29 April 2020 Published online: 13 May 2020

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