# Nonlinear fractional differential equation involving two mixed fractional orders with nonlocal boundary conditions and Ulam-Hyers stability 

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#### Abstract

In this paper, we study a nonlinear fractional differential equation involving two mixed fractional orders with nonlocal boundary conditions. By using some new techniques, we introduce a formula of solutions for above problem, which can be regarded as a novelty item. Moreover, under the weak assumptions and using Leray-Schauder degree theory, we obtain the existence result of solutions for above problem. Furthermore, we discuss the Ulam-Hyers stability of the above fractional differential equation. Three examples illustrate our results.


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## 1 Introduction

The fractional derivatives provide an excellent tool to describe the memory and hereditary properties of various materials and processes. The fractional differential equations can model some engineering and scientific disciplines in the fields of physics, chemistry, electrodynamics of complex medium, polymer rheology, etc. [1-12]. In particular, the forward and backward fractional derivatives provide an excellent tool for the description of some physical phenomena such as the fractional oscillator equations and the fractional Euler-Lagrange equations [3-7,13]. Recently, many researchers have focused on the existence of solutions for boundary value problems involving both the right Caputo and the left Riemann-Liouville fractional derivatives (see [6, 7, 13], and the references cited therein).

Moreover, the Ulam stability problem [14] has attracted many researchers (see [15, 16] and the references therein). Recently, the Ulam-Hyers stability of fractional differential equations has been gaining much importance and attention [17-20].

[^0]Sousa et al. [19] studied the $\psi$-Hilfer fractional derivative and the Hyers-Ulam-Rassias and Hyers-Ulam stability of the Volterra integrodifferential equation:

$$
\left\{\begin{array}{l}
{ }^{H} D_{0^{+}}^{\alpha, \beta ; \psi} u(t)=f(t, u(t))+\int_{0}^{t} K(t, s, u(t)) d s, \quad t \in[0, T], \\
I_{0^{+}}^{1-\gamma} u(0)=\sigma
\end{array}\right.
$$

where $\alpha \in(0,1), \beta \in[0,1], \gamma \in[0,1), \sigma$ is a constant, ${ }^{H} D_{0^{+}}^{\alpha, \beta ; \psi}$ is the $\psi$-Hilfer fractional derivative and $I_{0^{+}}^{1-\gamma}$ is the $\psi$-Riemann-Liouville fractional integral.
Chalishajar et al. [20] studied the existence, uniqueness, and Ulam-Hyers stability of solutions for the coupled system of fractional differential equations with integral boundary conditions:

$$
\left\{\begin{array}{lll}
{ }^{c} D_{0^{+}}^{\alpha} x(t)=f(t, y(t)), & t \in[0,1], & \\
{ }^{c} D_{0^{+}}^{\beta} y(t)=g(t, x(t)), & t \in[0,1], & \\
p x(0)+q x^{\prime}(0)=\int_{0}^{1} a_{1}(x(s)) d s, & p x(1)+q x^{\prime}(1)=\int_{0}^{1} a_{2}(x(s)) d s, \\
\tilde{p} y(0)+\tilde{q} y^{\prime}(0)=\int_{0}^{1} \tilde{a}_{1}(y(s)) d s, & \tilde{p} y(1)+\tilde{q} y^{\prime}(1)=\int_{0}^{1} \tilde{a}_{2}(y(s)) d s,
\end{array}\right.
$$

where $\alpha, \beta \in(1,2], p, q, \tilde{p}, \tilde{q} \geq 0$ are constants, $a_{1}, a_{2}, \tilde{a}_{1}, \tilde{a}_{2}$ are continuous functions.
In this paper, we study the following boundary value problem of fractional differential equation with two different fractional derivatives:

$$
\left\{\begin{array}{l}
{ }^{c} D_{1^{-}}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) u(t)=f(t, u(t)), \quad t \in J:=(0,1],  \tag{1.1}\\
\left(I_{0^{+}}^{1-\alpha} u\right)(0)=u_{0}, \quad \sum_{i=1}^{m} u\left(\xi_{i}\right)=\left({ }^{\rho} I_{0^{+}}^{\gamma} u\right)(\eta),
\end{array}\right.
$$

where $\alpha, \beta, \alpha+\beta \in(0,1), \lambda>0, \gamma>1, \rho>0, \alpha+\rho>1$ and $\xi_{i}, \eta \in(0,1](i=1,2, \ldots, m) .{ }^{c} D_{1^{-}}^{\beta}$ is the right-sided Caputo fractional derivative, ${ }^{L} D_{0^{+}}^{\alpha}$ is the left-sided Riemann-Liouville fractional derivative, $I_{0^{+}}^{1-\alpha}$ is the left-sided Riemann-Liouville fractional integral, ${ }^{\rho} I_{0^{+}}^{\gamma}$ is a Katugampola fractional integral.
Different from the previous results, the boundary conditions considered in this paper include the nonlocal Katugampola fractional integral, moreover, under the weak assumptions and using Leray-Schauder degree theory, we obtain the existence result of solutions for the above problem (Theorem 5.3). However, to the best of our knowledge, few papers can be found in the literature dealing with the existence result and the Ulam-Hyers stability of differential equation involving the forward and backward fractional derivatives.
The rest of this paper is organized as follows. In Sect. 2, we collect some concepts of fractional calculus. In Sect. 3, we prove some properties of classical and generalized MittagLeffler functions. In Sect. 4, we present the definition of solution to (1.1)-(1.2). In Sect. 5, we obtain the existence and uniqueness of solutions to problem (1.1)-(1.2). In Sect. 6, we present Ulam-Hyers stability result for Eq. (1.1). Three examples are given in Sect. 7 to demonstrate the applicability of our result.

## 2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus. Throughout this paper, we denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from $J$ to $\mathbb{R}$, by $A C([a, b], \mathbb{R})$ the space of absolutely continuous functions on $[a, b]$. $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ are the gamma and beta functions, respectively.

Definition $2.1([3,4])$ The left-sided and the right-sided fractional integrals of order $\delta$ for a function $x(t) \in L^{1}$ are defined by

$$
\left(I_{a^{+}}^{\delta} x\right)(t)=\frac{1}{\Gamma(\delta)} \int_{a}^{t}(t-s)^{\delta-1} x(s) d s, \quad t>a, \delta>0
$$

and

$$
\left(I_{b^{-}}^{\delta} x\right)(t)=\frac{1}{\Gamma(\delta)} \int_{t}^{b}(s-t)^{\delta-1} x(s) d s, \quad t<b, \delta>0
$$

respectively.
Definition $2.2([3,4])$ If $x(t) \in A C([a, b], \mathbb{R})$, then the left-sided Riemann-Liouville fractional derivative ${ }^{L} D_{a^{+}}^{\delta} x(t)$ of order $\delta$ exists almost everywhere on $[a, b]$ and can be written as

$$
\left({ }^{L} D_{a^{+}}^{\delta} x\right)(t)=\frac{1}{\Gamma(1-\delta)}\left(\frac{d}{d t}\right) \int_{a}^{t}(t-s)^{-\delta} x(s) d s=\frac{d}{d t}\left(I_{a^{+}}^{1-\delta} x\right)(t), \quad t>a, 0<\delta<1
$$

Definition 2.3 ([3, 4]) If $x(t) \in A C([a, b], \mathbb{R})$, then the right-sided Caputo fractional derivative ${ }^{c} D_{b^{-}}^{\delta} x(t)$ of order $\delta$ exists almost everywhere on $[a, b]$ and can be written as

$$
\left({ }^{c} D_{b^{-}}^{\delta} x\right)(t)=\left({ }^{L} D_{b^{-}}^{\delta}[x(s)-x(b)]\right)(t), \quad t<b, 0<\delta<1 .
$$

Definition 2.4 ([21]) For $\rho, q>0$, the Katugampola fractional integral of $y(t)$ is defined by

$$
\left({ }^{\rho} I_{a^{+}}^{q} y\right)(t)=\frac{\rho^{1-q}}{\Gamma(q)} \int_{a}^{t}\left(t^{\rho}-\tau^{\rho}\right)^{q-1} \tau^{\rho-1} y(\tau) d \tau, \quad t>a .
$$

## 3 Properties of the Mittag-Leffler functions

In this section, we prove some properties of the Mittag-Leffler functions.

Definition 3.1 ([3, 4]) For $\mu, v>0, z \in \mathbb{R}$, the classical Mittag-Leffler function $E_{\mu}(z)$ and the generalized Mittag-Leffler function $E_{\mu, \nu}(z)$ are defined by

$$
E_{\mu}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\mu k+1)}, \quad E_{\mu, v}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\mu k+v)}
$$

Clearly, $E_{\mu, 1}(z)=E_{\mu}(z)$.

Lemma $3.2([4,22])$ Let $\alpha \in(0,1)$. Then the nonnegative functions $E_{\alpha}, E_{\alpha, \alpha}, E_{\alpha, \alpha+1}$ have the following properties:
(i) For any $t \in J, E_{\alpha}\left(-\lambda t^{\alpha}\right) \leq 1, E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right) \leq \frac{1}{\Gamma(\alpha)}, E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right) \leq \frac{1}{\Gamma(\alpha+1)}$.
(ii) For any $t_{1}, t_{2} \in J$,

$$
\begin{aligned}
& \left|E_{\alpha}\left(-\lambda t_{2}^{\alpha}\right)-E_{\alpha}\left(-\lambda t_{1}^{\alpha}\right)\right|=O\left(\left|t_{2}-t_{1}\right|^{\alpha}\right), \quad \text { as } t_{2} \rightarrow t_{1} \\
& \left|E_{\alpha, \alpha}\left(-\lambda t_{2}^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda t_{1}^{\alpha}\right)\right|=O\left(\left|t_{2}-t_{1}\right|^{\alpha}\right), \quad \text { as } t_{2} \rightarrow t_{1} \\
& \left|E_{\alpha, \alpha+1}\left(-\lambda t_{2}^{\alpha}\right)-E_{\alpha, \alpha+1}\left(-\lambda t_{1}^{\alpha}\right)\right|=O\left(\left|t_{2}-t_{1}\right|^{\alpha}\right), \quad \text { as } t_{2} \rightarrow t_{1} .
\end{aligned}
$$

Lemma 3.3 Let $\gamma, \mu, v, \lambda>0, t>0,0<\alpha, \beta<1$, then the following formulas are valid:
(i) $\frac{d}{d t}\left[t^{\nu-1} E_{\mu, \nu}\left(-\lambda t^{\mu}\right)\right]=t^{\nu-2} E_{\mu, \nu-1}\left(-\lambda t^{\mu}\right)(\nu>1)$ and $\frac{d}{d t} E_{\mu}\left(-\lambda t^{\mu}\right)=-\lambda t^{\mu-1} E_{\mu, \mu}\left(-\lambda t^{\mu}\right)$;
(ii) $I_{0^{+}}^{\gamma}\left(s^{\nu-1} E_{\mu, \nu}\left(-\lambda s^{\mu}\right)\right)(t)=\frac{1}{\Gamma(\gamma)} \int_{0}^{t}(t-s)^{\gamma-1} s^{\nu-1} E_{\mu, \nu}\left(-\lambda s^{\mu}\right) d s=t^{\gamma+\nu-1} E_{\mu, \gamma+\nu}\left(-\lambda t^{\mu}\right)$;
(iii) $\left[{ }^{L} D_{0^{+}}^{v} s^{\beta-1} E_{\alpha, \beta}\left(-\lambda s^{\alpha}\right)\right](t)=t^{\beta-\nu-1} E_{\alpha, \beta-v}\left(-\lambda t^{\alpha}\right),(\beta>\nu)$;
(iv) $\left[{ }^{c} D_{1^{-}}^{\beta} D_{0^{+}}^{\alpha} S^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right](t)+\lambda\left[{ }^{c} D_{1^{-}}^{\beta} s^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right](t)=0$;
(v) $\left[{ }^{c} D_{1^{-}}^{\beta}{ }^{L} D_{0^{+}}^{\alpha} s^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda s^{\alpha}\right)\right](t)+\lambda\left[{ }^{c} D_{1-1}^{\beta} s^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda s^{\alpha}\right)\right](t)=0$;
(vi) $\left[{ }^{\rho} I_{0^{+}}^{\gamma} s^{\nu-1} E_{\alpha, \nu}\left(-\lambda s^{\alpha}\right)\right](t)=\frac{t^{\gamma \rho+v-1}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1} s^{\frac{v-1}{\rho}}(1-s)^{\gamma-1} E_{\alpha, v}\left(-\lambda t^{\alpha} s^{\frac{\alpha}{\rho}}\right) d s, v=\alpha$ or $v=\alpha+1$.

Proof It follows from the results in [3] that (i)-(iii) hold. Moreover,

$$
\begin{aligned}
{\left[{ }^{L} D_{0^{+}}^{\alpha} s^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda s^{\alpha}\right)\right](t) } & =\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t}(t-s)^{-\alpha} s^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda s^{\alpha}\right) d s \\
& =\frac{d}{d t}\left[E_{\alpha}\left(-\lambda t^{\alpha}\right)\right]=-\lambda t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)
\end{aligned}
$$

Similarly, we have $\left[{ }^{L} D_{0^{+}}^{\alpha} s^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right](t)=E_{\alpha}\left(-\lambda t^{\alpha}\right)$. Furthermore, we get

$$
\begin{aligned}
& {\left[{ }^{c} D_{1^{-}}^{\beta}{ }^{L} D_{0^{+}}^{\alpha} s^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right](t)+\lambda\left[{ }^{c} D_{1^{-}}^{\beta} s^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right](t)} \\
& \quad=\left[{ }^{c} D_{1^{-}}^{\beta}\left(E_{\alpha}\left(-\lambda s^{\alpha}\right)+\lambda s^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda s^{\alpha}\right)\right)\right](t) \\
& \quad=\left({ }^{c} D_{1^{-}}^{\beta} 1\right)(t)=0 .
\end{aligned}
$$

This yields (iv). (v) can be obtained in a similar way. Clearly, for $v=\alpha$ or $v=\alpha+1$, the integral $\int_{0}^{t}(1-\tau)^{\gamma-1} \tau^{\frac{\nu-1}{\rho}} E_{\alpha, v}\left(-\lambda t^{\alpha} \tau^{\frac{\alpha}{\rho}}\right) d \tau$ exists, then

$$
\begin{aligned}
{\left[{ }^{\rho} I_{0^{+}}^{\gamma}{ }^{\nu-1} E_{\alpha, \nu}\left(-\lambda s^{\alpha}\right)\right](t) } & =\frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{0}^{t}\left(t^{\rho}-s^{\rho}\right)^{\gamma-1} s^{\rho+\nu-2} E_{\alpha, \nu}\left(-\lambda s^{\alpha}\right) d s \\
& =\frac{t^{\rho \gamma+\nu-1}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-\tau)^{\gamma-1} \tau^{\frac{\nu-1}{\rho}} E_{\alpha, v}\left(-\lambda t^{\alpha} \tau^{\frac{\alpha}{\rho}}\right) d \tau
\end{aligned}
$$

Thus we have proved (vi).

## 4 Solutions for problem (1.1)-(1.2)

In this section, we present the formula of the solution to the problem (1.1)-(1.2).

Lemma 4.1 ([3]) For $\theta>0$, a general solution of the fractional differential equation ${ }^{c} D_{1^{-}}^{\theta} u(t)=0$ is given by

$$
u(t)=\sum_{i=0}^{n-1} c_{i}(1-t)^{i}
$$

where $c_{i} \in \mathbb{R}, i=0,1,2, \ldots, n-1(n=[\theta]+1)$, and $[\theta]$ denotes the integer part of the real number $\theta$.

Similar to the arguments in [3], we can obtain the following result.
Lemma 4.2 For $\alpha, \beta \in(0,1), h \in L^{1}(0,1)$, if ${ }^{c} D_{1^{-}}^{\beta}\left(D_{0^{+}}^{\alpha}+\lambda\right) u(t)=h(t), t \in J$, then

$$
u(t)=c_{0} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+c_{1} t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)
$$

$$
+\frac{1}{\Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right) h(\tau) d \tau d s, \quad t \in J .
$$

Formally, by Lemma 4.1, for $c_{0} \in \mathbb{R}$, we have $\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) u(t)=c_{0}+\left(I_{1^{-}}^{\beta} h\right)(t)$. Based on the arguments in Sect. 4.1.1 of [3], we obtain

$$
\begin{aligned}
u(t)= & c_{1} t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)+\int_{0}^{t}(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right)\left(c_{0}+\left(I_{1^{1}}^{\beta} h\right)(s)\right) d s \\
= & c_{0} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+c_{1} t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right) \\
& +\frac{1}{\Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right) h(\tau) d \tau d s .
\end{aligned}
$$

We define $C_{1-\alpha}([0,1], \mathbb{R})=\left\{u \in C(J, \mathbb{R}): t^{1-\alpha} u(t) \in C([0,1], \mathbb{R})\right\}$ with the norm $\|u\|_{1-\alpha}=$ $\max _{t \in[0,1]} t^{1-\alpha}|u(t)|$ and abbreviate $C_{1-\alpha}([0,1], \mathbb{R})$ to $C_{1-\alpha}$.

We need the following assumptions.
(H1) $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(\cdot, \omega): J \rightarrow \mathbb{R}$ is measurable for all $\omega \in \mathbb{R}$ and there exist $L_{f}>0$ and $\sigma \in[0,1)$ such that

$$
|f(t, \omega)-f(t, \widetilde{\omega})| \leq L_{f}|\omega-\widetilde{\omega}|^{\sigma} .
$$

(H2) $M_{f}:=\sup _{t \in J}|f(t, 0)|<\infty$.
For convenience of the following presentation, we set

$$
\begin{aligned}
& \left(F_{\beta} u\right)(s)=\left(I_{1-}^{\beta} f\right)(s)=\frac{1}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} f(\tau, u(\tau)) d \tau \\
& (G u)(t)=\int_{0}^{t}(t-s)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right)\left(F_{\beta} u\right)(s) d s, \\
& (\bar{G} u)(t)=\frac{t^{\rho \gamma}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}(G u)\left(t s^{\frac{1}{\rho}}\right) d s, \\
& A(\theta, t)=\frac{t^{\gamma \rho+\theta}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1} s^{\frac{\theta}{\rho}}(1-s)^{\gamma-1} E_{\alpha, \theta+1}\left(-\lambda t^{\alpha} s^{\frac{\alpha}{\rho}}\right) d s, \\
& \widetilde{A}(\theta, t)=A(\theta, t)-\sum_{i=1}^{m} \xi_{i}^{\theta} E_{\alpha, \theta+1}\left(-\lambda \xi_{i}^{\alpha}\right) .
\end{aligned}
$$

Since $\int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\alpha-1} d \tau<\frac{s^{\alpha-1}}{\beta}, \int_{0}^{t}(t-s)^{\alpha-1} s^{\alpha-1} d s=t^{2 \alpha-1} B(\alpha, \alpha) \leq t^{\alpha-1} B(\alpha, \alpha)$ and

$$
\begin{equation*}
|f(t, u(t))| \leq|f(t, u(t))-f(t, 0)|+|f(t, 0)| \leq L_{f} t^{\alpha-1}\|u\|_{\sigma, 1-\alpha}+M_{f}, \tag{4.1}
\end{equation*}
$$

where $\|u\|_{\sigma, 1-\alpha}=\max _{t \in[0,1]} t^{1-\alpha}|u(t)|^{\sigma}$, one can find

$$
\begin{align*}
\left|\left(F_{\beta} u\right)(s)\right| & \leq \frac{L_{f}\|u\|_{\sigma, 1-\alpha}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\alpha-1} d \tau+\frac{M_{f}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} d \tau \\
& \leq \frac{L_{f} s^{\alpha-1}\|u\|_{\sigma, 1-\alpha}+M_{f}}{\Gamma(\beta+1)},  \tag{4.2}\\
|(G u)(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)\right| d s \leq \frac{\alpha L_{f} t^{\alpha-1} B(\alpha, \alpha)\|u\|_{\sigma, 1-\alpha}+M_{f}}{\Gamma(\alpha+1) \Gamma(\beta+1)} . \tag{4.3}
\end{align*}
$$

Lemma 4.3 For $0<t_{1}<t_{2} \leq 1$,

$$
\begin{align*}
& \int_{0}^{t_{1}}\left[\left(t_{1}-s\right)^{\alpha-1}-\left(t_{2}-s\right)^{\alpha-1}\right]\left|\left(F_{\beta} u\right)(s)\right| d s \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1}  \tag{4.4}\\
& \int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)\right| d s \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1},  \tag{4.5}\\
& \int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\right| \cdot\left|\left(F_{\beta} u\right)(s)\right| d s \rightarrow 0, \\
& \text { as } t_{2} \rightarrow t_{1} \tag{4.6}
\end{align*}
$$

Proof By the mean value theorem and Lemma 3.2, we obtain

$$
\begin{align*}
& \int_{0}^{t_{1}}\left[\left(t_{1}-s\right)^{\alpha-1}-\left(t_{2}-s\right)^{\alpha-1}\right] s^{\alpha-1} d s \\
& \quad=\int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1} s^{\alpha-1} d s-\int_{0}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1} s^{\alpha-1} d s+\int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1} s^{\alpha-1} d s \\
& \quad \leq\left|t_{1}^{2 \alpha-1}-t_{2}^{2 \alpha-1}\right| B(\alpha, \alpha)+\frac{t_{1}^{\alpha-1}}{\alpha}\left(t_{2}-t_{1}\right)^{\alpha} \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1}  \tag{4.7}\\
& \int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\right| s^{\alpha-1} d s \\
& \quad=t_{1}^{2 \alpha-1} B(\alpha, \alpha) O\left(\left(t_{2}-t_{1}\right)^{\alpha}\right) \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1} . \tag{4.8}
\end{align*}
$$

Then, using the inequality $t_{2}^{\alpha}-t_{1}^{\alpha} \leq\left(t_{2}-t_{1}\right)^{\alpha}$ and Eqs. (4.2), (4.7) and (4.8), we get

$$
\begin{aligned}
& \int_{0}^{t_{1}}\left[\left(t_{1}-s\right)^{\alpha-1}-\left(t_{2}-s\right)^{\alpha-1}\right]\left|\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \leq \frac{L_{f}\|u\|_{\sigma, 1-\alpha}}{\Gamma(\beta+1)} \int_{0}^{t_{1}}\left[\left(t_{1}-s\right)^{\alpha-1}-\left(t_{2}-s\right)^{\alpha-1}\right] s^{\alpha-1} d s+\frac{2 M_{f}\left(t_{2}-t_{1}\right)^{\alpha}}{\alpha \Gamma(\beta+1)} \\
& \quad \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1} \\
& \int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \leq \frac{L_{f}\|u\|_{\sigma, 1-\alpha}}{\Gamma(\beta+1)} \int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1} s^{\alpha-1} d s+\frac{M_{f}\left(t_{2}-t_{1}\right)^{\alpha}}{\alpha \Gamma(\beta+1)} \\
& \quad \leq\left[L_{f}\|u\|_{\sigma, 1-\alpha} t_{1}^{\alpha-1}+M_{f}\right] \frac{\left(t_{2}-t_{1}\right)^{\alpha}}{\alpha \Gamma(\beta+1)} \\
& \quad \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\right| \cdot\left|\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \leq \frac{L_{f}\|u\|_{\sigma, 1-\alpha}}{\Gamma(\beta+1)} \int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\right| s^{\alpha-1} d s \\
& \quad+\frac{t_{1}^{\alpha} M_{f} \cdot O\left(\left(t_{2}-t_{1}\right)^{\alpha}\right)}{\alpha \Gamma(\beta+1)}
\end{aligned}
$$

$$
\rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1} .
$$

Lemma 4.4 Assume that (H1) and (H2) hold. For $u \in C_{1-\alpha}, t \in J$, we have
(i) $(G u)(t) \in A C(J, \mathbb{R})$;
(ii) $\left[{ }^{L} D_{0^{+}}^{\alpha}(G u)(s)\right](t)=-\lambda(G u)(t)+\left(I_{1-}^{\beta} f\right)(t)$;
(iii) $\left[{ }^{c} D_{1^{-}}^{\beta} D_{0^{+}}^{\alpha}(G u)(s)\right](t)+\lambda\left[{ }^{c} D_{1^{-}}^{\beta}(G u)(s)\right](t)=f(t, u(t))$;
(iv) $\left[{ }^{\rho} I_{0^{+}}^{\gamma}(G u)(s)\right](t)=\frac{t^{\gamma \rho}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}(G u)\left(t s^{\frac{1}{\rho}}\right) d s$.

Proof (i)-(iii) For every finite collection $\left\{\left(a_{j}, b_{j}\right)\right\}_{1 \leq j \leq n}$ on $J$ with $\sum_{j=1}^{n}\left(b_{j}-a_{j}\right) \rightarrow 0$, using the inequalities $b_{j}^{\alpha}-a_{j}^{\alpha} \leq\left(b_{j}-a_{j}\right)^{\alpha}, j=1,2, \ldots, n$, and Eqs. (4.4)-(4.6), we arrive at

$$
\begin{aligned}
& \sum_{j=1}^{n}\left|(G u)\left(b_{j}\right)-(G u)\left(a_{j}\right)\right| \\
& \quad=\sum_{j=1}^{n} \mid \int_{0}^{b_{j}}\left(b_{j}-s\right)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda\left(b_{j}-s\right)^{\alpha}\right)\left(F_{\beta} u\right)(s) d s \\
& \quad-\int_{0}^{a_{j}}\left(a_{j}-s\right)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda\left(a_{j}-s\right)^{\alpha}\right)\left(F_{\beta} u\right)(s) d s \mid \\
& \quad \leq \frac{1}{\Gamma(\alpha)}\left[\sum_{j=1}^{n} \int_{0}^{a_{j}}\left[\left(a_{j}-s\right)^{\alpha-1}-\left(b_{j}-s\right)^{\alpha-1}\right]\left|\left(F_{\beta} u\right)(s)\right| d s\right. \\
& \left.\quad+\sum_{j=1}^{n} \int_{a_{j}}^{b_{j}}\left(b_{j}-s\right)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)\right| d s\right] \\
& \quad+\sum_{j=1}^{n} \int_{0}^{a_{j}}\left(a_{j}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(a_{j}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(b_{j}-s\right)^{\alpha}\right)\right| \cdot\left|\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \rightarrow 0
\end{aligned}
$$

Then, $(G u)(t)$ is absolutely continuous on $J$. Hence, for almost all $t \in J,\left[{ }^{L} D_{0^{+}}^{\alpha}(G u)(s)\right](t)$ exists and from Lemma 3.3 it follows that

$$
\begin{aligned}
& {\left[{ }^{L} D_{0^{+}}^{\alpha}(G u)(s)\right](t) } \\
&=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t} \int_{0}^{s}(t-s)^{-\alpha}(s-\tau)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda(s-\tau)^{\alpha}\right)\left(I_{1}^{\beta}-f\right)(\tau) d \tau d s \\
&=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t}\left(I_{1-}^{\beta} f\right)(\tau) \int_{\tau}^{t}(t-s)^{-\alpha}(s-\tau)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda(s-\tau)^{\alpha}\right) d s d \tau \\
&=\frac{d}{d t} \int_{0}^{t}\left(I_{1}^{\beta}-f\right)(\tau) E_{\alpha}\left(-\lambda(t-\tau)^{\alpha}\right) d \tau \\
&=-\lambda(G u)(t)+\left(I_{1-}^{\beta} f\right)(t) .
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& {\left[{ }^{c} D_{1^{-}}^{\beta}{ }^{L} D_{0^{+}}^{\alpha}(G u)(s)\right](t)+\lambda\left[{ }^{c} D_{1^{-}}^{\beta}(G u)(s)\right](t)} \\
& \quad=\left[{ }^{c} D_{1^{-}}^{\beta}\left(-\lambda(G u)(s)+\left(I_{1^{-}}^{\beta} f\right)(s)\right)\right](t)+\lambda\left[{ }^{c} D_{1^{-}}^{\beta}(G u)(s)\right](t)
\end{aligned}
$$

$$
=f(t, u(t))
$$

(iv) It follows from (4.3) that ${ }^{\rho} I_{0^{+}}^{\gamma}(G u)(t)$ exists, and

$$
\begin{aligned}
{\left[{ }^{\rho} I_{0^{+}}^{\gamma}(G u)(s)\right](t) } & =\frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_{0}^{t}\left(t^{\rho}-s^{\rho}\right)^{\gamma-1} s^{\rho-1}(G u)(s) d s \\
& =\frac{t^{\gamma \rho}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}(G u)\left(t s^{\frac{1}{\rho}}\right) d s .
\end{aligned}
$$

Lemma 4.5 Assume that (H1) and (H2) hold. A function $u$ is a solution of the following fractional integral equation:

$$
\begin{equation*}
u(t)=(P u)(t)+(Q u)(t) \tag{4.9}
\end{equation*}
$$

if and only if $u$ is a solution of the problem (1.1)-(1.2), where

$$
\begin{aligned}
& (P u)(t)=\left[-\frac{\widetilde{A}(\alpha-1, \eta)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)\right] u_{0}, \\
& (Q u)(t)=-\frac{(\bar{G} u)(\eta)-\sum_{i=1}^{m}(G u)\left(\xi_{i}\right)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+(G u)(t) .
\end{aligned}
$$

Proof (Sufficiency) Let $u$ be the solution of (1.1)-(1.2), Lemma 3.3, Lemma 4.2 and Lemma 4.4 imply

$$
\begin{aligned}
& u(t)=c_{0} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+c_{1} t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)+(G u)(t), \quad t \in J, \\
& \left(I_{0^{+}}^{1-\alpha} u\right)(t)=c_{0} t E_{\alpha, 2}\left(-\lambda t^{\alpha}\right)+c_{1} E_{\alpha}\left(-\lambda t^{\alpha}\right)+\left[I_{0^{+}}^{1-\alpha}(G u)\right](t), \\
& \left({ }^{\rho} I_{0^{+}}^{\gamma} u\right)(t)=c_{0} A(\alpha, t)+c_{1} A(\alpha-1, t)+(\bar{G} u)(t),
\end{aligned}
$$

where $c_{0}, c_{1}$ are constants. Using the boundary value condition (1.2), we derive that $c_{1}=u_{0}$ and

$$
\begin{aligned}
& c_{0} \sum_{i=1}^{m} \xi_{i}^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda \xi_{i}^{\alpha}\right)+u_{0} \sum_{i=1}^{m} \xi_{i}^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda \xi_{i}^{\alpha}\right)+\sum_{i=1}^{m}(G u)\left(\xi_{i}\right) \\
& \quad=c_{0} A(\alpha, \eta)+u_{0} A(\alpha-1, \eta)+(\bar{G} u)(\eta),
\end{aligned}
$$

then

$$
c_{0}=-\frac{(\bar{G} u)(\eta)-\sum_{i=1}^{m}(G u)\left(\xi_{i}\right)+u_{0} \widetilde{A}(\alpha-1, \eta)}{\widetilde{A}(\alpha, \eta)} .
$$

Now we can see that (4.9) holds.
(Necessity) Let $u$ satisfy (4.9). From Lemma 3.3(iv), (v) and Lemma 4.4(iii), it follows that $\left[{ }^{c} D_{1^{-}}^{\beta}{ }^{L} D_{0^{+}}^{\alpha} u\right](t)$ exists and ${ }^{c} D_{1^{-}}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) u(t)=f(t, u(t))$ for $t \in J$. Clearly, the boundary value condition (1.2) holds and hence the necessity is proved.

## 5 Existence results for problem (1.1)-(1.2)

In this section, we deal with the existence and uniqueness of solutions to the problem (1.1)-(1.2).

Lemma 5.1 Let $v \in C([0,1], \mathbb{R})$ satisfy the following inequality:

$$
\begin{aligned}
|v(t)| \leq & a+b \int_{0}^{t} \int_{s}^{1} t^{1-\alpha}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|v(\tau)|^{\sigma} d \tau d s \\
& +c \sum_{i=1}^{m} \int_{0}^{\xi_{i}} \int_{s}^{1} t^{1-\alpha}\left(\xi_{i}-s\right)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|v(\tau)|^{\sigma} d \tau d s \\
& +d \int_{0}^{1} \int_{0}^{\eta s^{\frac{1}{\rho}}} \int_{\tau}^{1} t^{1-\alpha}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\sigma(\alpha-1)}|\nu(\zeta)|^{\sigma} d \zeta d \tau d s
\end{aligned}
$$

where $a, b, c, d>0$ are constants. Then $|v(t)| \leq M$, where $M$ is the only positive solution of the equation

$$
M=a+\left[b+c \sum_{i=1}^{m} \xi_{i}^{\alpha-\sigma}+d \eta^{\alpha-\sigma} B\left(\gamma, \frac{\alpha-\sigma+\rho}{\rho}\right)\right] \frac{B(\alpha, 1-\sigma)}{\beta} M^{\sigma} .
$$

Proof Let $\bar{m}=\max _{t \in[0,1]}|v(t)|$, using the following estimates:

$$
\begin{align*}
& \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)} d \tau \leq \frac{s^{\sigma(\alpha-1)}(1-s)^{\beta}}{\beta}<\frac{s^{\sigma(\alpha-1)}}{\beta}  \tag{5.1}\\
& \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)} d \tau d s \\
& \quad<\frac{1}{\beta} \int_{0}^{t}(t-s)^{\alpha-1} s^{\sigma(\alpha-1)} d s<\frac{t^{\alpha-\sigma}}{\beta} B(\alpha, 1-\sigma),  \tag{5.2}\\
& \int_{0}^{1} \int_{0}^{t s^{\frac{1}{\rho}}} \int_{\tau}^{1}(1-s)^{\gamma-1}\left(t s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\sigma(\alpha-1)} d \zeta d \tau d s \\
& \quad \leq \frac{B(\alpha, 1-\sigma)}{\beta} \int_{0}^{1}(1-s)^{\gamma-1}\left(t s^{\frac{1}{\rho}}\right)^{\alpha-\sigma} d s=\frac{B(\alpha, 1-\sigma) B\left(\gamma, \frac{\alpha-\sigma+\rho}{\rho}\right)}{\beta} t^{\alpha-\sigma}, \tag{5.3}
\end{align*}
$$

we conclude that

$$
\bar{m}<a+\left[b+c \sum_{i=1}^{m} \xi_{i}^{\alpha-\sigma}+d \eta^{\alpha-\sigma} B\left(\gamma, \frac{\alpha-\sigma+\rho}{\rho}\right)\right] \frac{B(\alpha, 1-\sigma)}{\beta} \bar{m}^{\sigma},
$$

thus $\bar{m} \leq M$.

Lemma 5.2 Let $\widetilde{v} \in C([0,1], \mathbb{R})$ satisfy the following inequality:

$$
\begin{aligned}
|\widetilde{v}(t)| \leq & \tilde{a} \int_{0}^{t} \int_{s}^{1} t^{1-\alpha}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\alpha-1}|\widetilde{v}(\tau)| d \tau d s \\
& +\widetilde{b} \sum_{i=1}^{m} \int_{0}^{\xi_{i}} \int_{s}^{1} t^{1-\alpha}\left(\xi_{i}-s\right)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\alpha-1}|\widetilde{\nu}(\tau)| d \tau d s
\end{aligned}
$$

$$
+\widetilde{c} \int_{0}^{1} \int_{0}^{\eta s^{\frac{1}{\rho}}} \int_{\tau}^{1} t^{1-\alpha}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\alpha-1}|\widetilde{\nu}(\zeta)| d \zeta d \tau d s
$$

where $\widetilde{a}, \widetilde{b}, \widetilde{c}>0$ are constants. If $\left(\widetilde{a}+\widetilde{b} \sum_{i=1}^{m} \xi_{i}^{\alpha-1}+\widetilde{c} \eta^{\alpha-1} B\left(\gamma, \frac{\alpha-1+\rho}{\rho}\right)\right) \frac{B(\alpha, \alpha)}{\beta}<1$, then $\widetilde{v}(t) \equiv 0$.
Proof Let $\tilde{m}=\max _{t \in[0,1]}|\widetilde{v}(t)|$, from the following inequalities:

$$
\begin{align*}
& \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\alpha-1} d \tau \leq \frac{s^{\alpha-1}}{\beta} \\
& \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\alpha-1} d \tau d s \leq \frac{t^{\alpha-1} B(\alpha, \alpha)}{\beta},  \tag{5.4}\\
& \int_{0}^{1} \int_{0}^{\eta^{\frac{1}{\rho}}} \int_{\tau}^{1}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\alpha-1} d \zeta d \tau d s \\
& \leq \eta^{\alpha-1} \int_{0}^{1}(1-s)^{\gamma-1} s^{\frac{\alpha-1}{\rho}} d s \frac{B(\alpha, \alpha)}{\beta} \\
& =\frac{\eta^{\alpha-1}}{\beta} B(\alpha, \alpha) B\left(\gamma, \frac{\alpha-1+\rho}{\rho}\right), \tag{5.5}
\end{align*}
$$

we deduce that

$$
\widetilde{m} \leq\left(\widetilde{a}+\widetilde{b} \sum_{i=1}^{m} \xi_{i}^{\alpha-1}+\widetilde{c} \eta^{\alpha-1} B\left(\gamma, \frac{\alpha-1+\rho}{\rho}\right)\right) \frac{\tilde{m} B(\alpha, \alpha)}{\beta}
$$

thus $\tilde{m}=0$.

Next, we study the existence result of solutions for (1.1)-(1.2). For convenience of the following presentation, we set

$$
\begin{aligned}
& M_{1}=\left[\frac{|\tilde{A}(\alpha-1, \eta)|}{|\widetilde{A}(\alpha, \eta)|}+\alpha\right] \frac{\left|u_{0}\right|}{\Gamma(\alpha+1)} ; \quad M_{2}=\frac{1}{|\widetilde{A}(\alpha, \eta)| \Gamma(\alpha+1)} \\
& M_{3}=\frac{1}{\rho^{\gamma} \Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)} ; \quad M_{4}=\frac{1}{\alpha \beta}\left[\frac{M_{2} M_{3}}{\gamma}+\frac{m M_{2}+1}{\Gamma(\alpha) \Gamma(\beta)}\right]
\end{aligned}
$$

Theorem 5.3 Assume that (H1) and (H2) are satisfied, then the problem (1.1)-(1.2) has at least one solution $u \in C_{1-\alpha}$.

Proof We consider an operator $\mathcal{F}: C_{1-\alpha} \rightarrow C_{1-\alpha}$ defined by

$$
(\mathcal{F} u)(t)=(P u)(t)+(Q u)(t) .
$$

Clearly, $\mathcal{F}$ is well defined, and the fixed point of $\mathcal{F}$ is the solution of the problem (1.1)(1.2).

Let $\left\{u_{n}\right\}$ be a sequence such that $u_{n} \rightarrow u$ in $C_{1-\alpha}$, then there exists $\varepsilon>0$ such that $\| u_{n}-$ $u \|_{1-\alpha}<\varepsilon$ for $n$ sufficiently large. By (H1), we have $\left|f\left(t, u_{n}(t)\right)-f(t, u(t))\right| \leq L_{f} t^{\sigma(\alpha-1)} \varepsilon^{\sigma}$. Moreover, from (5.1)-(5.3), we have

$$
\left|\left(F_{\beta} u_{n}\right)(s)-\left(F_{\beta} u\right)(s)\right|
$$

$$
\begin{aligned}
& \quad \leq \frac{1}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1}\left|f\left(\tau, u_{n}(\tau)\right)-f(\tau, u(\tau))\right| d \tau \\
& \quad \leq \frac{L_{f} \varepsilon^{\sigma}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)} d \tau \leq \frac{L_{f} s^{\sigma(\alpha-1)}}{\Gamma(\beta+1)} \varepsilon^{\sigma}, \\
& \left|\left(G u_{n}\right)(t)-(G u)(t)\right| \\
& \quad \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left|\left(F_{\beta} u_{n}\right)(s)-\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \leq \frac{L_{f} \varepsilon^{\sigma}}{\Gamma(\alpha) \Gamma(\beta+1)} \int_{0}^{t}(t-s)^{\alpha-1} s^{\sigma(\alpha-1)} d s \\
& \quad \leq \frac{t^{\alpha-\sigma} B(\alpha, 1-\sigma) L_{f}}{\Gamma(\alpha) \Gamma(\beta+1)} \varepsilon^{\sigma}, \\
& \left|\left(\bar{G} u_{n}\right)(t)-(\bar{G} u)(t)\right| \\
& \quad \leq \frac{1}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}\left|\left(G u_{n}\right)\left(t s^{\frac{1}{\rho}}\right)-(G u)\left(t s^{\frac{1}{\rho}}\right)\right| d s \\
& \quad \leq \frac{M_{3} t^{\alpha-\sigma} B(\alpha, 1-\sigma) L_{f} \varepsilon^{\sigma}}{\beta} \int_{0}^{1}(1-s)^{\gamma-1} s^{\frac{\alpha-\sigma}{\rho}} d s \\
& \quad=\frac{M_{3} t^{\alpha-\sigma} B(\alpha, 1-\sigma) B\left(\gamma, \frac{\alpha-\sigma+\rho}{\rho}\right) L_{f}}{\beta} \varepsilon^{\sigma},
\end{aligned}
$$

furthermore,

$$
\begin{aligned}
& t^{1-\alpha}\left|\left(\mathcal{F} u_{n}\right)(t)-(\mathcal{F} u)(t)\right| \leq t^{1-\alpha}\left|\left(Q u_{n}\right)(t)-(Q u)(t)\right| \\
& \quad \leq M_{2}\left[\left|\left(\bar{G} u_{n}\right)(\eta)-(\bar{G} u)(\eta)\right|+\sum_{i=1}^{m}\left|\left(G u_{n}\right)\left(\xi_{i}\right)-(G u)\left(\xi_{i}\right)\right|\right]+t^{1-\alpha}\left|\left(G u_{n}\right)(t)-(G u)(t)\right| \\
& \quad \rightarrow 0, \quad \text { as } n \rightarrow \infty .
\end{aligned}
$$

Now we see that $\mathcal{F}$ is continuous.
For $0<t_{1}<t_{2}<1$, from (4.4)-(4.6), we find

$$
\begin{aligned}
&\left|(G u)\left(t_{2}\right)-(G u)\left(t_{1}\right)\right| \\
&= \mid \int_{0}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\left(F_{\beta} u\right)(s) d s \\
&-\int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)\left(F_{\beta} u\right)(s) d s \mid \\
& \leq \frac{1}{\Gamma(\alpha)}\left[\int_{0}^{t_{1}}\left[\left(t_{1}-s\right)^{\alpha-1}-\left(t_{2}-s\right)^{\alpha-1}\right]\left|\left(F_{\beta} u\right)(s)\right| d s+\int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)\right| d s\right] \\
&+\int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1}\left|E_{\alpha, \alpha}\left(-\lambda\left(t_{1}-s\right)^{\alpha}\right)-E_{\alpha, \alpha}\left(-\lambda\left(t_{2}-s\right)^{\alpha}\right)\right| \cdot\left|\left(F_{\beta} u\right)(s)\right| d s \\
& \quad \rightarrow 0, \quad \text { as } t_{2} \rightarrow t_{1},
\end{aligned}
$$

moreover, according to (4.3) and Lemma 3.2, we know that $\left|(Q u)\left(t_{2}\right)-(Q u)\left(t_{1}\right)\right| \rightarrow 0$ and $\left|(P u)\left(t_{2}\right)-(P u)\left(t_{1}\right)\right| \rightarrow 0$ as $t_{2} \rightarrow t_{1}$. Hence the operator $\mathcal{F}$ is equicontinuous.

We just need to prove the existence of at least one solution $u \in C_{1-\alpha}$ satisfying $u=\mathcal{F} u$.
Hence, we show that $\mathcal{F}: \bar{B}_{R} \rightarrow C_{1-\alpha}$ satisfies the condition

$$
\begin{equation*}
u \neq \theta \mathcal{F} u, \quad \forall u \in \partial B_{R}, \forall \theta \in[0,1] \tag{5.6}
\end{equation*}
$$

where $B_{R}=\left\{u \in C_{1-\alpha}: t^{1-\alpha}|u(t)|<R, R>0\right\}$. We define $H(\theta, u)=\theta \mathcal{F} u, u \in C_{1-\alpha}, \theta \in[0,1]$. By the Arzela-Ascoli theorem, a continuous map $h_{\theta}$ defined by $h_{\theta}(u)=u-H(\theta, u)=u-$ $\theta \mathcal{F} u$ is completely continuous.

If (5.6) is true, then the Leray-Schauder degrees are well defined and from the homotopy invariance of topological degree, it follows that

$$
\begin{aligned}
\operatorname{deg}\left(h_{\theta}, B_{R}, 0\right) & =\operatorname{deg}\left(I-\theta \mathcal{F}, B_{R}, 0\right)=\operatorname{deg}\left(h_{1}, B_{R}, 0\right) \\
& =\operatorname{deg}\left(h_{0}, B_{R}, 0\right)=\operatorname{deg}\left(I, B_{R}, 0\right)=1 \neq 0, \quad 0 \in B_{R}
\end{aligned}
$$

Let $v(t)=t^{1-\alpha} u(t)$, we obtain the following estimate:

$$
\begin{aligned}
\left|\left(F_{\beta} u\right)(s)\right| & \leq \frac{1}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1}|f(\tau, u(\tau))| d \tau \\
& \leq \frac{L_{f}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|v(\tau)|^{\sigma} d \tau+\frac{M_{f}}{\Gamma(\beta+1)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
|(G u)(t)| \leq & \frac{L_{f}}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|v(\tau)|^{\sigma} d \tau d s \\
& +\frac{M_{f}}{\Gamma(\alpha+1) \Gamma(\beta+1)}
\end{aligned}
$$

then
$|(\bar{G} u)(\eta)|$

$$
\begin{aligned}
\leq & \frac{\eta^{\rho \gamma}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}\left|(G u)\left(\eta s^{\frac{1}{\rho}}\right)\right| d s \\
\leq & M_{3}\left[L_{f} \int_{0}^{1} \int_{0}^{\eta s^{\frac{1}{\rho}}} \int_{\tau}^{1}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\sigma(\alpha-1)}|\nu(\zeta)|^{\sigma} d \zeta d \tau d s\right. \\
& \left.+\frac{M_{f}}{\alpha \beta \gamma}\right]
\end{aligned}
$$

Applying (5.2) and (5.3), we obtain

$$
\begin{aligned}
|v(t)| & =\left|t^{1-\alpha} u(t)\right| \leq t^{1-\alpha}[|(P u)(t)|+|(Q u)(t)|] \\
& \leq M_{1}+t^{1-\alpha}\left[M_{2}\left(|(\bar{G} u)(\eta)|+\sum_{i=1}^{m}\left|(G u)\left(\xi_{i}\right)\right|\right)+|(G u)(t)|\right] \\
& \leq \bar{a}+\bar{b} \int_{0}^{t} \int_{s}^{1} t^{1-\alpha}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|v(\tau)|^{\sigma} d \tau d s
\end{aligned}
$$

$$
\begin{aligned}
& +\bar{c} \sum_{i=1}^{m} \int_{0}^{\xi_{i}} \int_{s}^{1} t^{1-\alpha}\left(\xi_{i}-s\right)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\sigma(\alpha-1)}|\nu(\tau)|^{\sigma} d \tau d s \\
& +\bar{d} \int_{0}^{1} \int_{0}^{\eta s^{\frac{1}{\rho}}} \int_{\tau}^{1} t^{1-\alpha}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\sigma(\alpha-1)}|\nu(\zeta)|^{\sigma} d \zeta d \tau d s
\end{aligned}
$$

where

$$
\bar{a}=M_{1}+M_{f} M_{4}, \quad \bar{b}=\frac{L_{f}}{\Gamma(\alpha) \Gamma(\beta)}, \quad \bar{c}=\frac{M_{2} L_{f}}{\Gamma(\alpha) \Gamma(\beta)}, \quad \bar{d}=M_{2} M_{3} L_{f}
$$

Then from Lemma 5.1, we find $\|u\|_{1-\alpha} \leq \widetilde{M}$, where $\widetilde{M}$ satisfies

$$
\tilde{M}=\bar{a}+\left[\bar{b}+\bar{c} \sum_{i=1}^{m} \xi_{i}^{\alpha-\sigma}+\bar{d} \eta^{\alpha-\sigma} B\left(\gamma, \frac{\alpha-\sigma+\rho}{\rho}\right)\right] \frac{B(\alpha, 1-\sigma)}{\beta} \tilde{M}^{\sigma} .
$$

Set $R=\widetilde{M}+1$, then (5.6) holds. This completes the proof.

Next, we study the uniqueness of solution, for this purpose, we give the following assumptions.
(H1') $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(\cdot, \omega): J \rightarrow \mathbb{R}$ is measurable for all $\omega \in \mathbb{R}$ and there exist $L_{f}^{\prime}, \widetilde{M}_{f}>0$ and $\widetilde{\sigma} \in[0,1)$ such that

$$
|f(t, \omega)| \leq L_{f}^{\prime}|\omega|^{\widetilde{\sigma}}+\widetilde{M}_{f} .
$$

(H2') There exists a constant $\widetilde{L}_{f}>0$ such that

$$
|f(t, \omega)-f(t, \widetilde{\omega})| \leq \widetilde{L}_{f}|\omega-\widetilde{\omega}|, \quad \text { for } \omega, \widetilde{\omega} \in \mathbb{R}, t \in J
$$

Theorem 5.4 Assume that ( $\mathrm{H}^{\prime}$ ) and ( $\mathrm{H}^{\prime}$ ) hold, then the problem (1.1)-(1.2) has a unique solution $u \in C_{1-\alpha}$, provided that

$$
\left(\frac{1+M_{2} \sum_{i=1}^{m} \xi_{i}^{\alpha-1}}{\Gamma(\alpha) \Gamma(\beta)}+M_{2} M_{3} \eta^{\alpha-1} B\left(\gamma, \frac{\alpha-1+\rho}{\rho}\right)\right) \frac{B(\alpha, \alpha) \tilde{L}_{f}}{\beta}<1
$$

Proof By ( $\mathrm{H1}^{\prime}$ ) and the proof of Theorem 5.3, it is not difficult to see that (1.1)-(1.2) has a solution $u(\cdot) \in C_{1-\alpha}$. Let $\tilde{u}(\cdot)$ be another solution of the problem (1.1)-(1.2). According to $\left(H 2^{\prime}\right)$, we find

$$
\begin{aligned}
& \left|\left(F_{\beta} u\right)(s)-\left(F_{\beta} \tilde{u}\right)(s)\right| \\
& \quad \leq \frac{\widetilde{L}_{f}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1}|u(\tau)-\widetilde{u}(\tau)| d \tau \\
& |(G u)(t)-(G \widetilde{u})(t)| \\
& \quad \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)-\left(F_{\beta} \widetilde{u}\right)(s)\right| d s \\
& \quad \leq \frac{\widetilde{L}_{f}}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1}|u(\tau)-\widetilde{u}(\tau)| d \tau d s
\end{aligned}
$$

$$
\begin{aligned}
& |(\bar{G} u)(t)-(\bar{G} \widetilde{u})(t)| \\
& \quad \leq \frac{1}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}\left|(G u)\left(t s^{\frac{1}{\rho}}\right)-(G \widetilde{u})\left(t s^{\frac{1}{\rho}}\right)\right| d s \\
& \quad \leq M_{3} \widetilde{L}_{f} \int_{0}^{1} \int_{0}^{t s^{\frac{1}{\rho}}} \int_{\tau}^{1}(1-s)^{\gamma-1}\left(t s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1}|u(\zeta)-\widetilde{u}(\zeta)| d \zeta d \tau d s .
\end{aligned}
$$

Let $\tilde{v}(t)=t^{1-\alpha}(u(t)-\widetilde{u}(t))$, then

$$
\begin{aligned}
&|\tilde{v}(t)| \\
&= t^{1-\alpha}|u(t)-\widetilde{u}(t)| \\
&= t^{1-\alpha}|(Q u)(t)-(Q \widetilde{u})(t)| \\
& \leq M_{2}\left[M_{3} \widetilde{L}_{f} \int_{0}^{1} \int_{0}^{\eta s^{\frac{1}{\rho}}} \int_{\tau}^{1} t^{1-\alpha}(1-s)^{\gamma-1}\left(\eta s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} \zeta^{\alpha-1}|\tilde{v}(\zeta)| d \zeta d \tau d s\right. \\
&\left.+\frac{\widetilde{L}_{f}}{\Gamma(\alpha) \Gamma(\beta)} \sum_{i=1}^{m} \int_{0}^{\xi_{i}} \int_{s}^{1} t^{1-\alpha}\left(\xi_{i}-s\right)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\alpha-1}|\tilde{v}(\tau)| d \tau d s\right] \\
&+\frac{\widetilde{L}_{f}}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{t} \int_{s}^{1} t^{1-\alpha}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} \tau^{\alpha-1}|\tilde{v}(\tau)| d \tau d s .
\end{aligned}
$$

Furthermore, from Lemma 5.2, it follows that $u(t)-\widetilde{u}(t) \equiv 0$. This yields the uniqueness of solution to the problem (1.1)-(1.2).

In order to obtain another result for uniqueness of the solutions, we make the following assumption:
(H3) There exists a constant $\bar{L}_{f}>0$ such that

$$
\left|f\left(t, t^{\alpha-1} \omega\right)-f\left(t, t^{\alpha-1} \widetilde{\omega}\right)\right| \leq \bar{L}_{f}|\omega-\widetilde{\omega}|, \quad t \in J, \omega, \widetilde{\omega} \in \mathbb{R}
$$

Theorem 5.5 Assume that (H3) holds, then the problem (1.1)-(1.2) has a unique solution $u \in C_{1-\alpha}$, provided that

$$
\bar{M}:=\frac{\bar{L}_{f}}{\Gamma(\beta+1) \Gamma(\alpha+1)}\left[1+\frac{M_{2}\left(1+m \rho^{\gamma} \Gamma(\gamma+1)\right)}{\rho^{\gamma} \Gamma(\gamma+1)}\right]<1 .
$$

Proof We consider an operator $\mathcal{F}: C_{1-\alpha} \rightarrow C_{1-\alpha}$ defined by

$$
(\mathcal{F} u)(t)=(P u)(t)+(Q u)(t) .
$$

Clearly, $\mathcal{F}$ is well defined. According to (H3), we find

$$
\begin{aligned}
& \left|\left(F_{\beta} u\right)(s)-\left(F_{\beta} \widetilde{u}\right)(s)\right| \\
& \quad \leq \frac{\bar{L}_{f}}{\Gamma(\beta)} \int_{s}^{1}(\tau-s)^{\beta-1} \tau^{1-\alpha}|u(\tau)-\widetilde{u}(\tau)| d \tau \leq \frac{\bar{L}_{f}}{\Gamma(\beta+1)}\|u-\widetilde{u}\|_{1-\alpha},
\end{aligned}
$$

$$
|(G u)(t)-(G \widetilde{u})(t)|
$$

$$
\begin{aligned}
& \quad \leq \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1}\left|\left(F_{\beta} u\right)(s)-\left(F_{\beta} \widetilde{u}\right)(s)\right| d s \leq \frac{\bar{L}_{f}}{\Gamma(\beta+1) \Gamma(\alpha+1)}\|u-\widetilde{u}\|_{1-\alpha} \\
& |(\bar{G} u)(t)-(\bar{G} \widetilde{u})(t)| \\
& \quad \leq \frac{1}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}\left|(G u)\left(t s^{\frac{1}{\rho}}\right)-(G \widetilde{u})\left(t s^{\frac{1}{\rho}}\right)\right| d s \\
& \quad \leq \frac{\bar{L}_{f}\|u-\widetilde{u}\|_{1-\alpha}}{\rho^{\gamma} \Gamma(\beta+1) \Gamma(\alpha+1) \Gamma(\gamma+1)} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \mid(\mathcal{F u ) ( t ) - ( \mathcal { F } \widetilde { u } ) ( t ) |} \\
& \quad=|(Q u)(t)-(Q \widetilde{u})(t)| \\
& \quad \leq \frac{|(\bar{G} u)(\eta)-(\bar{G} \widetilde{u})(\eta)|+\sum_{i=1}^{m}\left|(G u)\left(\xi_{i}\right)-(G \widetilde{u})\left(\xi_{i}\right)\right|}{|\widetilde{A}(\alpha, \eta)| \Gamma(\alpha+1)}+|(G u)(t)-(G \widetilde{u})(t)| \\
& \quad \leq \bar{M}\|u-\widetilde{u}\|_{1-\alpha},
\end{aligned}
$$

this means that $\mathcal{F}$ is a contraction, and by the Banach fixed point theorem there exists a unique solution $u \in \mathcal{C}_{1-\alpha}$.

## 6 Ulam-Hyers stability

Let $\tilde{\epsilon}$ be a positive real number. We consider Eq. (1.1) with inequality

$$
\begin{equation*}
\left|{ }^{c} D_{1^{-}-}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) x(t)-f(t, x(t))\right| \leq \tilde{\epsilon}, \quad t \in J . \tag{6.1}
\end{equation*}
$$

Definition 6.1 Equation (1.1) is Ulam-Hyers stable if there exists $c>0$ such that for each $\tilde{\epsilon}>0$ and for each solution $x(t)$ of the inequality (6.1) there exists a solution $y$ of Eq. (1.1) with

$$
|x(t)-y(t)| \leq \widetilde{\epsilon}, \quad t \in J .
$$

Remark 6.2 A function $x \in C_{1-\alpha}$ is a solution of the inequality (6.1) if and only if there exists a function $g \in C_{1-\alpha}$ such that (i) $|g(t)| \leq \tilde{\epsilon}$; (ii) ${ }^{c} D_{1^{-}}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) x(t)=f(t, x(t))+g(t)$.

Let

$$
\begin{aligned}
\tilde{x}(t)= & {\left[-\frac{\widetilde{A}(\alpha-1, \eta)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)\right] u_{0} } \\
& -\frac{(\bar{G} x)(\eta)-\sum_{i=1}^{m}(G x)\left(\xi_{i}\right)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+(G x)(t),
\end{aligned}
$$

where

$$
\begin{aligned}
& (G x)(t)=\frac{1}{\Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right) f(\tau, x(\tau)) d \tau d s, \\
& (\bar{G} x)(t)=\frac{t^{\rho \gamma}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}(G x)\left(t s^{\frac{1}{\rho}}\right) d s .
\end{aligned}
$$

Remark 6.3 Let $x \in C_{1-\alpha}$ be a solution of the inequality (6.1) with $\left(I_{0^{+}}^{1-\alpha} x\right)(0)=u_{0}$, $\sum_{i=1}^{m} x\left(\xi_{i}\right)=\left({ }^{\rho} I_{0^{+}}^{\gamma} x\right)(\eta)$. Then $x$ is a solution of the inequality $|x(t)-\tilde{x}(t)| \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma}$.

Indeed, by Remark 6.2, one can see

$$
\begin{aligned}
& { }^{c} D_{1^{-}}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) x(t)=f(t, x(t))+g(t), \\
& \left(I_{0^{+}}^{1-\alpha} x\right)(0)=u_{0}, \quad \sum_{i=1}^{m} x\left(\xi_{i}\right)=\left({ }^{\rho} I_{0^{+}}^{\gamma} x\right)(\eta) .
\end{aligned}
$$

Then we have

$$
\begin{aligned}
x(t)= & {\left[-\frac{\widetilde{A}(\alpha-1, \eta)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+t^{\alpha-1} E_{\alpha, \alpha}\left(-\lambda t^{\alpha}\right)\right] u_{0} } \\
& -\frac{(\bar{H} x)(\eta)-\sum_{i=1}^{m}(H x)\left(\xi_{i}\right)}{\widetilde{A}(\alpha, \eta)} t^{\alpha} E_{\alpha, \alpha+1}\left(-\lambda t^{\alpha}\right)+(H x)(t),
\end{aligned}
$$

where

$$
\begin{aligned}
& (H x)(t)=\frac{1}{\Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} E_{\alpha, \alpha}\left(-\lambda(t-s)^{\alpha}\right)[f(\tau, x(\tau))+g(\tau)] d \tau d s, \\
& (\bar{H} x)(t)=\frac{t^{\rho \gamma}}{\rho^{\gamma} \Gamma(\gamma)} \int_{0}^{1}(1-s)^{\gamma-1}(H x)\left(t s^{\frac{1}{\rho}}\right) d s .
\end{aligned}
$$

It is easy to check that $|x(t)-\tilde{x}(t)| \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma}$.

Theorem 6.4 Assume that (H3) is satisfied. If $\bar{M}<1$, then Eq. (1.1) is Ulam-Hyers stable.

Proof Let $x \in C_{1-\alpha}$ be a solution of the inequality (6.1) with $\left(I_{0^{+}}^{1-\alpha} x\right)(0)=u_{0}, \sum_{i=1}^{m} x\left(\xi_{i}\right)=$ $\left({ }^{\rho} I_{0^{+}}^{\gamma} x\right)(\eta) . y$ denotes the unique solution of the following problem:

$$
\left\{\begin{array}{l}
{ }^{c} D_{1^{-}}^{\beta}\left({ }^{L} D_{0^{+}}^{\alpha}+\lambda\right) y(t)=f(t, y(t)), \quad t \in J \\
\left(I_{0^{+}}^{1-\alpha} y\right)(0)=u_{0}, \quad \sum_{i=1}^{m} u\left(\xi_{i}\right)=\left({ }^{\rho} I_{0^{+}}^{\gamma} y\right)(\eta)
\end{array}\right.
$$

It follows from (H3) that

$$
\begin{aligned}
& |(G x)(t)-(G y)(t)| \\
& \quad \leq \frac{\bar{L}_{f}\|x-y\|_{1-\alpha}}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{t} \int_{s}^{1}(t-s)^{\alpha-1}(\tau-s)^{\beta-1} d \tau d s \leq \frac{\bar{L}_{f}\|x-y\|_{1-\alpha}}{\Gamma(\beta+1) \Gamma(\alpha+1)} \\
& |(\bar{G} x)(t)-(\bar{G} y)(t)| \\
& \quad \leq M_{3} \bar{L}_{f}\|x-y\|_{1-\alpha} \int_{0}^{1} \int_{0}^{t s} \int_{\tau}^{\frac{1}{\rho}} \int_{\tau}^{1}(1-s)^{\gamma-1}\left(t s^{\frac{1}{\rho}}-\tau\right)^{\alpha-1}(\zeta-\tau)^{\beta-1} d \zeta d \tau d s \\
& \quad \leq \frac{\bar{L}_{f}\|x-y\|_{1-\alpha}}{\rho^{\gamma} \Gamma(\beta+1) \Gamma(\alpha+1) \Gamma(\gamma+1)} .
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& |x(t)-y(t)| \\
& \quad \leq|x(t)-\tilde{x}(t)|+|\tilde{x}(t)-y(t)| \\
& \quad \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma}+\frac{|(\bar{G} x)(\eta)-(\bar{G} y)(\eta)|+\sum_{i=1}^{m}\left|(G x)\left(\xi_{i}\right)-(G y)\left(\xi_{i}\right)\right|}{|\widetilde{A}(\alpha, \eta)| \Gamma(\alpha+1)}+|(G x)(t)-(G y)(t)| \\
& \quad \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma}+\bar{M}\|x-y\|_{1-\alpha}
\end{aligned}
$$

which implies $\|x-y\|_{1-\alpha} \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma(1-\bar{M})}$, furthermore

$$
|x(t)-y(t)| \leq \frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma}+\frac{\bar{M} M_{3} \tilde{\epsilon}}{\alpha \beta \gamma(1-\bar{M})}=\frac{M_{3} \tilde{\epsilon}}{\alpha \beta \gamma(1-\bar{M})}
$$

that is, Eq. (1.1) is Ulam-Hyers stable.

## 7 Examples

In this section, we give two examples to illustrate our results.

Example 7.1 We consider the following boundary value problem:

$$
\left\{\begin{array}{l}
{ }^{c} D_{1^{-}}^{\frac{2}{5}}\left({ }^{L} D_{0^{+}}^{\frac{1}{5}}+2\right) u(t)=\frac{10}{\sqrt[5]{t}} \sin \left(7 t^{\frac{2}{3}}|u(t)|^{\frac{1}{2}}+3 t^{\frac{1}{4}}\right), \quad t \in J:=(0,1]  \tag{7.1}\\
\left(I_{0^{+}}^{\frac{4}{5}} u\right)(0)=u_{0}, \quad \sum_{i=1}^{100} u\left(\frac{1}{2^{i}}\right)=\left(\frac{7}{8} I_{0^{+}}^{2} u\right)\left(\frac{1}{3}\right) .
\end{array}\right.
$$

Corresponding to (1.1)-(1.2), we have $\alpha=\frac{1}{5}, \beta=\frac{2}{5}, \lambda=2, \rho=\frac{7}{8}, \gamma=2, \xi_{i}=\frac{1}{2^{i}}(i=$ $1,2, \ldots, 100), \eta=\frac{1}{3}$ and

$$
f(t, u(t))=\frac{10}{\sqrt[5]{t}} \sin \left(7 t^{\frac{2}{3}}|u(t)|^{\frac{1}{2}}+3 t^{\frac{1}{4}}\right)
$$

We define the space $C_{\frac{4}{5}}=\left\{u \in C(J, \mathbb{R}): t^{\frac{4}{5}} u(t) \in C([0,1], \mathbb{R})\right\}$ with the norm $\|u\|_{\frac{4}{5}}=$ $\max _{t \in[0,1]} t^{\frac{4}{5}}|u(t)|$.
Obviously, $|f(t, u(t))-f(t, \tilde{u}(t))| \leq 70|u(t)-\tilde{u}(t)|^{\frac{1}{2}}$ and $|f(t, 0)|=\frac{10}{\sqrt[5]{t}}\left|\sin 3 t^{\frac{1}{4}}\right| \leq 30$. By Theorem 5.3, the problem (7.1)-(7.2) has at least one solution.

Example 7.2 We consider the following boundary value problem for nonlinear fractional differential equation:

$$
\left\{\begin{array}{l}
{ }^{c} D_{1^{-}}^{\frac{1}{5}}\left({ }^{L} D_{0^{+}}^{\frac{3}{5}}+3\right) u(t)=\frac{1}{1+99 e^{\cdot}} \cdot \frac{u(t)}{|u(t)|+\sqrt{1+|u(t)|}}+\frac{1}{10}, \quad t \in J:=(0,1]  \tag{7.3}\\
\left(I_{0^{+}}^{\frac{2}{5}} u\right)(0)=u_{0}, \quad u\left(\frac{1}{5}\right)+u\left(\frac{1}{4}\right)=\left({ }^{2} I_{0^{+}}^{3} u\right)\left(\frac{1}{2}\right) .
\end{array}\right.
$$

Set

$$
f(t, u(t))=\frac{1}{1+99 e^{t}} \cdot \frac{u(t)}{|u(t)|+\sqrt{1+|u(t)|}}+\frac{1}{10} .
$$

For $t \in J$, we have

$$
\begin{aligned}
& |f(t, u(t))| \\
& \quad \leq \frac{1}{100} \frac{|u(t)|}{|u(t)|+(1+|u(t)|)^{\frac{1}{2}}}+\frac{1}{10}<\frac{1}{100}|u(t)|^{\frac{1}{2}}+\frac{1}{10}, \\
& |f(t, u)-f(t, \tilde{u})| \\
& \quad \leq \frac{1}{100}\left|\frac{u}{|u|+\sqrt{1+|u|}}-\frac{\tilde{u}}{|\tilde{u}|+\sqrt{1+|\tilde{u}|} \mid}\right| \\
& \quad \leq \frac{1}{100}\left[\frac{2|u-\tilde{u}||\tilde{u}|+|u-\tilde{u}| \sqrt{1+|\tilde{u}|}+|\tilde{u}| \mid \sqrt{1+|u|}-\sqrt{1+|\tilde{u}| \mid}}{(|u|+\sqrt{1+|u|})(|\tilde{u}|+\sqrt{1+|\tilde{u}|})}\right] \\
& \quad \leq \frac{1}{25}|u-\tilde{u}| .
\end{aligned}
$$

Let $\alpha=\frac{3}{5}, \beta=\frac{1}{5}, \lambda=3, \rho=2, \gamma=3, \xi_{1}=\frac{1}{5}, \xi_{2}=\frac{1}{4}, \eta=\frac{1}{2}, L_{f}^{\prime}=\frac{1}{100}, \widetilde{\sigma}=\frac{1}{2}, \tilde{M}_{f}=\frac{1}{10}$ and $\widetilde{L}_{f}=\frac{1}{25}$. We define the space $C_{\frac{2}{5}}=\left\{u \in C(J, \mathbb{R}): t^{\frac{2}{5}} u(t) \in C([0,1], \mathbb{R})\right\}$ with the norm $\|u\|_{\frac{2}{5}}=\max _{t \in[0,1]} t^{\frac{2}{5}}|u(t)|$.

By direct computation, we have

$$
\begin{aligned}
& A\left(\frac{3}{5}, \frac{1}{2}\right)=\frac{1}{2^{\frac{53}{5}}} \int_{0}^{1} s^{\frac{3}{10}}(1-s)^{2} E_{\frac{3}{5}, \frac{8}{5}}\left(-\frac{3 s^{\frac{3}{10}}}{2^{\frac{3}{5}}}\right) d s \approx 6.7 \times 10^{-5} ; \\
& \widetilde{A}\left(\frac{3}{5}, \frac{1}{2}\right)=A\left(\frac{3}{5}, \frac{1}{2}\right)-\left[\left(\frac{1}{5}\right)^{\frac{3}{5}} E_{\frac{3}{5}, \frac{8}{5}}\left(-3 \times\left(\frac{1}{5}\right)^{\frac{3}{5}}\right)+\left(\frac{1}{4}\right)^{\frac{3}{5}} E_{\frac{3}{5}, \frac{8}{5}}\left(-3 \times\left(\frac{1}{4}\right)^{\frac{3}{5}}\right)\right] \\
& \quad \approx-0.43 ; \\
& M_{2}=\frac{1}{\left|\widetilde{A}\left(\frac{3}{5}, \frac{1}{2}\right)\right| \Gamma\left(\frac{8}{5}\right)} \approx 2.61 ; \quad M_{3}=\frac{1}{8 \Gamma\left(\frac{3}{5}\right) \Gamma\left(\frac{1}{5}\right) \Gamma(3)} \approx 9.14 \times 10^{-3} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \left(\frac{1+M_{2} \sum_{i=1}^{m} \xi_{i}^{\alpha-1}}{\Gamma(\alpha) \Gamma(\beta)}+M_{2} M_{3} \eta^{\alpha-1} B\left(\gamma, \frac{\alpha-1+\rho}{\rho}\right)\right) \frac{B(\alpha, \alpha) \tilde{L}_{f}}{\beta} \\
& \quad=\frac{1}{5}\left[\frac{1+M_{2}\left(5^{\frac{2}{5}}+4^{\frac{2}{5}}\right)}{\Gamma\left(\frac{3}{5}\right) \Gamma\left(\frac{1}{5}\right)}+2^{\frac{2}{5}} M_{2} M_{3} B\left(3, \frac{4}{5}\right)\right] B\left(\frac{3}{5}, \frac{3}{5}\right) \approx 0.75<1 .
\end{aligned}
$$

Thus by Theorem 5.4, the problem (7.3)-(7.4) has a unique solution.

Example 7.3 We consider the following boundary value problem for nonlinear fractional differential equation:

$$
\left\{\begin{array}{l}
{ }^{c} D_{1^{-}}^{\frac{1}{6}}\left({ }^{L} D_{0^{+}}^{\frac{2}{3}}+2\right) u(t)=\frac{1}{10 \sqrt[9]{t}} \sin \left(1+t^{\frac{1}{2}} u(t)\right), \quad t \in J:=(0,1]  \tag{7.5}\\
\left(I_{0^{+}}^{\frac{1}{3}} u\right)(0)=u_{0}, \quad u\left(\frac{1}{2}\right)=\left({ }^{\frac{2}{3}} I_{0^{+}}^{5} u\right)\left(\frac{1}{10}\right) .
\end{array}\right.
$$

Set

$$
f(t, u(t))=\frac{1}{10 \sqrt[9]{t}} \sin \left(1+t^{\frac{1}{2}} u(t)\right)
$$

For $t \in J$, we have

$$
\left|f\left(t, t^{-\frac{1}{3}} u(t)\right)-f\left(t, t^{-\frac{1}{3}} \tilde{u}(t)\right)\right| \leq \frac{t^{\frac{1}{6}}}{10 \sqrt[9]{t}}|u(t)-\tilde{u}(t)| \leq \frac{1}{10}|u(t)-\tilde{u}(t)|
$$

Let $\alpha=\frac{2}{3}, \beta=\frac{1}{6}, \lambda=2, \rho=\frac{2}{3}, \gamma=5, \xi_{1}=\frac{1}{2}, \eta=\frac{1}{10}, \bar{L}_{f}=\frac{1}{10}$. We define the space $C_{\frac{1}{3}}=\{u \in$ $\left.C(J, \mathbb{R}): t^{\frac{1}{3}} u(t) \in C([0,1], \mathbb{R})\right\}$ with the norm $\|u\|_{\frac{1}{3}}=\max _{t \in[0,1]} t^{\frac{1}{3}}|u(t)|$.

By direct computation, we get

$$
\begin{aligned}
& A\left(\frac{2}{3}, \frac{1}{10}\right)=\frac{1}{10^{4} \Gamma(5)\left(\frac{2}{3}\right)^{5}} \int_{0}^{1} s(1-s)^{4} E_{\frac{2}{3}, \frac{5}{3}}\left(-\frac{2 s}{10^{\frac{2}{3}}}\right) d s \approx 1.07 \times 10^{-6} ; \\
& \widetilde{A}\left(\frac{2}{3}, \frac{1}{10}\right)=A\left(\frac{2}{3}, \frac{1}{10}\right)-\left(\frac{1}{2}\right)^{\frac{2}{3}} E_{\frac{2}{3}, \frac{5}{3}}\left(-2^{\frac{1}{3}}\right) \approx-0.33 ; M_{2}=\frac{1}{\left|\widetilde{A}\left(\frac{2}{3}, \frac{1}{10}\right)\right| \Gamma\left(\frac{5}{3}\right)} \approx 3.34 ; \\
& \bar{M}=\frac{\bar{L}_{f}}{\Gamma(\beta+1) \Gamma(\alpha+1)}\left[1+\frac{M_{2}\left(1+m \rho^{\gamma} \Gamma(\gamma+1)\right)}{\rho^{\gamma} \Gamma(\gamma+1)}\right] \\
& \quad=\frac{1}{10 \Gamma\left(\frac{7}{6}\right) \Gamma\left(\frac{5}{3}\right)}\left[1+\frac{M_{2}\left(1+\left(\frac{2}{3}\right)^{5} \Gamma(6)\right)}{\left(\frac{2}{3}\right)^{5} \Gamma(6)}\right] \approx 0.54<1 .
\end{aligned}
$$

Then by Theorem 6.4, the problem (7.5)-(7.6) has a unique solution and Eq. (7.5) is UlamHyers stable.

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The authors declare that they have no competing interests.

## Authors' contributions

The authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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