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Positive periodic solution for prescribed mean curvature generalized Liénard equation with a singularity

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Abstract
The main purpose of this paper is to investigate the existence of a positive periodic solution for a prescribed mean curvature generalized Liénard equation with a singularity (weak and strong singularities of attractive type, or weak and strong singularities of repulsive type). Our proof is based on an extension of Mawhin's continuation theorem.

MSC: 34B16; 34B18; 34C25

Keywords: Positive periodic solution; Prescribed mean curvature; Weak and strong; Attractive and repulsive; Liénard equation

1 Introduction

In this paper, we consider the following p-Laplacian prescribed mean curvature Liénard equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + f\big(t,u(t)\big)u'(t) + g\big(u(t)\big) = e(t), \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, p is a positive constant, and $p > 1$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function and $f(t+T,\cdot) \equiv f(t,\cdot)$, $g : (0,+\infty) \to \mathbb{R}$ is the continuous function and has a singularity at the origin, $e \in L^\sigma(\mathbb{R})$ is T-periodic function and $1 \le \sigma < \infty$, T is a positive constant.

During the past 30 years, the problem of existence of positive periodic solutions to Liénard equations with singularity was extensively studied by many researchers [1–9]. In [9], Zhang discussed the existence of a positive periodic solution to equation (1.1), where $\frac{u'(t)}{\sqrt{1+(u'(t))^2}} = u'(t)$, $p = 2$, $f(t,u) = f(u)$ and $e(t) \equiv 0$, g satisfies a semilinear condition and has a strong singularity of repulsive type, i.e.,

$$\lim_{u\to 0^+} g(u) = -\infty \quad \text{and} \quad \lim_{u\to 0^+} \int_u^1 g(v)\,dv = +\infty. \tag{1.2}$$

After that, Yu and Lu [7] improved the results of [9], showing in their Theorem 2.1 (see [7]) that g may possess weak and strong singularities. Zhang and Yu's proof was based on coincidence degree theory.

Compared with Liénard equations, only a few works focus on prescribed mean curvature Liénard equations, especially p-Laplacian prescribed mean curvature Liénard equations. As far as we know, prescribed mean curvature $\frac{u'(t)}{\sqrt{1+(u'(t))^2}}$ of $u(t)$ appears in different geometry and physics problems [10–16]. Using coincidence degree theory, Feng [17] and Lu [18] et al. investigated respectively the existence of a positive periodic solution for equation (1.1) without singularity and with a strong singularity of repulsive type, where $p=2$, $f(t,u)=f(u)$, and g satisfying a semilinear condition.

Inspired by [7, 9, 17, 18], in this paper, we further consider the existence of a positive periodic solution for equation (1.1) by means of an extension of Mawhin's continuation theorem due to Ge and Ren [19]. It is worth mentioning that conditions on f, g and the work for estimating *a priori bounds* of positive periodic solutions for equation (1.1) are more complex than in [7, 9, 17, 18]. Firstly, the friction term $f(u(t))u'(t)$ in [7, 9, 17, 18] satisfies $\int_0^T f(u(t))u'(t)\,dt=0$, which is crucial to estimating *a priori bounds* of positive periodic solutions for these equations. However, the friction term of this paper, $f(t,u(t))u'(t)$, may not satisfy $\int_0^T f(t,u(t))u'(t)\,dt=0$. Secondly, g of this paper possesses weak and strong singularities of attractive type (or weak and strong singularities of repulsive type) at the origin. Thirdly, g of this paper may satisfy sublinearity, semilinearity, or superlinearity conditions at infinity. Therefore, we extend and improve the results in [7, 9, 17, 18].

2 Positive periodic solution for equation (1.1) when $p>1$

In this section, we study the existence of a positive periodic solution to equation (1.1). Since $(\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}))'$ is a nonlinear term, coincidence degree theory does not apply directly. The traditional study method is to translate equation (1.1) into the following two-dimensional system:

$$\begin{cases} u_1'(t)=\frac{\phi_q(u_2(t))}{\sqrt{1-\phi_q^2(u_2(t))}}, \\ u_2'(t)=-f(t,u_1(t))u_1'(t)-g(u_1(t))+e(t), \end{cases}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for which coincidence degree theory can be applied. However, from the first equation of the above system it is obvious that $\|u_2\|<1$, where $\|u_2\|:=\max_{t\in\mathbb{R}}|u'(t)|$. Therefore, estimating an upper bound of $u_2(t)$ is very complicated; in order to get around this difficulty, we find other methods to study equation (1.1). We first investigate the following second-order prescribed mean curvature equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)'=\tilde{f}\big(t,u(t),u'(t)\big), \tag{2.1}$$

where $\tilde{f}:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Applying the extension of Mawhin's continuous theorem due to Ge and Ren [19, Theorem 2.1], we get the following conclusion.

Lemma 2.1 *Assume Ω is an open bounded set in $C_T^1:=\{u\in C^1(\mathbb{R},\mathbb{R}):u(t+T)\equiv u(t)$ and $u'(t+T)\equiv u'(t),\forall t\in\mathbb{R}\}$. Suppose the following conditions hold*:

(i) *For each* $\lambda \in (0,1)$, *the equation*

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' = \lambda \tilde{f}\big(t, u(t), u'(t)\big)$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(t, a, 0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then equation (2.1) *has at least one* T*-periodic solution on* $\bar{\Omega}$.

Proof First, operators M and N_λ are defined by

$$M : \operatorname{dom} M \cap X \to Z, \quad (Mu)(t) = \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)', \quad t \in \mathbb{R},$$

$$N_\lambda : X \to Z, (N_\lambda u)(t) = \lambda \tilde{f}\big(t, u(t), u'(t)\big).$$

Obviously, equation (2.1) can be converted into

$$Mu = N_\lambda u, \quad \lambda \in (0,1).$$

By [20, Lemmas 3.1 and 3.2], we know that M is a quasilinear operator, N_λ is M-compact. From assumption (i), one finds

$$Mu \neq N_\lambda u, \lambda \in (0,1) \quad \text{and} \quad u \in \partial\Omega,$$

and assumptions (ii) and (iii) imply that $\deg\{JQN, \Omega \cap \ker M, \theta\}$ is valid and

$$\deg\{JQN, \Omega \cap \ker M, \theta\} \neq 0.$$

Therefore, applying the extension of Mawhin's continuous theorem, equation (2.1) has at least one T-periodic solution. □

In the following, applying Lemma 2.1, we prove the existence of a positive periodic solution for equation (1.1) with a singularity of repulsive type.

Theorem 2.1 *Assume that equation* (1.2) *holds. Furthermore, suppose the following conditions hold*:

(H_1) *There exists a positive constant* γ *such that* $\inf_{(t,u)\in[0,T]\times\mathbb{R}} |f(t,u)| \geq \gamma > 0$.

(H_2) *There exist two positive constants* d_1, d_2 *with* $d_1 < d_2$ *such that* $g(u) - e(t) < 0$ *for* $(t,u) \in [0,T] \times (0, d_1)$ *and* $g(u) - e(t) > 0$ *for* $(t,u) \in [0,T] \times (d_2, +\infty)$.

Then equation (1.1) *has at least one positive periodic solution*.

Proof We embed equation (1.1) into the following family of equations:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \lambda f\big(t,u(t)\big)u'(t) + \lambda g\big(u(t)\big) = \lambda e(t), \tag{2.2}$$

where $\lambda \in (0,1]$. Firstly, we claim that there exist two points $\tau, \xi \in (0,T)$ such that

$$u(\tau) \geq d_1 \quad \text{and} \quad u(\xi) \leq d_2. \tag{2.3}$$

In fact, since $\int_0^T u'(t)\,dt = 0$, it is easy to verify that there exist two point $t_1, t_2 \in (0,T)$ such that

$$u'(t_1) \leq 0 \quad \text{and} \quad u'(t_2) \geq 0.$$

Therefore, we get

$$\phi_p\left(\frac{u'(t_1)}{\sqrt{1+(u'(t_1))^2}}\right) \leq 0 \quad \text{and} \quad \phi_p\left(\frac{u'(t_2)}{\sqrt{1+(u'(t_2))^2}}\right) \geq 0.$$

Letting $\bar{t}, \underline{t} \in (0,T)$ be maximum and minimum points of the prescribed mean curvature term $\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})$, the above inequalities imply

$$\phi_p\left(\frac{u'(\bar{t})}{\sqrt{1+(u'(\bar{t}))^2}}\right) \geq 0 \quad \text{and} \quad \left(\phi_p\left(\frac{u'(\bar{t})}{\sqrt{1+(u'(\bar{t}))^2}}\right)\right)' = 0; \tag{2.4}$$

and

$$\phi_p\left(\frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}}\right) \leq 0 \quad \text{and} \quad \left(\phi_p\left(\frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}}\right)\right)' = 0. \tag{2.5}$$

Applying equations (2.5) into (2.2), we deduce

$$g\big(u(\underline{t})\big) - e(\underline{t}) = -f\big(\underline{t}, u(\underline{t})\big)u'(\underline{t}). \tag{2.6}$$

By condition (H_1), we know that f may not change sign (i.e., $f(t,u) > 0$ or $f(t,u) < 0$, for $(t,u) \in [0,T] \times \mathbb{R}$). Without loss of generality, suppose $f(t,u) > 0$, for $(t,u) \in [0,T] \times \mathbb{R}$. Besides, since

$$\phi_p\left(\frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}}\right) = \left|\frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}}\right|^{p-2} \frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}} \leq 0,$$

then it is clear that $u'(\underline{t}) \leq 0$. By condition (H_2) and since $g(u(\underline{t})) - e(\underline{t}) \leq 0$, we get that $u(\underline{t}) \geq d_1$.

Similarly, by condition (H_2) and equation (2.4), we obtain that $u(\bar{t}) \leq d_2$. Taking $\tau = \underline{t}$ and $\xi = \bar{t}$, (2.3) is proved.

Multiplying both sides of equation (2.2) by $u'(t)$ and integrating from 0 to T, we have

$$\begin{aligned}
&\int_0^T \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' u'(t)\,dt + \lambda\int_0^T f\bigl(t,u(t)\bigr)\bigl|u'(t)\bigr|^2\,dt + \lambda\int_0^T g\bigl(u(t)\bigr)u'(t)\,dt \\
&\quad = \lambda\int_0^T e(t)u'(t)\,dt.
\end{aligned} \tag{2.7}$$

Substituting

$$\begin{aligned}
&\int_0^T \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' u'(t)\,dt \\
&\quad = \phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)u'(t)\bigg|_0^T - \int_0^T \phi_p\left(\frac{u'}{\sqrt{1+(u')^2}}\right)du' = 0
\end{aligned}$$

and $\int_0^T g(u(t))u'(t)\,dt = 0$ into (2.7), it is clear that

$$\left|\int_0^T f(t,u)\bigl|u'(t)\bigr|^2\,dt\right| = \left|\int_0^T e(t)u'(t)\,dt\right|.$$

By condition (H_1) and Hölder inequality, the above equality implies

$$\begin{aligned}
\gamma\int_0^T \bigl|u'(t)\bigr|^2\,dt &\le \left|\int_0^T f\bigl(t,u(t)\bigr)\bigl|u'(t)\bigr|^2\,dt\right| \\
&\le \int_0^T \bigl|e(t)\bigr|\bigl|u'(t)\bigr|\,dt \\
&\le \left(\int_0^T \bigl|e(t)\bigr|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \bigl|u'(t)\bigr|^2\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\int_0^T |u'(t)|^2\,dt \neq 0$ and $\gamma > 0$, we arrive at

$$\left(\int_0^T \bigl|u'(t)\bigr|^2\,dt\right)^{\frac{1}{2}} \le \frac{\|e\|_2}{\gamma}, \tag{2.8}$$

where $\|e\|_2 := (\int_0^T |e(t)|^2\,dt)^{\frac{1}{2}}$. From equations (2.3) and (2.8), using Hölder inequality, we get

$$\begin{aligned}
u(t) &\le d_2 + \int_0^T \bigl|u'(t)\bigr|\,dt \\
&\le d_2 + T^{\frac{1}{2}}\left(\int_0^T \bigl|u'(t)\bigr|^2\,dt\right)^{\frac{1}{2}} \\
&\le d_2 + \frac{T^{\frac{1}{2}}\|e\|_2}{\gamma} := M_1.
\end{aligned} \tag{2.9}$$

From equation (2.8) and using Hölder inequality, we deduce

$$\|u'\| = \frac{1}{T}\int_0^T \|u'\|\,dt \le T^{-\frac{1}{2}}\left(\int_0^T \|u'\|^2\,dt\right)^{\frac{1}{2}} \le T^{-\frac{1}{2}}\frac{\|e\|_2}{\gamma} := M_2. \tag{2.10}$$

On the other hand, let $\tau \in (0,T)$ be as in equation (2.3). Multiplying both sides of equation (2.2) by $u'(t)$ and integrating over the interval $[\tau,t]$, where $t \in [\tau,T]$, we see that

$$\begin{aligned}\lambda \int_{u(\tau)}^{u(t)} g(u)\,du &= \lambda \int_{\tau}^{t} g\bigl(u(s)\bigr)u'(s)\,ds \\ &= -\int_{\tau}^{t} \biggl(\phi_p\biggl(\frac{u'(s)}{\sqrt{1+(u'(s))^2}}\biggr)\biggr)' u'(s)\,ds - \lambda \int_{\tau}^{t} f\bigl(s,u(s)\bigr)\bigl|u'(s)\bigr|^2\,ds \\ &\quad + \lambda \int_{\tau}^{t} e(s)u'(s)\,ds.\end{aligned}$$

Furthermore, from equations (2.2), (2.9) and (2.10), applying Hölder inequality, the above equation implies

$$\begin{aligned}\lambda \biggl|\int_{u(\tau)}^{u(t)} g(u)\,du\biggr| &\le \int_0^T \biggl|\biggl(\phi_p\biggl(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\biggr)\biggr)'\biggr|\bigl|u'(t)\bigr|\,dt \\ &\quad + \lambda \int_0^T \bigl|f\bigl(t,u(t)\bigr)\bigr|\bigl|u'(t)\bigr|^2\,dt + \lambda \int_0^T \bigl|e(t)\bigr|\bigl|u'(t)\bigr|\,dt \\ &\le \lambda M_2\biggl(\int_0^T \bigl|f\bigl(t,u(t)\bigr)\bigr|\bigl|u'(t)\bigr|\,dt + \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt\biggr) \\ &\quad + \lambda M_2^2 \int_0^T \bigl|f\bigl(t,u(t)\bigr)\bigr|\,dt + \lambda M_1 T^{\frac{1}{2}}\|e\|_2 \\ &\le 2\lambda M_2\bigl(M_2 T\|f_{M_1}\| + T^{\frac{1}{2}}\|e\|_2\bigr) + \lambda M_2 \int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt, \end{aligned} \tag{2.11}$$

where $\|f_{M_1}\| := \max_{(t,u)\in[0,T]\times(0,M_1]} |f(t,u)|$.

Next, we consider $\int_0^T |g(u(t))|\,dt$. Integrating equation (2.2) over the interval $[0,T]$, we obtain

$$\int_0^T \bigl(f\bigl(t,u(t)\bigr)u'(t) + g\bigl(u(t)\bigr) - e(t)\bigr)\,dt = 0. \tag{2.12}$$

From equation (2.12), we see that

$$\begin{aligned}\int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &= \int_{g(u(t))\ge 0} g\bigl(u(t)\bigr)\,dt - \int_{g(u(t))\le 0} g\bigl(u(t)\bigr)\,dt \\ &= 2\int_{g(u(t))\ge 0} g^+\bigl(u(t)\bigr)\,dt + \int_0^T f\bigl(t,u(t)\bigr)u'(t)\,dt - \int_0^T e(t)\,dt \\ &\le 2\int_0^T g^+\bigl(u(t)\bigr)\,dt + \int_0^T \bigl|f\bigl(t,u(t)\bigr)\bigr|\bigl|u'(t)\bigr|\,dt + \int_0^T \bigl|e(t)\bigr|\,dt, \end{aligned} \tag{2.13}$$

where $g^+(u) := \max\{g(u),0\}$. Since $g^+(u(t)) \ge 0$, form conditions (H_2) and equation (1.2), we know that there exists a positive constant d_2^* with $d_2^* > d_1$ such that $u(t) \ge d_2^*$. Therefore, from equations (2.9) and (2.10), equation (2.13) implies

$$\begin{aligned}\int_0^T \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &\le 2T\bigl\|g_{M_1}^+\bigr\| + \int_0^T \bigl|f\bigl(t,u(t)\bigr)\bigr|\bigl|u'(t)\bigr| + \int_0^T \bigl|e(t)\bigr|\,dt \\ &\le 2T\bigl\|g_{M_1}^+\bigr\| + M_2 T\|f_{M_1}\| + T^{\frac{1}{2}}\|e\|_2, \end{aligned} \tag{2.14}$$

where $\|g^+_{M_1}\| := \max_{d_2^* \le u \le M_1} g^+(u)$. Applying equations (2.14) into (2.11), we have

$$\lambda\left|\int_{u(\tau)}^{u(t)} g(u)\,du\right| \le 3\lambda M_2\bigl(M_2T\|f_{M_1}\| + T^{\frac{1}{2}}\|e\|_2\bigr) + 2\lambda M_2 T\left\|g^+_{M_1}\right\|.$$

According to equation (1.2), we see that there exists a positive constant M_3' such that

$$u(t) \ge M_3', \quad \text{for } t \in [\tau, T]. \tag{2.15}$$

If $t \in [0,\tau]$, we can handle this case similarly.

From equations (2.9), (2.10), and (2.15), we obtain that a periodic solution u of equation (2.2) satisfies

$$M_3 < u(t) < M_1, \qquad \left\|u'\right\| < M_2,$$

where $M_3 := \min\{d_1, M_3'\}$. Then condition (1) of Lemma 2.1 is satisfied. For a possible solution C to equation

$$g(C) - \frac{1}{T}\int_0^T e(t)\,dt = 0,$$

we have $C \in [M_3, M_1]$. Therefore, condition (2) of Lemma 2.1 holds. Finally, by condition (H_2), we arrive at

$$g(M_3) - \frac{1}{T}\int_0^T e(t)\,dt < 0 \quad \text{and} \quad g(M_1) - \frac{1}{T}\int_0^T e(t)\,dt > 0.$$

So condition (3) of Lemma 2.1 is also satisfied. By Theorem 2.1, equation (1.1) has at least one positive periodic solution. □

In equation (1.2), the nonlinear term g requires a strong singularity of repulsive type (i.e., $\lim_{u\to 0^+}\int_u^1 g(\nu)\,d\nu = +\infty$). It is clear that the method of Theorem 2.1 is no longer applicable to estimate a lower bound on a periodic solution $u(t)$ of equation (1.1) in the case of a weak singularity of repulsive type (i.e., $\lim_{u\to 0^+}\int_u^1 g(\nu)\,d\nu < +\infty$). Therefore, we need to find another method to consider equation (1.1) in the case of a weak singularity of repulsive type.

Theorem 2.2 *Assume that conditions (H_1) and (H_2) hold. If $\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma} < d_1$, here d_1 is defined in Theorem 2.1, then equation (1.1) has at least one positive periodic solution.*

Proof We follow the same strategy and notation as in the proof of Theorem 2.1. Next, we consider the lower bound on a periodic solution $u(t)$ of equation (1.1). From equations (2.3) and (2.8), applying Hölder inequality, we get

$$\begin{aligned} u(t) &= \frac{1}{2}\bigl(u(t) + u(t-T)\bigr) \\ &= \frac{1}{2}\left(u(\tau) + \int_\tau^t u'(s)\,ds + u(\tau) - \int_{t-T}^\tau u'(s)\,ds\right) \end{aligned}$$

$$
\begin{aligned}
&\geq u(\tau)-\frac{1}{2}\left|\int_{\tau}^{t} u'(s)\,ds-\int_{t-T}^{\tau} u'(s)\,ds\right| \\
&\geq u(\tau)-\frac{1}{2}\left(\int_{\tau}^{t}\left|u'(s)\right|ds+\int_{t-T}^{\tau}\left|u'(s)\right|ds\right) \\
&= u(\tau)-\frac{1}{2}\int_{t-T}^{t}\left|u'(s)\right|ds \\
&\geq d_1-\frac{1}{2}\int_{0}^{T}\left|u'(s)\right|ds \\
&\geq d_1-\frac{1}{2}T^{\frac{1}{2}}\left(\int_{0}^{T}\left|u'(s)\right|ds\right)^{\frac{1}{2}} \\
&\geq d_1-\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma_2}:=M_3>0,
\end{aligned}
$$

since $\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma}<d_1$. The remaining part of the proof is the same as that of Theorems 2.1. □

Comparing Theorems 2.1 to 2.2, Theorem 2.2 is applicable to weak as well as strong singularities, whereas Theorem 2.1 is only applicable to a strong singularity. Besides, equation (1.2) is relatively weaker than condition $\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma}<d_1$. On the other hand, Theorems 2.1 and 2.2 require that g possesses a singularity of repulsive type (i.e., $\lim_{u\to 0^+} g(u)=-\infty$). In the following, we consider that g possesses a singularity of attractive type (i.e., $\lim_{u\to 0^+} g(u)=+\infty$). It is obvious that the attractivity condition and equation (1.2) with (H_2) contradict each other. Therefore, we have to find other conditions to consider equation (1.1) with a singularity of attractive type.

Theorem 2.3 *Assume that* (H_1) *holds. Furthermore, suppose the following conditions hold*:

(H_3) *There exist two positive constants* d_3, d_4 *with* $d_3<d_4$ *such that* $g(u)-e(t)>0$ *for* $(t,u)\in[0,T]\times(0,d_3)$ *and* $g(u)-e(t)<0$ *for* $(t,u)\in[0,T]\times(d_4,+\infty)$.

(H_4) (*Strong singularity of attractive type*)

$$
\lim_{u\to 0^+} g(u)=+\infty \quad \text{and} \quad \lim_{u\to 0^+}\int_{u}^{1} g(v)\,dv=-\infty.
$$

Then equation (1.1) *has at least one positive periodic solution.*

Proof We follow the same strategy and notation as in the proof of Theorem 2.1. Next, we consider $\int_0^T |g(u(t))|\,dt$. From equations (2.12) and (2.13), we see that

$$
\begin{aligned}
\int_{0}^{T}\left|g\big(u(t)\big)\right|dt &= \int_{g(u(t))\geq 0} g\big(u(t)\big)\,dt-\int_{g(u(t))\leq 0} g\big(u(t)\big)\,dt \\
&= -2\int_{g(u(t))\leq 0} g^{-}\big(u(t)\big)\,dt-\int_{0}^{T} f\big(t,u(t)\big)u'(t)\,dt+\int_{0}^{T} e(t)\,dt \\
&\leq 2\int_{0}^{T}\left|g^{-}\big(u(t)\big)\right|dt+\int_{0}^{T}\left|f\big(t,u(t)\big)\right|\left|u'(t)\right|dt+\int_{0}^{T}\left|e(t)\right|dt, \qquad (2.16)
\end{aligned}
$$

where $g^{-}(u):=\min\{g(u),0\}$. Since $g^{-}(u(t))\leq 0$, form conditions (H_3) and (H_4), we know that there exists a positive constant d_4^* with $d_4^*>d_3$ such that $u(t)\geq d_4^*$. Therefore, from equations (2.9) and (2.10), equation (2.16) implies

$$\int_0^T \left|g\bigl(u(t)\bigr)\right| dt \le 2T\left\|g_{M_1}^-\right\| + M_2T\|f_{M_1}\| + T^{\frac{1}{2}}\|e\|_2,$$

where $\|g_{M_1}^-\| := \max_{d_4^* \le u \le M_1} |g^-(u)|$. The remaining part of the proof is the same as that of Theorem 2.1. □

By Theorems 2.2 and 2.3, we obtain the following conclusion.

Theorem 2.4 *Assume that conditions (H_1) and (H_3) hold. If $\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma} < d_3$, then equation (1.1) has at least one positive periodic solution.*

Finally, we illustrate our results with two numerical examples.

Example 2.1 Consider the following prescribed mean curvature Liénard equation with a strong singularity of repulsive type

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + \bigl((\sin t + 3)u^4(t) + 1\bigr)u'(t) + \sum_{i=1}^{n} u^i(t) = \frac{6}{u^\mu(t)} + e^{\cos t}, \tag{2.17}$$

where μ is a positive constant and $\mu \ge 1$, n is a positive integer.

It is clear that $T = 2\pi$, $f(t,u) = (\sin t + 3)u^4 + 1$, $g(u) = \sum_{i=1}^n u^i - \frac{6}{u^\mu}$, $e(t) = e^{\cos t}$. We know that $|f(t,u)| = (\sin t + 3)u^4 + 1 \ge 1$. Take $\gamma = 1$, $d_1 = 0.01$, $d_2 = 3$. Then conditions (H_1) and (H_2) hold. Since $\lim_{u\to 0^+}\int_u^1 g(v)\,dv = \lim_{u\to 0^+}\int_u^1(\sum_{i=1}^n v^i - \frac{6}{v^\mu})\,dv = +\infty$, equation (1.2) is satisfied. Therefore, by Theorem 2.1, equation (2.17) has at least one positive 2π-periodic solution.

Example 2.2 Consider the following prescribed mean curvature Liénard equation with a weak singularity of attractive type:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' - \bigl((\cos 2t + 5)u^6(t) + 100\bigr)u'(t) - u^5(t) + \frac{4}{u^{\frac{1}{2}}(t)} = \sin 2t, \tag{2.18}$$

where $p > 1$.

It is obvious that $T = \pi$, $f(t,u) = -(\cos 2t + 5)u^6 - 100$, $g(u) = -u^5 + \frac{4}{u^{\frac{1}{2}}}$, $e(t) = \sin 2t$. Taking $\gamma = 100$, $d_3 = 0.09$, $d_4 = 4$, conditions (H_1) and (H_3) are satisfied. Furthermore, we consider

$$\frac{T^{\frac{1}{2}}\|e\|_2}{2\gamma} = \frac{\pi}{200} < 0.09.$$

Hence, applying Theorem 2.4, equation (2.18) has at least one positive π-periodic solution.

3 Positive periodic solution for equation (1.1) when $p > 1$ and $p \neq 2$

In the following, by Lemma 2.1 and Theorem 2.1, we prove the existence of a positive periodic solution for equation (1.1) with a singularity of repulsive type.

Theorem 3.1 *Assume that conditions* (H_1), (H_2) *and* $p \neq 2$ *hold. Then equation* (1.1) *has at least one positive periodic solution.*

Proof Let $t^*, t_* \in (0, T)$ be the maximum and minimum points of $u(t)$, and $u'(t^*) = u'(t_*) = 0$. Besides, we claim that there exists a positive constant ε such that

$$u'(t) \geq 0, \quad \text{for } t \in \left(t^* - \varepsilon, t^* + \varepsilon\right). \tag{3.1}$$

Assume, by way of contradiction, that inequality (3.1) does not hold. Then $u'(t) < 0$ for $t \in (t^* - \varepsilon, t^* + \varepsilon)$. Therefore, $u(t)$ is strictly decreasing for $t \in (t^* - \varepsilon, t^* + \varepsilon)$, this contradicts the definition of t^*. Hence, equation (3.1) is true. Since

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' = \left(\left|\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right|^{p-2}\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)'. \tag{3.2}$$

Applying equations (3.1) into (3.2), we get

$$\begin{aligned}\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' &= \left(\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)^{p-1}\right)' \\ &= (p-1)\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)^{p-2}\left(\frac{2u''(t)+u''(t)(u'(t))^2}{\sqrt{1+(u'(t))^2}}\right),\end{aligned} \tag{3.3}$$

for $t \in (t^* - \varepsilon, t^* + \varepsilon)$. From equation (3.3) and $p \neq 2$, we obtain

$$\left(\phi_p\left(\frac{u'(t^*)}{\sqrt{1+(u'(t^*))^2}}\right)\right)' = 0. \tag{3.4}$$

From equations (2.2) and (3.4), we have

$$g\left(t^*, u\left(t^*\right)\right) - e\left(t^*\right) = 0.$$

By condition (H_2), we get

$$d_1 \leq u\left(t^*\right) \leq d_2. \tag{3.5}$$

Similarly, by condition (H_2), we obtain

$$d_1 \leq u(t_*) \leq d_2. \tag{3.6}$$

Therefore, from equations (3.5) and (3.6), we see that

$$d_1 \leq u(t) \leq d_2, \quad \text{for } t \in \mathbb{R}. \tag{3.7}$$

By Theorem 2.1, we get that there exist a positive constant M_2^* such that

$$\|u'\| \leq M_2^*.$$

The remaining part of the proof is the same as that of Theorem 2.1. □

Comparing Theorems 2.1 and 3.1, Theorem 3.1 is applicable to weak and strong singularities. Theorem 2.1 is only applicable to a strong singularity. However, Theorem 3.1 does not cover the case of $p = 2$, while Theorem 2.1 covers the case of $p = 2$. Therefore, Theorem 2.1 can be more general. Besides, Theorem 3.1 requires that g possesses a singularity of repulsive type. In the following, we consider that g possesses a singularity of attractive type. It is obvious that attractivity condition and (H_2) contradict each other. By Theorems 2.3 and 3.1, we obtain the following conclusion.

Theorem 3.2 *Assume that conditions* (H_1), (H_3) *and* $p \neq 2$ *hold. Then equation* (1.1) *has at least one positive periodic solution.*

It is worth mentioning that the method of Theorem 3.1 is also applicable to the case where g is nonautonomous, i.e., $g(u(t)) = g(t, u(t))$. Then equation (1.1) is rewritten as the following form:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + f\bigl(t, u(t)\bigr)u'(t) + g\bigl(t, u(t)\bigr) = e(t). \tag{3.8}$$

Applying Lemma 2.1 and Theorem 3.1, we obtain the following conclusion.

Theorem 3.3 *Assume that conditions* (H_1) *and* $p \neq 2$ *hold. Furthermore, suppose the following condition holds*:

(H_5) *There exist two positive constants* d_5, d_6 *with* $d_5 < d_6$ *such that* $g(t,u) - e(t) < 0$ *for* $(t,u) \in [0,T] \times (0, d_5)$ *and* $g(t,u) - e(t) > 0$ *for* $(t,u) \in [0,T] \times (d_6, +\infty)$.

Then equation (3.8) *has at least one positive periodic solution.*

Proof Consider the following equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \lambda f\bigl(t, u(t)\bigr)u'(t) + \lambda g\bigl(t, u(t)\bigr) = \lambda e(t), \tag{3.9}$$

where $\lambda \in (0,1)$. From equation (3.7) and (H_5), we get

$$d_1 \leq u(t) \leq d_2, \quad \text{for } t \in \mathbb{R}. \tag{3.10}$$

Multiplying both sides of equation (3.9) by $u'(t)$ and integrating from 0 to T, we have

$$\begin{aligned}
&\int_0^T \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' u'(t)\,dt + \lambda \int_0^T f\bigl(t,u(t)\bigr)u'(t)\,dt + \lambda \int_0^T g\bigl(t,u(t)\bigr)u'(t)\,dt \\
&\quad = \lambda \int_0^T e(t)u'(t)\,dt.
\end{aligned} \tag{3.11}$$

Substituting $\int_0^T (\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}))'u'(t)\,dt = 0$ into equation (3.11), it is clear that

$$\left|\int_0^T f\bigl(t,u(t)\bigr)u'(t)\,dt\right| = \left|-\int_0^T g\bigl(t,u(t)\bigr)u'(t)\,dt + \int_0^T e(t)u'(t)\,dt\right|.$$

By condition (H_1) and equation (3.10), the above equation imply

$$
\begin{aligned}
\gamma \int_0^T \left|u'(t)\right|^2 dt &\le \left|\int_0^T f\big(t,u(t)\big)u'(t)\,dt\right| \\
&\le \int_0^T \left|g\big(t,u(t)\big)\right|\left|u'(t)\right| dt + \int_0^T \left|e(t)\right|\left|u'(t)\right| dt \\
&\le \|g_1\| T^{\frac{1}{2}} \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \left|e(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\le \left(\|g_1\| T^{\frac{1}{2}} + \|e\|_2\right)\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\|g_1\| := \max_{d_1 \le u(t) \le d_2} |g(t,u)|$. Since $\int_0^T |u'(t)|^2\,dt \ne 0$ and $\gamma > 0$, we arrive at

$$
\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{\|g_1\| T^{\frac{1}{2}} + \|e\|_2}{\gamma} := M_2'', \tag{3.12}
$$

From equation (3.12), using Hölder inequality, we get

$$
\left\|u'\right\| = \frac{1}{T}\int_0^T \left\|u'\right\| dt \le T^{-\frac{1}{2}} \left(\int_0^T \left\|u'\right\|^2 dt\right)^{\frac{1}{2}} \le T^{-\frac{1}{2}} M_2'' := M_2^{**}. \tag{3.13}
$$

The remaining part of the proof is the same as that of Theorem 2.1. □

Theorem 3.3 requires that g of equation (3.8) possesses a singularity of repulsive type. In the following, by Theorems 2.3 and 3.3, we discuss equation (3.8) with a singularity of attractive type.

Theorem 3.4 *Assume that conditions (H_1) and $p \ne 2$ hold. Furthermore, suppose the following condition holds*:

(H_6) *There exist two positive constants d_7, d_8 with $d_7 < d_8$ such that $g(t,u) - e(t) > 0$ for $(t,u) \in [0,T] \times (0,d_7)$ and $g(t,u) - e(t) < 0$ for $(t,u) \in [0,T] \times (d_8,+\infty)$.*

Then equation (3.8) has at least one positive periodic solution.

Finally, we illustrate our results with one numerical example.

Example 3.1 Consider the following prescribed mean curvature Liénard equation with a weak singularity of repulsive type

$$
\begin{aligned}
&\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \big((\sin^2 t)u^8(t) + 1\big)u'(t) + \big(\cos^2 t + 2\big)u^3(t) - \frac{\sin^2 t + 1}{u^{\frac{1}{5}}(t)} \\
&\quad = e^{\cos 2t},
\end{aligned} \tag{3.14}
$$

where $p = 5$.

It is clear that $T = \pi$, $f(t,u) = (\sin^2 t)u^8(t) + 1$, $g(t,u) = (\cos^2 t + 2)u^3(t) - \frac{\sin^2 t + 1}{u^{\frac{1}{5}}(t)}$, $e(t) = e^{\cos 2t}$. Take $\gamma = 1$, $d_5 = 0.01$, $d_6 = 3$. Then conditions (H_1) and (H_5) hold. Therefore, by Theorem 3.3, equation (3.14) has at least one positive π-periodic solution.

4 Conclusions

In this paper, applying an extension of Mawhin's continuation theorem, we first investigate the existence of a periodic solution for equation (1.1) in the case that $p > 1$, where g possesses weak and strong singularities of attractive type, or weak and strong singularities of repulsive type, and g may satisfy sublinearity, semilinearity, or superlinearity conditions at infinity. After that, we consider the existence of a periodic solution for equation (1.1) when $p > 1$ and $p \neq 2$. Note that the conditions which f, g satisfy and the work of estimating *a priori bounds* of positive periodic solutions for equation (1.1) are more complex than in [7, 9, 17, 18]. Firstly, the friction term $f(u(t))u'(t)$ in [7, 9, 17, 18] satisfies $\int_0^T f(u(t))u'(t)\,dt = 0$, which is crucial to estimate *a priori bounds* of positive periodic solutions for these equations. However, the friction term of this paper, $f(t, u(t))u'(t)$, may not satisfy $\int_0^T f(t, u(t))u'(t)\,dt = 0$. Secondly, g of this paper possesses weak and strong singularities of attractive type (or weak and strong singularities of repulsive type) at the origin. Thirdly, g of this paper may satisfy sublinearity, semilinearity, or superlinearity conditions at infinity. Therefore, we extend and improve the results in [7, 9, 17, 18].

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