Li and Lai *Boundary Value Problems* (2020) 2020:102
https://doi.org/10.1186/s13661-020-01400-w

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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The entropy weak solution to a generalized Fornberg–Whitham equation

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Abstract

We investigate a nonlinear generalized Fornberg–Whitham equation. The key element is that we derive an $L^2(\mathbb{R})$ conservation law for solutions of the equation. We establish several estimates by utilizing the $L^2(\mathbb{R})$ conservation law. These estimates lead to the proof of the existence and uniqueness of entropy weak solution of the equation in the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

MSC: 35G25; 35L05

Keywords: Entropy weak solutions; Existence and uniqueness; The Fornberg–Whitham equation

1 Introduction

Consider the nonlinear partial differential equation

$$V_t - V_{txx} + kV_x + mVV_x = \frac{9}{2}V_xV_{xx} + \frac{3}{2}VV_{xxx}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1}$$

where $m > 0$ and k are constants. Assume that $V_0(x) = V(0,x)$ is an initial value to Eq. (1). We establish the inequality

$$c_1\|V_0\|_{L^2(\mathbb{R})} \leq \|V\|_{L^2(\mathbb{R})} \leq c_2\|V_0\|_{L^2(\mathbb{R})}, \tag{2}$$

where $c_1 > 0$ and $c_2 > 0$ are constants independent of t.

If $k = -1$ and $m = \frac{3}{2}$, then Eq. (1) becomes the Fornberg–Whitham equation [1, 2]

$$V_t - V_{txx} - V_x + \frac{3}{2}VV_x = \frac{9}{2}V_xV_{xx} + \frac{3}{2}VV_{xxx}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3}$$

Recently, Holmes and Thompson [3] proved the well-posedness of Eq. (3) in the Besov space in the periodic and nonperiodic cases and established a Cauchy–Kowalevski-type theorem for Eq. (3) to show the existence and uniqueness of analytic solutions. The blow-up criterion for the solutions is given in [3]. Using several estimates derived from the Fornberg–Whitham equation itself and the conclusions in [4], Haziot [5] found sufficient conditions on the initial data to guarantee the wave breaking of solutions of Eq. (3). Gao et al. [6] proved the L^1 local stability of strong solutions of Eq. (3).

We know that the dynamic properties of the Fornberg–Whitham model are related to those of the Cammassa–Holm equation[7], Degasperis–Processi equation [8], and Novikov equation[9], which have peakon solutions (see[10–13]). Other dynamical properties of the Camassa–Holm, Degasperis–Processi, and Novikov equations can be found in [14–21] and the references therein.

We write the Cauchy problem for Eq. (1):

$$\begin{cases} V_t - V_{txx} = -kV_x - mVV_x + \frac{9}{2}V_xV_{xx} + \frac{3}{2}VV_{xxx} \\ \qquad\qquad = -kV_x - (\frac{m}{2}V^2)_x + \frac{3}{4}\partial^3_{xxx}V^2, \\ V(0,x) = V_0(x), \end{cases} \tag{4}$$

which is equivalent to

$$\begin{cases} V_t + \frac{3}{2}VV_x + \partial_x Q(t,x) = 0, \\ V(0,x) = V_0(x), \end{cases} \tag{5}$$

where $m > 0$ is a constant, $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, and $Q(t,x) = [kV + (\frac{m}{2} - \frac{3}{4})V^2(t,x)]$.

Motivated by the desire to further investigate the Fornberg–Whitham equation (3), the objective of this work is to establish the existence and uniqueness of entropy solutions for Eq. (1). Using the viscous approximation techniques and assuming that the initial value $V_0(x)$ belongs to the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we prove the well-posedness of the entropy solutions. The novelty is that we derive a new $L^2(\mathbb{R})$ conservation law for Eq. (1). The ideas for obtaining our main result come from those in [22].

The structure of this paper is as follows. In Sect. 2, we establish several estimates for the viscous approximations of problem (5), and in Sect. 3, we present our main results and their proofs.

2 Estimates of viscous approximations

Set

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

$\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ with $0 < \varepsilon < 1$, and $V_{0,\varepsilon} = \phi_\varepsilon \star V_0 = \int_{\mathbb{R}} \phi_\varepsilon(x-y)V_0(y)\,dy$. We have $V_{0,\varepsilon} \in C^\infty$ for any $V_0 \in H^s$ with $s \geq 0$.

For conciseness in this paper, we let c denote an arbitrary positive constant, which is independent of parameter ε and time t.

For a smooth function $V_{0,\varepsilon}$ and $s \geq 0$, we have

$$\|V_{0,\varepsilon}\|_{L^p(\mathbb{R})} \leq c\|V_0\|_{L^p(\mathbb{R})} \quad \text{for } 1 \leq p < \infty,$$

$$V_{0,\varepsilon} \to V_0 \quad (\varepsilon \to 0) \text{ in } L^p(\mathbb{R}) \text{ for } 1 \leq p < \infty,$$

$$\|V_{0,\varepsilon}\|_{H^q} \leq c\|V_0\|_{H^s} \quad \text{if } q \leq s.$$

For problem (4), we will discuss the limiting behavior of a sequence of smooth functions $\{V_\varepsilon\}_{\varepsilon>0}$, where each function V_ε satisfies the viscous problem

$$\begin{cases} \partial_t V_\varepsilon - \partial_{txx}^3 V_\varepsilon + k\partial_x V_\varepsilon + mV_\varepsilon \partial_x V_\varepsilon \\ \quad = \frac{9}{2} V_\varepsilon \partial_{xxx}^3 V_\varepsilon + \frac{3}{2}\partial_x V_\varepsilon \partial_{xx}^2 V_\varepsilon + \varepsilon \partial_{xx}^2 V_\varepsilon - \varepsilon \partial_{xxxx}^4 V_\varepsilon, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ V_\varepsilon(0,x) = V_{0,\varepsilon}(x), \quad x \in \mathbb{R}, \end{cases} \tag{6}$$

or, in the equivalent form,

$$\begin{cases} \partial_t V_\varepsilon + \frac{3}{4}\partial_x(V_\varepsilon^2) + \partial_x Q_\varepsilon(t,x) = \varepsilon \partial_{xx}^2 V_\varepsilon, \\ Q_\varepsilon(t,x) = \Lambda^{-2}[kV_\varepsilon + (\frac{m}{2} - \frac{3}{4})V_\varepsilon^2], \\ V_\varepsilon(0,x) = V_{0,\varepsilon}(x), \end{cases} \tag{7}$$

where

$$Q_\varepsilon(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left[kV_\varepsilon(t,y) + \left(\frac{m}{2} - \frac{3}{4}\right)V_\varepsilon^2(t,y)\right] dy. \tag{8}$$

Lemma 2.1 *If $V_0 \in L^2(\mathbb{R})$, then for any fixed $\varepsilon > 0$, there exists a unique global smooth solution $V_\varepsilon = V_\varepsilon(t,x)$ to the Cauchy problem (6) belonging to $C([0,\infty); H^s(\mathbb{R}))$ with $s \geq 0$.*

Proof Using Theorem 2.3 in [23], we directly get the result of this lemma. □

Now we give the following lemma, which plays a key role in our investigation of Eq. (1).

Lemma 2.2 *Suppose that V_ε is a solution of problem (7), $V_0 \in L^2(\mathbb{R})$, and $t > 0$. Then*

$$c_1\|V_0\|_{L^2(\mathbb{R})} \leq \big\|V_\varepsilon(t,\cdot)\big\|_{L^2(\mathbb{R})} \leq c_2\|V_0\|_{L^2(\mathbb{R})}, \tag{9}$$

$$\varepsilon\int_0^t \big\|\partial_x V_\varepsilon(\tau,\cdot)\big\|_{L^2(\mathbb{R})}^2 d\tau \leq c_3\|V_0\|_{L^2(\mathbb{R})}^2, \tag{10}$$

where c_1, c_2, and c_3 are positive constants independent of ε and t.

Proof Let $g_\varepsilon = (\frac{2m}{3} - \partial_{xx}^2)^{-1}V_\varepsilon$. We have

$$\frac{2m}{3}g_\varepsilon - \partial_{xx}^2 g_\varepsilon = V_\varepsilon. \tag{11}$$

Multiplying the first equation of problem (7) by $g_\varepsilon - \partial_{xx}^2 g_\varepsilon$ and integrating over $\mathbb{R}$ yields

$$\begin{aligned} &\int_{\mathbb{R}} \partial_t V_\varepsilon\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx - \varepsilon\int_{\mathbb{R}} \partial_{xx}^2 V_\varepsilon\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx \\ &\quad = -\frac{3}{2}\int_{\mathbb{R}} V_\varepsilon \partial_x V_\varepsilon\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx - \int_{\mathbb{R}} \partial_x Q_\varepsilon(t,x)\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx. \end{aligned} \tag{12}$$

We have

$$
\begin{aligned}
&\int_{\mathbb{R}} \partial_t V_\varepsilon \big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 V_\varepsilon \big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx \\
&\quad = \int_{\mathbb{R}} \left(\frac{2m}{3}\partial_t g_\varepsilon - \partial_{txx}^3 g_\varepsilon\right)\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx - \varepsilon \int_{\mathbb{R}} \left(\frac{2m}{3}\partial_{xx}^2 g_\varepsilon - \partial_{xxxx}^4 g_\varepsilon\right)\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx \\
&\quad = \int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon \partial_t g_\varepsilon - g_\varepsilon \partial_{txx}^3 g_\varepsilon - \frac{2m}{3}\partial_t g_\varepsilon \partial_{xx}^2 g_\varepsilon + \partial_{xx}^2 g_\varepsilon \partial_{txx}^3 g_\varepsilon\right)dx \\
&\qquad - \varepsilon \int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon \partial_{xx}^2 g_\varepsilon - \frac{2m}{3}\big(\partial_{xx}^2 g_\varepsilon\big)^2 - g_\varepsilon \partial_{xxxx}^4 g_\varepsilon + \partial_{xx}^2 g_\varepsilon \partial_{xxxx}^4 g_\varepsilon\right)dx \\
&\quad = \int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon \partial_t g_\varepsilon - \left(\frac{2m}{3}+1\right) g_\varepsilon \partial_{txx}^3 g_\varepsilon + \partial_{xx}^2 g_\varepsilon \partial_{txx}^3 g_\varepsilon\right)dx \\
&\qquad - \varepsilon \int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon \partial_{xx}^2 g_\varepsilon - \left(\frac{2m}{3}+1\right) g_\varepsilon \partial_{xxxx}^4 g_\varepsilon + \partial_{xx}^2 g_\varepsilon \partial_{xxxx}^4 g_\varepsilon\right)dx \\
&\quad = \int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon \partial_t g_\varepsilon + \left(\frac{2m}{3}+1\right) \partial_x g_\varepsilon \partial_{tx}^2 g_\varepsilon + \partial_{xx}^2 g_\varepsilon \partial_{txx}^3 g_\varepsilon\right)dx \\
&\qquad - \varepsilon \int_{\mathbb{R}} \left(-\frac{2m}{3} \partial_x g_\varepsilon \partial_x g_\varepsilon - \left(\frac{2m}{3}+1\right) \partial_{xx}^2 g_\varepsilon \partial_{xx}^2 g_\varepsilon - \partial_{xxx}^3 g_\varepsilon \partial_{xxx}^3 g_\varepsilon\right)dx \\
&\quad = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon^2 + \left(\frac{2m}{3}+1\right)(\partial_x g_\varepsilon)^2 + \big(\partial_{xx}^2 g_\varepsilon\big)^2\right)dx \\
&\qquad + \varepsilon \int_{\mathbb{R}} \left(\frac{2m}{3}(\partial_x g_\varepsilon)^2 + \left(\frac{2m}{3}+1\right)\big(\partial_{xx}^2 g_\varepsilon\big)^2 + \big(\partial_{xxx}^3 g_\varepsilon\big)^2\right)dx.
\end{aligned}
\tag{13}
$$

For the right-hand side of (13), integrating by parts and using (11) result in

$$
\begin{aligned}
&-\frac{3}{2}\int_{\mathbb{R}} V_\varepsilon \partial_x V_\varepsilon \big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx - \int_{\mathbb{R}} \partial_x Q_\varepsilon(t,x)\big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx \\
&\quad = -\frac{3}{2}\int_{\mathbb{R}} V_\varepsilon \partial_x V_\varepsilon \big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx + \int_{\mathbb{R}} \big(Q_\varepsilon - \partial_{xx}^2 Q_\varepsilon\big)(t,x)\partial_x g_\varepsilon\,dx \\
&\quad = -\frac{3}{2}\int_{\mathbb{R}} V_\varepsilon \partial_x V_\varepsilon \big(g_\varepsilon - \partial_{xx}^2 g_\varepsilon\big)\,dx + \int_{\mathbb{R}} \left[kV_\varepsilon + \left(\frac{m}{2}-\frac{3}{4}\right)V_\varepsilon^2\right]\partial_x g_\varepsilon\,dx \\
&\quad = \frac{3}{4}\int_{\mathbb{R}} \partial_x\big(V_\varepsilon^2\big)\partial_{xx}^2 g_\varepsilon\,dx + \frac{m}{2}\int_{\mathbb{R}} V_\varepsilon^2 \partial_x g_\varepsilon\,dx + k\int_{\mathbb{R}} \left(\frac{2m}{3} g_\varepsilon - \partial_{xx} g_\varepsilon\right)\partial_x g_\varepsilon\,dx \\
&\quad = \frac{3}{4}\int_{\mathbb{R}} \partial_x\big(V_\varepsilon^2\big)\left[\frac{2m}{3} g_\varepsilon - V_\varepsilon\right]dx + \frac{m}{2}\int_{\mathbb{R}} V_\varepsilon^2 \partial_x g_\varepsilon\,dx + 0 \\
&\quad = -\frac{3}{4}\int_{\mathbb{R}} V_\varepsilon^2 \partial_x V_\varepsilon\,dx = 0.
\end{aligned}
\tag{14}
$$

From (12), (13), and (14) we conclude that

$$
\begin{aligned}
&\frac{2m}{3}\|g_\varepsilon\|_{L^2}^2 + \left(\frac{2m}{3}+1\right)\|\partial_x g_\varepsilon\|_{L^2}^2 + \big\|\partial_{xx}^2 g_\varepsilon\big\|_{L^2}^2 \\
&\qquad + 2\varepsilon\int_0^t \left(\frac{2m}{3}\|\partial_x g_\varepsilon\|_{L^2}^2 + \left(\frac{2m}{3}+1\right)\big\|\partial_{xx}^2 g_\varepsilon\big\|_{L^2}^2 + \big\|\partial_{xxx}^3 g_\varepsilon\big\|_{L^2}^2\right)d\tau \\
&\quad = \frac{2m}{3}\big\|g_\varepsilon(0,\cdot)\big\|_{L^2}^2 + \left(\frac{2m}{3}+1\right)\big\|\partial_x g_\varepsilon(0,\cdot)\big\|_{L^2}^2 + \big\|\partial_{xx}^2 g_\varepsilon(0,\cdot)\big\|_{L^2}^2.
\end{aligned}
\tag{15}
$$

Using the smoothness of the function $V_{0,\varepsilon}$, we have

$$\|g_\varepsilon(0,\cdot)\|_{L^2}, \|\partial_x g_\varepsilon(0,\cdot)\|_{L^2}, \|\partial_{xx}^2 g_\varepsilon(0,\cdot)\|_{L^2} \le c\|V_{0,\varepsilon}\|_{L^2} \le c\|V_0\|_{L^2}.$$

It follows from (11) that

$$\begin{aligned}\|V_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}}\left(-\partial_{xx}^2 g_\varepsilon + \frac{2m}{3}g_\varepsilon\right)^2 dx\\ &= \int_{\mathbb{R}}\left(\partial_{xx}^2 g_\varepsilon\right)^2 dx - \frac{4m}{3}\int_{\mathbb{R}} g_\varepsilon \partial_{xx}^2 g_\varepsilon\, dx + \frac{4m^2}{9}\int_{\mathbb{R}} g_\varepsilon^2\, dx\\ &= \int_{\mathbb{R}}\left(\partial_{xx}^2 g_\varepsilon\right)^2 dx + \frac{4m}{3}\int_{\mathbb{R}} (\partial_x g_\varepsilon)^2\, dx + \frac{4m^2}{9}\int_{\mathbb{R}} g_\varepsilon^2\, dx.\end{aligned} \tag{16}$$

Using (15) and (16), we derive that there exist constants c_1 and c_2 such that

$$c_1\|V_0\|_{L^2(\mathbb{R})} \le \|V_\varepsilon\|_{L^2(\mathbb{R})} \le c_2\|V_0\|_{L^2(\mathbb{R})} \tag{17}$$

and

$$\begin{aligned}\varepsilon\int_0^t \|\partial_x V_\varepsilon\|_{L^2}^2\, d\tau &\le 2\varepsilon\int_0^t \left(\|\partial_{xxx}^3 g_\varepsilon\|_{L^2}^2 + 2\left(\frac{2m}{3}\right)^2 \varepsilon\|\partial_x g_\varepsilon\|_{L^2}^2\right) d\tau\\ &\le \varepsilon c\int_0^t \left(\frac{2m}{3}\|\partial_x g_\varepsilon\|_{L^2}^2 + \left(\frac{2m}{3}+1\right)\|\partial_{xx}^2 g_\varepsilon\|_{L^2}^2 + \|\partial_{xxx}^3 g_\varepsilon\|_{L^2}^2\right) d\tau\\ &\le \varepsilon c\left(\|g_\varepsilon(0,\cdot)\|_{L^2}^2 + \|\partial_x g_\varepsilon(0,\cdot)\|_{L^2}^2 + \|\partial_{xx}^2 g_\varepsilon(0,\cdot)\|_{L^2}^2\right)\\ &\le c\|V_{0,\varepsilon}\|_{L^2}^2\\ &\le c\|V_0\|_{L^2}^2.\end{aligned} \tag{18}$$

The proof of Lemma 2.2 follows from (17) and (18). □

Letting $\varepsilon = 0$ in the proof of Lemma 2.2, for Eq. (1), we obtain inequality (2). Using Lemma 2.2, we give the following conclusion for the term $Q_\varepsilon(t,x)$.

Lemma 2.3 *If $V_0 \in L^2(\mathbb{R})$, then*

$$\|Q_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}, \quad \|\partial_x Q_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le c\left(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2\right), \tag{19}$$

$$\|Q_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}, \quad \|\partial_x Q_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \le c\left(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2\right). \tag{20}$$

Proof We have

$$Q_\varepsilon(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left[kV_\varepsilon(t,y) + \left(\frac{m}{2}-\frac{3}{4}\right)V_\varepsilon^2(t,y)\right] dy \tag{21}$$

and

$$\partial_x Q_\varepsilon(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}sign(x-y)\left[kV_\varepsilon(t,y) + \left(\frac{m}{2}-\frac{3}{4}\right)V_\varepsilon^2(t,y)\right] dy. \tag{22}$$

Using the Schwarz inequality leads to

$$\begin{aligned}\left|\int_{\mathbb{R}} e^{-\frac{1}{2}|x-y|} V_\varepsilon(t,y)\,dy\right| &\le \left(\int_{\mathbb{R}} e^{-|x-y|}\,dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} V_\varepsilon^2(t,y)\,dy\right)^{\frac{1}{2}} \\ &\le c\|V_0\|_{L^2}.\end{aligned} \tag{23}$$

Utilizing the Tonelli theorem and (23), we get

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-|x-y|} V_\varepsilon(t,y)\,dy\right| dx &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-\frac{1}{2}|x-y|} V_\varepsilon(t,y)\,dy\right| e^{-\frac{1}{2}|x-y|}\,dx \\ &\le \|V_0\|_{L^2}\int_{\mathbb{R}} e^{-\frac{1}{2}|x-y|}\,dx \le c\|V_0\|_{L^2}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-|x-y|} V_\varepsilon^2(t,y)\,dy\right| dx &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-|x-y|} V_\varepsilon^2(t,y)\,dy\right| dx \\ &\le c\|V_0\|_{L^2}^2.\end{aligned} \tag{25}$$

From (21)–(25) and Lemma 2.2 we derive that (19) and (20) hold. The proof is finished. □

If $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then we derive $V_0 \in L^2(\mathbb{R})$.

Lemma 2.4 *If $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then*

$$\left\|V_\varepsilon(t,\cdot)\right\|_{L^\infty} \le \|V_0\|_{L^\infty} + ct\left(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2\right). \tag{26}$$

Proof Using the first equation of problem (7), we have

$$\partial_t V_\varepsilon + \frac{3}{2} V_\varepsilon \partial_x V_\varepsilon - \varepsilon \partial_{xx} V_\varepsilon = -\partial_x Q_\varepsilon. \tag{27}$$

Applying Lemma 2.3 yields

$$\|\partial_x Q_\varepsilon\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \le c\left(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2\right). \tag{28}$$

Setting $K(t) = \|V_0\|_{L^\infty(\mathbb{R})} + ct(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2)$, we get

$$\frac{dK}{dt} = c\left(\|V_0\|_{L^2} + \|V_0\|_{L^2}^2\right). \tag{29}$$

Since $\|V_\varepsilon(0,x)\|_{L^\infty(\mathbb{R})} \le K(0)$, using the comparison principle, we derive that (26) holds. □

Applying Lemma 2.4 and the methods presented in [22], we obtain the following result.

Lemma 2.5 (Oleinik-type estimate) *Let $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $T > 0$. Then*

$$\partial_x V_\varepsilon(t,x) \le \frac{1}{t} + C_T, \quad x \in R, 0 < t \le T, \tag{30}$$

where the constant C_T depends on T.

We omit the proof of this lemma since it is similar to that of Lemma 2.11 in [22].

We state the concepts of weak solution and entropy weak solution (see [22, 24]).

Definition 2.6 (Weak solution) A function $V:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is called a weak solution of the Cauchy problem (5) if

(i) $V\in L^\infty(\mathbb{R}_+;L^2(\mathbb{R}))$, and

(ii) $\partial_t V+\frac{3}{4}\partial_x(V^2)+\partial_x Q(t,x)=0$ in $D'([0,\infty)\times\mathbb{R})$, that is, for all $f\in C_c^\infty([0,\infty)\times\mathbb{R})$, we have the identity

$$\int_{R_+}\int_{\mathbb{R}}\left(V\partial_t f+\frac{3V^2}{4}\partial_x f-\partial_x Q(t,x)f\right)dx\,dt+\int_{\mathbb{R}}V_0(x)f(0,x)\,dx=0. \tag{31}$$

Definition 2.7 (Entropy weak solution) We call a function $V:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ an entropy weak solution of Cauchy problem (5) if

(i) V is a weak solution in the sense of Definition 2.6,

(ii) $V\in L^\infty([0,T]\times\mathbb{R})$ for any $T>0$, and

(iii) for any convex C^2 entropy function $\eta:\mathbb{R}\to\mathbb{R}$ with corresponding entropy flux $q:\mathbb{R}\to\mathbb{R}$ defined by $q'(V)=\frac{3}{4}\eta'(V)V$, we have

$$\partial_t\eta(V)+\partial_x q(V)+\eta'(V)\partial_x Q\le 0\quad\text{in } D'\big([0,\infty)\times\mathbb{R}\big), \tag{32}$$

that is, for all $f\in C_c^\infty([0,\infty)\times\mathbb{R})$, $f(t,x)\ge 0$, we have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\big(\eta(V)\partial_t f+q(V)\partial_x f-\eta'(V)\partial_x Qf\big)\,dx\,dt+\int_{\mathbb{R}}\eta\big(V_0(x)\big)f(0,x)\,dx\ge 0. \tag{33}$$

Remark 2.8 As stated by Coclite and Karsen [22], by a standard argument we get that the Kruzkov entropies/entropy fluxes

$$\eta(V)=|V-k_1|,\qquad q(V):=\frac{3}{4}\operatorname{sign}(V-k_1)\big(V^2-k_1^2\big), \tag{34}$$

where k_1 is an arbitrary constant, satisfy (33).

3 Main results

We state the following $L^1(\mathbb{R})$ stability result of entropy weak solutions for Eq. (1).

Theorem 3.1 (L^1-stability) *Assume that $V_1(t,x)$ and $V_2(t,x)$ are two entropy weak solutions of problem (5) with initial data $V_{01}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $V_{02}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, respectively. Let $T>0$ be the maximal existence time of solutions $V_1(t,x)$ and $V_2(t,x)$. Then*

$$\big\|V_1(t,\cdot)-V_2(t,\cdot)\big\|_{L^1(\mathbb{R})}\le Ce^{Ct}\int_{-\infty}^{\infty}\big|V_{01}(x)-V_{02}(x)\big|\,dx,\quad t\in[0,T], \tag{35}$$

where C depends on $V_{01}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $V_{02}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ and T.

The proof of Theorem 3.1 is the standard argument presented in Gao et al. [6]. We omit its proof.

We employ the compensated compactness method in [25, 26] to discuss the strong convergence of a subsequence of the viscosity approximations.

Lemma 3.2 *Let $\{V_\varepsilon\}_{\varepsilon>0}$ be a family of functions defined on $(0,\infty)\times\mathbb{R}$ such that*

$$\|V_\varepsilon\|_{L^\infty}\le C_T,$$

where the constant $C_T>0$ depends on T, and the family

$$\left\{\partial_t\eta(V_\varepsilon)+\partial_x q(V_\varepsilon)\right\}_{\varepsilon>0}$$

is compact in $H^{-1}_{\mathrm{loc}}((0,\infty)\times\mathbb{R})$ for any convex $\eta\in C^2(\mathbb{R})$, where $q(V)=aV\eta'(V)$ with constant $a>0$. Then there exist a sequence $\{\varepsilon_n\}_{n\in N}$, $\varepsilon_n\to 0$, and a function $V\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that

$$V_{\varepsilon_n}\to V \quad \textit{a.e. and in } L^p_{\mathrm{loc}}\big((0,\infty)\times\mathbb{R}\big), 1\le p<\infty.$$

Lemma 3.2 can be found in [25] or [26].

Lemma 3.3 ([27]) *Suppose that Ω is a bounded open subset of R^H, $H\ge 2$. Assume that the sequence $\{M_n\}_{n=1}^\infty$ of distributions is bounded in $W^{-1,\infty}(\Omega)$ and*

$$M_n=M_n^{(1)}+M_n^{(2)},$$

where $\{M_n^{(1)}\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$, and $\{M_n^{(2)}\}_{n=1}^\infty$ lies in a bounded subset of $L^1_{\mathrm{loc}}(\Omega)$. Then $\{M_n\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

Lemma 3.4 *Let $V_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then there exists a subsequence V_{ε_n}, $n\in 1,2,3,\dots$, of $\{V_\varepsilon\}_{\varepsilon>0}$ and a limit function*

$$V\in L^\infty\big(\mathbb{R}_+;L^2(\mathbb{R})\big)\cap L^\infty\big((0,T);L^\infty\cap L^1(\mathbb{R})\big) \tag{36}$$

such that

$$V_{\varepsilon_k}\to V \quad \textit{in } L^p\big((0,T]\times\mathbb{R}\big), \forall p\in[1,\infty). \tag{37}$$

Proof Suppose that $\eta:\mathbb{R}\to\mathbb{R}$ is an arbitrary convex C^2 entropy function that is compactly supported, and $q:\mathbb{R}\to\mathbb{R}$ is the corresponding entropy flux defined by $q'(V)=\frac{3}{4}\eta'(V)V$. We set

$$\partial_t\eta(V_\varepsilon)+\partial_x q(V_\varepsilon)=M_\varepsilon^{(1)}+M_\varepsilon^{(2)}, \tag{38}$$

where

$$\begin{cases} M_\varepsilon^{(1)}=\varepsilon\partial_{xx}^2\eta(V_\varepsilon),\\ M_\varepsilon^{(2)}=-\varepsilon\eta''(V_\varepsilon)(\partial_x V_\varepsilon)^2-\eta'(V_\varepsilon)\partial_x Q_\varepsilon(t,x).\end{cases} \tag{39}$$

We claim that

$$\begin{cases} M_\varepsilon^{(1)}\to 0 \quad \text{in } H^{-1}([0,T]\times\mathbb{R}), T>0,\\ M_\varepsilon^{(2)} \quad \text{is uniformly bounded in } L^1([0,T]\times\mathbb{R}).\end{cases} \tag{40}$$

Using Lemmas 2.2–2.5 yields

$$\left\|\varepsilon \partial_{xx}^2 \eta(V_\varepsilon)\right\|_{H^{-1}(\mathbb{R}_+\times\mathbb{R})} \le \sqrt{\varepsilon} c \left\|\eta'\right\|_{L^\infty} \|V_0\|_{L^2(\mathbb{R})} \to 0, \tag{41}$$

$$\left\|\varepsilon \eta''(V_\varepsilon)(\partial_x V_\varepsilon)^2\right\|_{L^1(\mathbb{R}_+\times\mathbb{R})} \le c\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\|V_0\|_{L^2(\mathbb{R})}, \tag{42}$$

$$\left\|\eta'(V_\varepsilon)\right\|_{L^1((0,T)\times\mathbb{R})} \le c\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\|V_0\|_{L^2(\mathbb{R})}. \tag{43}$$

Therefore we know that (40) holds. Using Lemmas 3.2 and 3.3, we confirm that there exists a subsequence $\{V_{\varepsilon_n}\}$ and a limit function V satisfying (36) such that, as $n \to \infty$,

$$V_{\varepsilon_n} \to V \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}) \text{ for any } p \in [1,\infty), \tag{44}$$

$$\text{and} \quad V_{\varepsilon_n} \to V \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}. \tag{45}$$

Using Lemma 2.5, from (44) and (45) we obtain (37). The proof is finished. □

Lemma 3.5 *If $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then there exists a function $Q(t,x) = [kV + (\frac{m}{2} - \frac{3}{4})V^2(t,x)]$ such that*

$$Q_{\varepsilon_n} \to Q \quad \textit{in } L^p\big([0,T); W^{1,p}(\mathbb{R})\big), T > 0, \forall p \in [1,\infty), \tag{46}$$

where the sequence ε_n, and the function V are constructed in Lemma 3.4.

We omit the proof of Lemma 3.5 since it is similar to that of Lemma 4.4 in [22].

Theorem 3.6 *Let $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then there exists at least one entropy weak solution to problem (5).*

Proof If $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, then from (31) we get

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\left(V_\varepsilon \partial_t f + \frac{3}{4}V_\varepsilon^2 \partial_x f - \partial_x Q_\varepsilon f + \varepsilon V_\varepsilon \partial_{xx}^2 f\right) dx\,dt + \int_{\mathbb{R}} V_{0,\varepsilon} f(0,x)\,dx = 0. \tag{47}$$

Using Lemmas 3.4, we make sure that the function V presented in Lemma 3.4 is a weak solution of problem (5) in the sense of Definition 2.6. We have to verify that V satisfies the entropy inequalities in Definition 2.7. Let $\eta \in C^2(\mathbb{R})$ be a convex entropy with flux q defined by $q'(V) = \frac{3}{4}V\eta'(V)$. Using the convexity of η and problem (7) results in

$$\partial_t \eta(V_\varepsilon) + \partial_x q(V_\varepsilon) + \eta'(V_\varepsilon)\partial_x Q_\varepsilon = \varepsilon \partial_{xx}^2 \eta(V_\varepsilon) - \varepsilon \eta''(V_\varepsilon)(\partial_x V_\varepsilon)^2 \le \varepsilon \partial_{xx}^2 \eta(V_\varepsilon). \tag{48}$$

Thus by Lemmas 3.4 and 3.5 it follows that the entropy inequality holds. The proof is finished. □

From Theorems 3.1 and 3.6 we have the following:

Theorem 3.7 *Let $V_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the Cauchy problem (5) has a unique entropy weak solution in the sense of Definition 2.7.*

Funding
This work is supported by the National Natural Science Foundation of China (No. 11471263).

List of abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The authors contributed equally to the writing of this paper. They read and approved the final manuscript.

Publisher's Note

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Received: 16 December 2019 Accepted: 15 May 2020 Published online: 24 May 2020

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