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Existence of least energy nodal solution for Kirchhoff–Schrödinger–Poisson system with potential vanishing

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Abstract

This paper deals with the following Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u + \phi u = K(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where a, b are positive constants, $K(x), V(x)$ are positive continuous functions vanishing at infinity, and $f(u)$ is a continuous function. Using the Nehari manifold and variational methods, we prove that this problem has a least energy nodal solution. Furthermore, if f is an odd function, then we obtain that the equation has infinitely many nontrivial solutions.

Keywords: Potential vanishing; Nehari manifold; Least energy nodal solution

1 Introduction

In this paper, we discuss the existence of a least energy nodal solution of the following Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} -(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u + \phi u = K(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where a, b are positive constants, $K(x), V(x)$ are positive continuous functions vanishing at infinity, and $f(u)$ is a continuous function.

When $a = 1$ and $b = 0$, system (1.1) stems from the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \quad (1.2)$$

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Recently, many authors paid their attentions to finding nodal solutions to problems like (1.2), and indeed some interesting results were obtained; see, for example, [2–4, 7, 17, 18, 22, 23, 30, 36–38, 41, 50] and the references therein.

Note that system (1.1) is also related to the following Kirchhoff-type equations:

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^3. \quad (1.3)$$

Equation (1.3) has arisen the interest of many mathematicians. Especially, there are many papers on nodal solutions to problems like (1.3) [6, 9, 11–13, 15, 20, 21, 25, 26, 29, 31, 33, 34, 39, 40, 46–48].

Since there are both nonlocal operator and nonlocal nonlinear terms, the study of system (1.1) becomes more complicated. In recent years, some scholars began to show interest to problem like (1.1); see [8, 10, 14, 19, 24, 27, 43, 44, 49] and references therein. However, to the best of our knowledge, few papers considered nodal solutions to problem like (1.1). Via a gluing method, Deng and Yang [10] studied the nodal solutions for system (1.1) with $f(u) = |u|^{p-2}u$, $p \in (4, 6)$. Wang, Li, and Hao [35] studied the existence and asymptotic behavior of a least energy nodal solution for system (1.1) by using the constraint variation methods.

Inspired by the works mentioned, especially by [10, 13, 15, 35, 45], in this paper, we find the nodal solutions to system (1.1) under some weaker assumptions on f . As in [1], we say that $(V, K) \in \mathcal{K}$ if continuous functions $V, K : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfy the following conditions:

(VK₀) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$;

(VK₁) If $\{A_n\}_n \subset \mathbb{R}^3$ is a sequence of Borel sets such that their Lebesgue measures $|A_n| \leq R$ for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r \rightarrow +\infty} \int_{A_n \cap B_r^c(0)} K(x) = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

Moreover, one of the following two conditions holds:

(VK₂) $\frac{K}{V} \in L^\infty(\mathbb{R}^3)$;

or

(VK₃) there exists $p \in (2, 6)$ so that

$$\frac{K(x)}{V(x)^{\frac{6-p}{4}}} \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

As for the function f , we assume that $f \in C(\mathbb{R}, \mathbb{R})$ and satisfies the following conditions:

(f₁) $\lim_{t \rightarrow 0} \frac{f(t)}{t^3} = 0$ if (VK₂) holds, and $\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{p-1}} = 0$ for some $p \in (4, 6)$ if (VK₃) holds;

(f₂) $\lim_{|t| \rightarrow +\infty} \frac{f(t)}{t^5} = 0$;

(f₃) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{t^4} = +\infty$, where $F(t) = \int_0^t f(s) ds$;

(f₄) $\frac{f(t)}{t^3}$ is nondecreasing for $t > 0$, and $\frac{f(t)}{t^3}$ is nonincreasing for $t < 0$.

Since the potential function $V(x)$ vanishes at infinity, as in [5], we consider following function space:

$$W = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy < \infty \right\}$$

with the norm

$$\|u\|_W^2 = \int_{\mathbb{R}^3} a|\nabla u|^2 dx + \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy \right)^{\frac{1}{2}}.$$

Then it follows from [16] that W is a uniformly convex Banach space.

According to the Lax–Milgram theorem, for any $u \in W$, there exists unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$. Furthermore,

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy. \quad (1.4)$$

By (1.4) we know that system (1.1) is a single equation on u :

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u + \phi_u u = K(x)f(u) \quad \text{in } \mathbb{R}^3. \quad (1.5)$$

Since $(V, K) \in \mathcal{K}$, we know that the space

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty \right\}$$

with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) dx$$

is compactly embedded into the weighted Lebesgue space $L_k^q(\mathbb{R}^3)$ for some $q \in (2, 6)$ (see Lemma 2.2), where

$$L_k^q(\mathbb{R}^3) = \left\{ u : u \text{ is measurable on } \mathbb{R}^3 \text{ and } \int_{\mathbb{R}^3} K(x)|u|^q dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_K = \left(\int_{\mathbb{R}^3} K(x)|u|^q dx \right)^{\frac{1}{q}}.$$

In this paper, we discuss our problem on the new space

$$H := \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty, \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy < \infty \right\}$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) dx + \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy \right)^{\frac{1}{2}}.$$

Since $(W, \|\cdot\|)$ and $(E, \|\cdot\|)$ are Banach spaces, $(H, \|\cdot\|)$ is also a Banach space. Denote the usual norm in $L^p(\mathbb{R}^3)$ by $\|\cdot\|_p$. So, it follows that $H \hookrightarrow E \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is a

continuous embedding. Let

$$S_6 = \inf_{u \in E, u \neq 0} \frac{\int_{\mathbb{R}^3} (a|\nabla u|^2 + Vu^2) dx}{\left(\int_{\mathbb{R}^3} u^6 dx\right)^{\frac{1}{3}}}. \quad (1.6)$$

The energy functional associated with system (1.1) is defined by

$$\begin{aligned} J(u) = & \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \\ & - \int_{\mathbb{R}^3} K(x)F(u) dx, \quad \forall u \in H. \end{aligned} \quad (1.7)$$

Under assumptions (f_1) – (f_4) , $J \in C^1(H, \mathbb{R})$, and

$$\begin{aligned} \langle J'(u), v \rangle = & \int_{\mathbb{R}^3} (a\nabla u \nabla v + V(x)uv) dx + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla v dx \\ & + \int_{\mathbb{R}^3} \phi_u uv dx - \int_{\mathbb{R}^3} K(x)f(u)v dx, \quad \forall u, v \in H. \end{aligned} \quad (1.8)$$

The solution of system (1.1) is the critical point of the functional J . Especially, we call u a nodal solution to (1.1) if u is a solution of (1.1) with $u^\pm \neq 0$, where

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$

The following two lemmas are useful for dealing with the compactness of (PS) sequence of the functional J ; the details can be found in [16, 28].

Lemma 1.1

- (1) If $\{u_n\} \subset W$, then $u_n \rightarrow u$ in W if and only if $u_n \rightarrow u$ and $\phi_{u_n} \rightarrow \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.
Moreover, $u_n \rightharpoonup u$ in W if and only if $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy$ is bounded, and then $\phi_{u_n} \rightarrow \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.
- (2) $\phi_{tu}(x) = t^2 \phi_u(x) \geq 0$, $\forall u \in D^{1,2}(\mathbb{R}^3)$, $\forall t \in \mathbb{R}$.

For simplicity, we define $T : W^4 \rightarrow \mathbb{R}$ by

$$T(u, v, w, z) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x)v(x)w(y)z(y)}{4\pi|x-y|} dx dy.$$

Lemma 1.2 Suppose that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, $w_n \rightharpoonup w$, and $z \in W$. Then

$$T(u_n, v_n, w_n, z) \rightarrow T(u, v, w, z).$$

The following theorem is the main result of this paper.

Theorem 1.1 Let $(V, K) \in \mathcal{K}$. Under assumptions (f_1) – (f_4) , system (1.1) has at least one energy nodal solution. In addition, if f is an odd function, that is, $f(-t) = -f(t)$ for $t \in \mathbb{R}$, then system (1.1) has infinitely many nontrivial solutions.

2 The variation framework and basic conclusions

Lemma 2.1 ([1]) *Let $(V, K) \in \mathcal{K}$. If (VK_2) holds, then E is continuously embedded in $L_K^q(\mathbb{R}^3)$ for every $q \in [2, 6]$; if (VK_3) holds, then E is continuously embedded in $L_K^p(\mathbb{R}^3)$.*

Lemma 2.2 ([1]) *Let $(V, K) \in \mathcal{K}$. If (VK_2) holds, then E is compactly embedded in $L_K^q(\mathbb{R}^3)$ for every $q \in (2, 6)$; if (VK_3) holds, then E is compactly embedded in $L_K^p(\mathbb{R}^3)$.*

Lemma 2.3 ([1]) *Suppose that $(V, K) \in \mathcal{K}$ and (f_1) – (f_2) hold. If $\{v_n\} \subset E$ and $v_n \rightharpoonup v$ in E , then*

$$\int_{\mathbb{R}^3} K(x)F(v_n) dx \rightarrow \int_{\mathbb{R}^3} K(x)F(v) dx, \quad \int_{\mathbb{R}^3} K(x)f(v_n)v_n dx \rightarrow \int_{\mathbb{R}^3} K(x)f(v)v dx.$$

The following results are very important, because they allow us to overcome the non-differentiability of $\mathcal{N} := \{u \in H \setminus \{0\} : \langle I'_b(u), u \rangle = 0\}$.

Lemma 2.4 *Suppose that $(V, K) \in \mathcal{K}$ and (f_1) – (f_4) hold. Then:*

- (1) *Define $h_u : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $h_u(t) = J(tu)$, $u \in H \setminus \{0\}$. Then there exists a unique $t_u > 0$ such that $h'_u(t) > 0$, $t \in (0, t_u)$. However, $h'_u(t) < 0$, $t \in (t_u, \infty)$.*
- (2) *$t_u \geq \tau$, $u \in S$, where τ is independent of $u \in S$. Moreover, for any compact set $Q \subset S$, there is a constant $C_Q > 0$ such that $t_u \leq C_Q$, $u \in Q$.*
- (3) *The map $\hat{m} : H \setminus \{0\} \rightarrow \mathcal{N}$ is continuous for $\hat{m}(u) = t_u u$, and $m := \hat{m}|_S$ is a homeomorphic mapping between sphere S and $\mathcal{N}$.*

Proof Firstly, suppose that (VK_2) holds. According to (f_1) and (f_2) , for a given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon |s| + C_\varepsilon |s|^5, \quad s \in \mathbb{R}. \quad (2.1)$$

Hence we have that

$$\begin{aligned} J(tu) &= \frac{1}{2} \|tu\|_E^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla(tu)|^2 dx \right)^2 + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} K(x)F(tu) dx \\ &\geq \frac{t^2}{2} \|u\|_E^2 - \varepsilon \int_{\mathbb{R}^3} K(x)t^2 u^2 dx - C_\varepsilon \int_{\mathbb{R}^3} K(x)t^6 u^6 dx \\ &\geq \left(\frac{1}{2} - \varepsilon \|KV^{-1}\|_\infty \right) t^2 \|u\|_E^2 - S_6^{-3} C_\varepsilon t^6 \|K\|_\infty \|u\|_E^6. \end{aligned}$$

Choosing $\varepsilon < \frac{1}{2} \|KV^{-1}\|_\infty$, there exists $t_0 > 0$ sufficiently small such that

$$0 < h_u(t) = J(tu), \quad \forall t \in (0, t_0). \quad (2.2)$$

If (VK_3) holds, then according to discussion in [1], there is a constant $C_p > 0$ such that, for each $\varepsilon \in (0, C_p)$, there is $R > 0$ satisfying

$$\int_{B_R(0)^c} K(x)|u|^p dx \leq \varepsilon \int_{B_R(0)^c} (V(x)u^2 + u^6) dx, \quad \forall u \in H. \quad (2.3)$$

On the other hand, by (f_1) and (f_2) we derive that

$$J(tu) \geq \frac{t^2}{2} \|u\|_E^2 - C_1 t^p \int_{\mathbb{R}^3} K(x) |u|^p dx - C_2 t^6 \int_{\mathbb{R}^3} K(x) u^6 dx.$$

Thanks to (2.3), the Hölder inequality, and (VK_0) , we get

$$\begin{aligned} J(tu) &\geq \frac{t^2}{2} \|u\|_E^2 - C_1 \varepsilon t^p \int_{B_R(0)^c} (V(x)u^2 + u^6) dx - C_1 t^p \|K\|_{\frac{6}{6-p}(B_R(0))} \left(\int_{B_R(0)} u^6 dx \right)^{\frac{p}{6}} \\ &\quad - C_2 \|K\|_{\infty} t^6 \int_{\mathbb{R}^3} u^6 dx \\ &\geq \frac{t^2}{2} \|u\|_E^2 - C_2 t^6 \|K\|_{\infty} S_6^{-3} \|u\|_E^6 - C_1 (\varepsilon \|u\|_E^2 \\ &\quad + S_6^{-3} \varepsilon \|u\|_E^6 + S_6^{-\frac{p}{2}} \|K\|_{\frac{6}{6-p}(B_R(0))} \|u\|_E^{\frac{p}{6}}) t^p. \end{aligned}$$

Since $p > 2$, we get that (2.2) also holds.

From (f_1) and (f_4) we have that $F(t) \geq 0$, $t \in \mathbb{R}$. Then

$$J(tu) \leq \frac{1}{2} \|tu\|_E^2 + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_A K(x) F(tu) dx,$$

where $A \subset \text{supp } u$ is a measurable set with finite measure. By Lemma 1.1, (f_3) , and Fatou's lemma we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{J(tu)}{\|tu\|^4} &\leq \limsup_{t \rightarrow \infty} \frac{1}{2\|tu\|^2} + \frac{b+1}{4} \\ &\quad - \liminf_{t \rightarrow \infty} \left\{ \int_A K(x) \left[\frac{F(tu)}{(tu)^4} \right] \left(\frac{u}{\|u\|} \right)^4 dx \right\} \\ &\rightarrow -\infty. \end{aligned} \quad (2.4)$$

So, choosing $R > 0$ large enough, we have

$$h_u(R) = J(Ru) < 0. \quad (2.5)$$

Thanks to (2.2) and (2.5), from the continuity of h_u and (f_4) it follows that $t_u > 0$ is the global maximum point of h_u and $t_u u \in \mathcal{N}$.

We assert that t_u is the unique critical point of h_u . Indeed, if there are $t_1 > t_2 > 0$ such that $h'_u(t_1) = h'_u(t_2)$, then from (f_4) we have that

$$0 > \frac{1}{\|t_1 u\|_E^2} - \frac{1}{\|t_2 u\|_E^2} = \frac{1}{\|u\|_E^4} \int_{\mathbb{R}^3} K(x) \left[\frac{f(t_1 u)}{(t_1 u)^3} - \frac{f(t_2 u)}{(t_2 u)^3} \right] u^4 dx \geq 0,$$

which is a contradiction. So t_u is the unique critical point of h_u .

Now we prove (2).

For any $u \in S$, by (1) and (2.1) we have that

$$\begin{aligned} t_u^2 \|u\|_E^2 &\leq t_u^2 \|u\|_E^2 + b t_u^2 \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + t_u^4 \int_{\mathbb{R}^3} \phi_u u^2 dx \\ &= \int_{\mathbb{R}^3} K(x) f(t_u u) t_u u dx \\ &\leq \int_{\mathbb{R}^3} K(x) [\varepsilon |t_u u|^2 + C_\varepsilon |t_u u|^6] dx \\ &\leq \varepsilon t_u^2 \|KV^{-1}\|_\infty \|u\|_E^2 + S_6^{-3} t_u^6 C_\varepsilon \|K\|_\infty \|u\|_E^6. \end{aligned} \quad (2.6)$$

So, there exists $\tau > 0$, independent on u , such that $t_u \geq \tau$.

On the other hand, let $Q \subset S$ be a compact set. Suppose that there are a sequence $\{u_n\} \subset Q$ such that $t_n := t_{u_n} \rightarrow \infty$ and $u \in Q$ such that $u_n \rightarrow u$ in H . Then, by (2.4),

$$J(t_u u_n) \rightarrow -\infty. \quad (2.7)$$

From (f_4) we have that $\Theta(t) := \frac{1}{4}f(t)t - F(t) \geq 0$ is nondecreasing for $t \geq 0$ and nonincreasing for $t \leq 0$. Then, for each $u \in \mathcal{N}$,

$$\begin{aligned} J(u) &= J(u) - \frac{1}{4} \langle J'(u), u \rangle \\ &= \frac{1}{4} \|u\|^2 - \int_{\mathbb{R}^3} K(x) F(u) dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) f(u) u dx \\ &= \frac{1}{4} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x) (f(u)u - 4F(u)) dx \\ &\geq 0. \end{aligned} \quad (2.8)$$

Since $t_n u_n \in \mathcal{N}$, setting $u = t_n u_n$ in (2.8), we get a contradiction with (2.7). Thus (2) holds.

Next, we prove (3). We assert that $\hat{m}, m, m^{-1}$ are well defined. Indeed, for each $u \in H \setminus \{0\}$, by (1) there exists a unique $\hat{m}(u) \in \mathcal{N}$. On the other hand, if $u \in \mathcal{N}$, then $u \neq 0$, and thus $m^{-1}(u) = \frac{u}{\|u\|} \in S$ and $m^{-1}(u)$ are well defined. Furthermore,

$$\begin{aligned} m^{-1}(m(u)) &= m^{-1}(t_u u) = \frac{t_u u}{t_u \|u\|} = u, \quad \forall u \in S, \\ m^{-1}(m^{-1}(u)) &= m\left(\frac{u}{\|u\|}\right) = t_{(\frac{u}{\|u\|})} \frac{u}{\|u\|} = u, \quad \forall u \in \mathcal{N}. \end{aligned}$$

So m is bijective, and m^{-1} continuous.

We now prove that $\hat{m} : H \setminus \{0\} \rightarrow \mathcal{N}$ is continuous. Let $\{u_n\} \subset H \setminus \{0\}$ and $u \in H \setminus \{0\}$ be such that $u_n \rightarrow u$ in $H \subset E$. From (2) we have that there is $t_0 > 0$ such that $\|u_n\| t_{u_n} = t_{(\frac{u_n}{\|u_n\|})} \rightarrow t_0$. Hence $t_{u_n} \rightarrow \frac{t_0}{\|u\|} := t_*$. Since $t_{u_n} u_n \in \mathcal{N}$, we have that

$$t_{u_n}^2 \|u_n\|_E^2 + b t_{u_n}^4 \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 + t_{u_n}^4 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} K(x) f(t_{u_n} u_n) t_{u_n} u_n dx.$$

So, passing to the limit as $n \rightarrow \infty$, we get

$$t_*^2 \|u\|_E^2 + bt_*^4 \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \int_{\mathbb{R}^3} \phi_{t_*u}(t_*u)^2 dx = \int_{\mathbb{R}^3} K(x)f(t_*u)t_*u dx.$$

Hence $t_*u \in \mathcal{N}$ and $t_u = t_*$, that is, $\hat{m}(u_n) \rightarrow \hat{m}(u)$, and $\hat{m}$ and m are continuous. $\square$

Define $\hat{\psi} : E \rightarrow \mathbb{R}$ and $\psi : \mathcal{S} \rightarrow \mathbb{R}$ by

$$\hat{\psi}(u) = J(\hat{m}(u)) \quad \text{and} \quad \psi := \hat{\psi}|_{\mathcal{S}}.$$

By Lemma 2.4 we have following results; for details, see the book [32].

Proposition 2.1 Suppose $(V, K) \in \mathcal{K}$ and f satisfies $(f_1) - (f_4)$. Then:

- $\hat{\psi} \in C^1(H \setminus \{0\}, \mathbb{R})$, and $\hat{\psi}'(u)v = \frac{\|\hat{m}(u)\|}{\|u\|} J'(\hat{m}(u))v$, $v \in H$, $u \in H \setminus \{0\}$.
- $\psi \in C^1(\mathcal{S}, \mathbb{R})$, $\psi'(u)v = \|m(u)\| J'(m(u))v$, $v \in T_u \mathcal{S}$.
- If $\{u_n\}$ is a $(PS)_d$ sequence for ψ , then $\{m(u_n)\}$ is a $(PS)_d$ sequence for J . If $\{u_n\} \subset \mathcal{N}$ is a bounded $(PS)_d$ sequence for J , then $\{m^{-1}(u_n)\}$ is a $(PS)_d$ sequence for ψ .
- u is a critical point of ψ if and only if $m(u)$ is a nontrivial point of J . Moreover, the corresponding critical values coincide, and $\inf_{\mathcal{S}} \psi = \inf_{\mathcal{N}} J$.

Proposition 2.2 If $(f_1) - (f_4)$ hold, then

$$d_\infty := \inf_{u \in \mathcal{N}} J(u) = \inf_{u \in H \setminus \{0\}} \max_{t > 0} J(tu) = \inf_{u \in \mathcal{S}} \max_{t > 0} J(tu) > 0. \quad (2.9)$$

3 Technical lemmas

In this section, we give some technical lemmas related to the existence of a least energy nodal solution.

Set $\mathcal{M} = \{u \in H, u^\pm \neq 0, \langle J'(u), u^\pm \rangle = 0\}$. For $u \in H$ with $u^\pm \neq 0$, let

$$G_u(t, s) = J(tu^+ + su^-), \quad s > 0, t > 0.$$

Lemma 3.1 Assume that $(V, K) \in \mathcal{K}$ and f satisfies $(f_1) - (f_4)$. If $u \in H$ with $u^\pm \neq 0$, then:

- (t, s) is a critical point of G_u with $t, s > 0$ if and only if $tu^+ + su^- \in \mathcal{M}$.
- G_u has a unique critical point (t_+, s_-) with $t_+ = t_+(u) > 0$ and $s_- = s_-(u) > 0$, which is the unique maximum global point of G_u .
- The maps $\alpha_+(r) = \frac{\partial G_u}{\partial t}(r, s_-)$ and $\alpha_-(r) = \frac{\partial G_u}{\partial s}(t_+, s)$ are such that $\alpha_+(r) > 0$ for $r \in (0, t_+)$ and $\alpha_+(r) < 0$ for $r \in (t_+, \infty)$, and $\alpha_-(r) > 0$ for $r \in (0, s_-)$ and $\alpha_-(r) < 0$ for $r \in (s_-, \infty)$.

Proof By direct calculation we have

$$\begin{aligned} \nabla G_u(t, s) &= \left(\frac{\partial G_u}{\partial t}(t, s), \frac{\partial G_u}{\partial s}(t, s) \right) \\ &= (\langle J'(tu^+ + su^-), u^+ \rangle, \langle J'(tu^+ + su^-), u^- \rangle) \\ &= \left(\frac{1}{t} \langle J'(tu^+ + su^-), tu^+ \rangle, \frac{1}{s} \langle J'(tu^+ + su^-), su^- \rangle \right), \end{aligned}$$

from which (1) directly follows.

Now we prove (2). Firstly, we prove the existence of a critical point for G_u . Fixing $u \in H$ with $u^\pm \neq 0$ and $t_0 \geq 0$, we define $g_1 : [0, \infty) \rightarrow \mathbb{R}$ by $g_1(t) = G_u(t, s_0)$. Similarly as in the proof (1) of Lemma 2.3, we get that g_1 has a positive maximum point. Suppose $t_1 > t_2 > 0$ are all critical points of $g_1(t)$, that is, $g'_1(t_1) = g'_1(t_2) = 0$. Then

$$\begin{aligned} & t_i^2 \|u^+\|_E^2 + b t_i^4 \left(\int_{\mathbb{R}^3} |\nabla u^+|^2 dx \right)^2 + \int_{\mathbb{R}^3} \phi_{t_i u^+ + s_0 u^-} (t_i u^+)^2 dx \\ & + b s_0^2 t_i^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx \\ & = \int_{\mathbb{R}^3} K(x) f(t_i u^+) t_i u^+ dx, \quad i = 1, 2. \end{aligned}$$

From these equalities we have that

$$\begin{aligned} & \left(\frac{1}{t_1^2} - \frac{1}{t_2^2} \right) \left(\|u^+\|_E^2 + b s_0^2 \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + s_0^2 \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 dx \right) \\ & = \int_{\mathbb{R}^3} K(x) \left[\frac{f(t_1 u^+)}{(t_1 u^+)^3} - \frac{f(t_2 u^+)}{(t_2 u^+)^3} \right] (u^+)^4 dx, \end{aligned} \quad (3.1)$$

a contradiction thanks to (f_4) and $0 < t_2 < t_1$. So there is a unique $t_0 = t_0(u, s_0) > 0$ such that

$$g'_1(t) > 0, \quad t \in (0, t_0); \quad g'_1(t_0) = 0; \quad g'_1(t) < 0, \quad t \in (t_0, \infty).$$

Then we define the map $\varphi_1 : [0, \infty) \rightarrow [0, \infty)$ by $\varphi_1(s) = t(u, s)$, where $t(u, s)$ has properties similar to $t_0 = t_0(u, s_0)$ mentioned before.

By the definition of φ_1 we get

$$g'_1(\varphi_1(s)) = \frac{\partial G_u}{\partial t}(\varphi_1(s), s) = 0, \quad \forall s \geq 0.$$

Furthermore, φ_1 has the following properties:

(P₁) φ_1 is continuous;

(P₂) $\varphi_1(0) > 0$;

(P₃) $\varphi_1(s) \leq s$ for s large.

Similarly, we can define a map $\varphi_2(t)$ satisfying properties (P₁)–(P₃).

Fix $s_1 > 0$ and set

$$t_n = \varphi_1(s_n) \quad \text{and} \quad s_{n+1} = \varphi_2(t_n). \quad (3.2)$$

Then by the definitions of φ_1 and φ_2 we have that

$$\frac{\partial G_u}{\partial t}(t_n, s_n) = \frac{\partial G_u}{\partial s}(t_n, s_{n+1}) = 0, \quad \forall n \in \mathbb{N}. \quad (3.3)$$

We claim that $\{t_n\}$ and $\{s_n\}$ are bounded. Suppose, by contradiction, there is a subsequence $t_n \rightarrow \infty$. By (P₃) there is $C_1 > 0$ such that when $t_n > C_1$ for some n , we have $s_{n+1} = \varphi_2(t_n) \leq t_n$.

In the same way, there exists $C_2 > 0$ such that $s \geq C_2$ implies $\varphi_1(s) \leq s$. So, if $s_{n+1} \geq C_2$, then we get

$$t_{n+1} = \varphi_1(s_{n+1}) \leq s_{n+1} \leq t_n. \quad (3.4)$$

On the other hand, by (P_1) , if $s_{n+1} \leq C_2$, then we derive that

$$t_{n+1} = \varphi_1(s_{n+1}) \leq C_3 := \max_{s \in [0, C_2]} \varphi_1(s). \quad (3.5)$$

From (3.4) and (3.5) we deduce that $\{t_n\}$ is bounded. By applying (P_3) again we can prove that $\{s_n\}$ is also bounded.

Therefore, in subsequence sense, there are $t_+ \geq 0$ and $s_- \geq 0$ such that $t_n \rightarrow t_+$ and $s_n \rightarrow s_-$. Therefore from (3.2) and the continuity of φ_i , $i = 1, 2$, we have

$$t_+ = \varphi_1(s_-) \quad \text{and} \quad s_- = \varphi_2(t_+). \quad (3.6)$$

Furthermore, since $\varphi_i > 0$, we get $t_+ > 0$ and $s_- > 0$. Hence by (3.3) we obtain that (t_+, s_-) is a critical point for G_u .

Now we prove the uniqueness of (t_+, s_-) . By standard arguments here we only prove the uniqueness in the case $w \in \mathcal{M}$.

For any $w \in \mathcal{M}$, we obtain

$$\begin{aligned} \nabla G_w(1, 1) &= \left(\frac{\partial G_w}{\partial t}(1, 1), \frac{\partial G_w}{\partial s}(1, 1) \right) \\ &= (\langle J'(w^+ + w^-), w^+ \rangle, \langle J'(w^+ + w^-), w^- \rangle) \\ &= (0, 0). \end{aligned}$$

Obviously, $(1, 1)$ is a critical point of G_w . Now we prove that $(1, 1)$ is a unique critical point of G_w with positive coordinates. Assume that (t_0, s_0) are critical points of G_w with $0 < t_0 \leq s_0$. Then

$$\begin{aligned} &s_0^2 \|w^-\|_E^2 + bs_0^4 \left(\int_{\mathbb{R}^3} |\nabla w^-|^2 dx \right)^2 + \int_{\mathbb{R}^3} \phi_{t_0 w^+ + s_0 w^-} (s_0 w^-)^2 dx \\ &\quad + bs_0^2 t_0^2 \int_{\mathbb{R}^3} |\nabla w^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla w^-|^2 dx \\ &= \int_{\mathbb{R}^3} K(x) f(s_0 w^-) s_0 w^- dx, \end{aligned} \quad (3.7)$$

$$\begin{aligned} &t_0^2 \|w^+\|_E^2 + bt_0^4 \left(\int_{\mathbb{R}^3} |\nabla w^+|^2 dx \right)^2 + \int_{\mathbb{R}^3} \phi_{t_0 w^+ + s_0 w^-} (t_0 w^+)^2 dx \\ &\quad + bs_0^2 t_0^2 \int_{\mathbb{R}^3} |\nabla w^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla w^-|^2 dx \\ &= \int_{\mathbb{R}^3} K(x) f(t_0 w^+) t_0 w^+ dx. \end{aligned} \quad (3.8)$$

From (3.7) and $0 < t_0 \leq s_0$ we derive that

$$\begin{aligned} & \frac{\|w^-\|_E^2}{s_0^2} + b \left(\int_{\mathbb{R}^3} |\nabla w^-|^2 dx \right)^2 + b \int_{\mathbb{R}^3} |\nabla w^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla w^-|^2 dx + \int_{\mathbb{R}^3} \phi_w (w^-)^2 dx \\ & \geq \frac{\|w^-\|_E^2}{s_0^2} + b \left(\int_{\mathbb{R}^3} |\nabla w^-|^2 dx \right)^2 + \frac{bt_0^2}{s_0^2} \int_{\mathbb{R}^3} |\nabla w^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla w^-|^2 dx \\ & \quad + \frac{t_0^2}{s_0^2} \int_{\mathbb{R}^3} \phi_{w^+} (w^-)^2 dx + \int_{\mathbb{R}^3} \phi_{w^-} (w^-)^2 dx \\ & = \int_{\text{supp}(w^-)} K(x) \frac{f(s_0 w^-)}{(s_0 w^-)^3} (w^-)^4 dx. \end{aligned} \quad (3.9)$$

On the other hand, since $w \in \mathcal{M}$, we obtain

$$\begin{aligned} & \|w^-\|_E^2 + b \left(\int_{\mathbb{R}^3} |\nabla w^-|^2 dx \right)^2 + b \int_{\mathbb{R}^3} |\nabla w^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla w^-|^2 dx + \int_{\mathbb{R}^3} \phi_w (w^-)^2 dx \\ & = \int_{\mathbb{R}^3} K(x) f(w^-) w^- dx. \end{aligned} \quad (3.10)$$

From (3.9) with (3.10) we get

$$\left(\frac{1}{s_0^2} - 1 \right) \|w^-\|_E^2 \geq \int_{\text{supp}(w^-)} K(x) \left(\frac{f(s_0 w^-)}{(s_0 w^-)^3} - \frac{f(w^-)}{(w^-)^3} \right) (w^-)^4 dx. \quad (3.11)$$

Hence, according to (f_4) , we obtain that $0 < t_0 \leq s_0 < 1$.

Similarly, thanks to (3.8) and (f_4) , we can prove that $t_0 \geq 1$. Consequently, $t_0 = s_0 = 1$, that is, $(1, 1)$ is a unique critical point of G_w with positive coordinates.

In the following, we prove that the map G_u has a maximum global point $(s_0, t_0) \in (0, \infty) \times (0, \infty)$. Indeed, from (f_3) and $F \geq 0$ it follows that

$$\lim_{|(t,s)| \rightarrow \infty} G_u(t, s) = -\infty. \quad (3.12)$$

Since G_u is a continuous function, from (3.12) we conclude that G_u has a global maximum at some point $(s_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, which is a critical point of G_u . Furthermore, for any $u \in H$ with $u^\pm \neq 0$, since $J(tu^+) + J(su^-) \leq J(tu^+ + su^-)$, $t, s \geq 0$, we get

$$G_u(t, 0) + G_u(0, s) \leq G_u(t, s), \quad t, s \geq 0.$$

So we derive that

$$\max_{t \geq 0} G_u(t, 0) \leq \max_{t, s > 0} G_u(t, s), \quad \max_{s \geq 0} G_u(0, s) \leq \max_{t, s > 0} G_u(t, s),$$

which implies that $(s_0, t_0) \in (0, \infty) \times (0, \infty)$.

Finally, we prove (3). From (1) of Lemma 2.3 we have $\frac{\partial G_u}{\partial t}(r, s_-) > 0$ for $r \in (0, t_+)$ and $\frac{\partial G_u}{\partial t}(t_+, s_-) = 0$ and $\frac{\partial G_u}{\partial t}(r, s_-) < 0$ for $r \in (t_+, \infty)$. Thus α_+ and α_- have the same behavior. The proof is complete. $\square$

Now we define

$$c_\infty = \inf_{u \in \mathcal{M}} J(u).$$

Since $\mathcal{M} \subset \mathcal{N}$, we have $c_\infty \geq d_\infty$. On the other hand, for any $v \in H$ with $v^\pm \neq 0$, according to (1) of Lemma 2.4, there are $t^+, s^- > 0$ such that $t^+v^+, s^-v^- \in \mathcal{N}$. So, by Lemma 3.1 we derive that

$$J(v) \geq J(t^+v^+ + s^-v^-) \geq J(t^+v^+) + J(s^-v^-) \geq 2d_\infty, \quad v \in \mathcal{M},$$

which implies

$$2d_\infty \leq c_\infty. \quad (3.13)$$

Lemma 3.2 *The infimum c_∞ is achieved.*

Proof Let $\{u_n\}$ be a sequence in $\mathcal{M}$ such that

$$J(u_n) \rightarrow c_\infty. \quad (3.14)$$

Firstly, we prove that $\{u_n\}$ is bounded in H . By contradiction there is a subsequence, still denoted $\{u_n\}$, such that

$$\|u_n\| \rightarrow \infty. \quad (3.15)$$

Set $v_n := \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Thus there is $v \in H$ such that

$$v_n \rightharpoonup v \quad \text{as } n \rightarrow \infty. \quad (3.16)$$

From Lemma 2.2, up to a subsequence, we deduce that

$$v_n(x) \rightarrow v(x) \quad \text{a.e. in } \mathbb{R}^3. \quad (3.17)$$

Thanks to Lemma 3.1 and $\{u_n\} \subset \mathcal{M}$, we have that $t_+(v_n) = s_-(v_n) = \|u_n\|$. Thus, using Lemma 3.1 again, we obtain

$$\begin{aligned} J(\|u_n\|v_n) &\geq J(tv_n) \\ &= \frac{t^2}{2} \|v_n\|_E^2 + \frac{bt^4}{4} \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 + \frac{t^4}{4} \int_{\mathbb{R}^3} \phi_{v_n}(v_n)^2 dx \\ &\quad - \int_{\mathbb{R}^3} K(x)F(tv_n) dx, \quad \forall t > 0, n \in \mathbb{N}. \end{aligned} \quad (3.18)$$

Suppose that $v = 0$. Then by (3.16) and Lemma 2.3 we have that

$$\int_{\mathbb{R}^3} K(x)F(tv_n) dx \rightarrow 0, \quad \forall t > 0. \quad (3.19)$$

Passing to the limit as $n \rightarrow \infty$ in (3.18), it follows from (3.19) and $\|v_n\| = 1$ that there exists $\alpha_0 > 0$ such that

$$c_\infty \geq \frac{t^2}{2} \lim_{n \rightarrow \infty} \left[\|v_n\|_E^2 + \frac{b}{2} \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{v_n}(v_n)^2 dx \right] := \frac{t^2}{2} \alpha_0, \quad \forall t \geq 1,$$

which is a contradiction. Hence we obtain $v \neq 0$.

On the other hand, we obtain

$$\begin{aligned} \frac{J(u_n)}{\|u_n\|^4} &= \frac{\|u_n\|_E^2}{2\|u_n\|^4} + \frac{b}{4\|u_n\|^4} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \\ &\quad + \frac{1}{4\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} K(x) \frac{F(u_n)}{\|u_n\|^4} dx \\ &\leq \frac{1}{2\|u_n\|^2} + \frac{b+1}{4} - \int_{\mathbb{R}^3} K(x) \frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4} v_n^4 dx. \end{aligned} \quad (3.20)$$

Thanks to $v \neq 0$, from (3.15), (3.16), (f_3) , and Fatou's lemma we obtain

$$\int_{\mathbb{R}^3} K(x) \frac{F(v_n\|u_n\|)}{(v_n\|u_n\|)^4} v_n^4 dx \rightarrow \infty. \quad (3.21)$$

Thus by (3.15), (3.16), and (3.21), passing to the limit as $n \rightarrow \infty$ in (3.20), we get a contradiction. Therefore $\{u_n\}$ is bounded in H . So there is $u \in H$ such that

$$u_n \rightharpoonup u \quad \text{in } H, \quad u_n \rightarrow u \quad \text{a.e. in } \mathbb{R}^3. \quad (3.22)$$

Next, we prove that $u^\pm \neq 0$.

Firstly, we claim that there exists $\tau > 0$ such that

$$\|u^\pm\| \geq \tau, \quad \forall u \in \mathcal{M}. \quad (3.23)$$

In fact, since $u \in \mathcal{M}$, we have that

$$\begin{aligned} \|u^\pm\|_E^2 &\leq \|u^\pm\|_E^2 + b \left(\int_{\mathbb{R}^3} |\nabla u^\pm|^2 dx \right)^2 \\ &\quad + b \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + \int_{\mathbb{R}^3} \phi_u (u^\pm)^2 dx \\ &= \int_{\mathbb{R}^3} K(x) f(u^\pm) u^\pm dx. \end{aligned} \quad (3.24)$$

So, similarly as in proof of (2.6), we can prove that there is $\tau > 0$ such that

$$\|u^\pm\| \geq \|u^\pm\|_E \geq \tau.$$

Since $\{u_n\} \subset \mathcal{M}$, we have

$$\int_{\mathbb{R}^3} K(x) f(u_n^\pm) u_n^\pm dx \geq \|u_n^\pm\|_E^2 \geq \tau^2. \quad (3.25)$$

Combining Lemma 2.3 with (3.25), we have

$$\tau^2 \leq \int_{\mathbb{R}^3} K(x)f(u^\pm)u^\pm dx,$$

which implies that $u^\pm \neq 0$.

So, according to Lemma 3.1, there exist $t_+, s_- > 0$ such that

$$t_+u^+ + s_-u^- \in \mathcal{M}. \quad (3.26)$$

We claim that $0 < t_+, s_- \leq 1$. In fact, according to (3.22), Lemma 2.3, and Fatou's lemma, we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{u_n}(u_n^\pm)^2 dx \geq \int_{\mathbb{R}^3} \phi_u(u^\pm)^2 dx, \quad (3.27)$$

$$\int_{\mathbb{R}^3} K(x)f(u_n^\pm)u_n^\pm dx \rightarrow \int_{\mathbb{R}^3} K(x)f(u^\pm)u^\pm dx. \quad (3.28)$$

By the weak semicontinuity of norm in $D^{1,2}(\mathbb{R}^3)$ and E we have

$$\begin{aligned} \int_{\mathbb{R}^3} (a \nabla u u^\pm + V(x)u u^\pm) dx &= \|u^\pm\|_E^2 \\ &\leq \lim_{n \rightarrow \infty} \|u_n^\pm\|_E^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} (a \nabla u_n u_n^\pm + V(x)u_n u_n^\pm) dx, \\ \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla u^\pm dx &\leq \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla u_n^\pm dx \right). \end{aligned}$$

From (3.22), (3.27), and (3.28) we obtain

$$\langle J'(u), u^\pm \rangle \leq \liminf_{n \rightarrow \infty} \langle J'(u_n), u_n^\pm \rangle = 0. \quad (3.29)$$

According to (3.29), we have that

$$\begin{aligned} \|u^-\|_E^2 + b \left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx \right)^2 + b \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + \int_{\mathbb{R}^3} \phi_u(u^-)^2 dx \\ \leq \int_{\mathbb{R}^3} K(x) \frac{f(u^-)}{(u^-)^3} (u^-)^4 dx. \end{aligned} \quad (3.30)$$

On the other hand, suppose that $0 < t_+ \leq s_-$. Then from (3.26) we have that

$$\begin{aligned} \frac{\|u^-\|_E^2}{s_-^2} + b \left(\int_{\mathbb{R}^3} |\nabla u^-|^2 dx \right)^2 + b \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \cdot \int_{\mathbb{R}^3} |\nabla u^-|^2 dx + \int_{\mathbb{R}^3} \phi_u(u^-)^2 dx \\ \geq \int_{\mathbb{R}^3} K(x) \frac{f(s_- u^-)}{(s_- u^-)^3} (u^-)^4 dx. \end{aligned} \quad (3.31)$$

Combining (3.30) with (3.31), we derive that

$$\left(\frac{1}{s_-^2} - 1 \right) \|u^-\|_E^2 \geq \int_{\mathbb{R}^3} K(x) \left[\frac{f(s_- u^-)}{(s_- u^-)^3} - \frac{f(u^-)}{(u^-)^3} \right] (u^-)^4 dx.$$

By (f_4) we conclude that $0 < s_- \leq 1$, that is, $0 < t_+, s_- \leq 1$.

Now we prove that

$$J(t_+u^+ + s_-u^-) = c_\infty.$$

Let $u_\pm := t_+u^+ + s_-u^-$. By the definition of c_∞ and Fatou's lemma we have

$$\begin{aligned} c_\infty &\leq J(u_\pm) \\ &= J(u_\pm) - \frac{1}{4} \langle J'(u_\pm), u_\pm \rangle \\ &= \frac{1}{4} \|u_\pm\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)(f(u_\pm)u_\pm - 4F(u_\pm)) dx \\ &\leq \frac{1}{4} \|u\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)(f(u)u - 4F(u)) dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{1}{4} \|u_n\|_E^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)(f(u_n)u_n - 4F(u_n)) dx \right] \\ &= \liminf_{n \rightarrow \infty} \left[J(u_n) - \frac{1}{4} \langle J'(u_n), u_n \rangle \right] \\ &= c_\infty. \end{aligned}$$

So $t^+ = s^- = 1$, and c_∞ is achieved by $u := u^+ + u^- \in \mathcal{M}$. $\square$

4 Proof of main results

Proof According to Lemma 3.2, we just prove that the minimizer u for c_∞ is indeed a nodal solution for system (1.1). By contradiction we suppose that $J'(u) \neq 0$. By continuity there are $\delta, \mu > 0$ such that

$$\|J'(v)\|_{H^{-1}} > \mu \quad \text{if } \|v - u\| < 3\delta. \quad (4.1)$$

Choose $\rho \in (0, \min\{\frac{1}{2}, \frac{\delta}{\sqrt{2}\|u\|}\})$ and let $D = (1 - \rho, 1 + \rho) \times (1 - \rho, 1 + \rho)$, $g(t, s) = tu^+ + su^-$, $(t, s) \in D$. By Lemma 3.1 we get

$$\beta := \max_{(t,s) \in \partial D} J(g(t, s)) < c_\infty. \quad (4.2)$$

Let $\varepsilon \in (0, \min\{\frac{c_\infty - \beta}{4}, \frac{\mu\delta}{8}\})$. By Lemma 2.3 in [42] there is a deformation $\eta \in C([0, 1] \times H, H)$ such that

- (a) $\eta(t, v) = v$ if $v \notin J^{-1}([c_\infty - 2\varepsilon, c_\infty + 2\varepsilon])$;
- (b) $J(\eta(1, v)) \leq c_\infty - \varepsilon$ for each $v \in H$ such that $\|v - u\| \leq \delta$, and $J(v) \leq c_\infty + \varepsilon$;
- (c) $J(\eta(1, v)) \leq J(v)$ for $v \in H$;
- (d) $\|\eta(t, v) - u\| \leq \delta$ for $v \in H$ and $t \in [0, 1]$.

It follows from (4.2) and (b) that

$$\max_{(t,s) \in D} J(\eta(1, g(t, s))) < c_\infty.$$

Then by arguments similar to those in [13, 15], using Brouwer degree theory, we can prove that

$$\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset,$$

a contradiction. Thus $u = u^+ + u^-$ is a critical point of J , which is a least energy nodal solution of system (1.1).

Furthermore, if f is odd, then functional ψ is even. According to (2.9) and (3.13), we conclude that ψ is bounded from below in S . By Lemmas 2.2 and 2.3 we can prove that ψ satisfies the Palais–Smale condition on S . Hence by Lemma 2.4, Proposition 2.1, and [32] the functional J has infinitely many critical points. $\square$

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Authors' contributions

Both authors have the same contribution. Both authors read and approved the final manuscript.

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References

1. Alves, C.O., Souto, M.A.S.: Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity. *J. Differ. Equ.* **254**, 1977–1991 (2013)
2. Alves, C.O., Souto, M.A.S.: Existence of least nodal solution for a Schrödinger–Poisson system in bounded domains. *Z. Angew. Math. Phys.* **65**, 1153–1166 (2014)
3. Alves, C.O., Souto, M.A.S., Soares, S.H.M.: A sign-changing solution for the Schrödinger–Poisson equation in $\mathbb{R}^3$. *Rocky Mt. J. Math.* **47**, 1–25 (2017)
4. Batista, A.M., Furtado, M.F.: Positive and nodal solutions for a nonlinear Schrödinger–Poisson system with sign-changing potentials. *Nonlinear Anal., Real World Appl.* **39**, 142–156 (2018)
5. Bonheure, D., Van Schaftingen, J.: Ground states for nonlinear Schrödinger equation with potential vanishing at infinity. *Ann. Mat. Pura Appl.* **189**, 273–301 (2010)
6. Cassani, D., Liu, Z., Tarsi, C., Zhang, J.: Multiplicity of sign-changing solutions for Kirchhoff-type equations. *Nonlinear Anal.* **186**, 145–161 (2019)
7. Chen, S., Tang, X.H.: Ground state sign-changing solutions for a class of Schrödinger–Poisson type problems in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* **67**, 102 (2016)
8. Chen, S., Zhang, B.L., Tang, X.H.: Existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity. *Adv. Nonlinear Anal.* **9**, 148–167 (2020)
9. Cheng, B., Tang, X.H.: Ground state sign-changing solutions for asymptotically 3-linear Kirchhoff-type problems. *Complex Var. Elliptic Equ.* **62**, 1093–1116 (2017)
10. Deng, J., Yang, J.F.: Nodal solutions for Schrödinger–Poisson type equations in $\mathbb{R}^3$. *Electron. J. Differ. Equ.* **2016**, 277 (2016)
11. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. *J. Funct. Anal.* **269**, 3500–3527 (2015)
12. Deng, Y.B., Shuai, W.: Sign-changing multi-bump solutions for Kirchhoff-type equations in $\mathbb{R}^3$. *Discrete Contin. Dyn. Syst., Ser. A* **38**, 3139–3168 (2018)
13. Figueiredo, G.M., Santos Júnior, J.R.: Existence of a least energy nodal solution for a Schrödinger–Kirchhoff equation with potential vanishing at infinity. *J. Math. Phys.* **56**, 051506 (2015)
14. Fiscella, A., Pucci, P., Zhang, B.L.: p -fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities. *Adv. Nonlinear Anal.* **8**, 1111–1131 (2019)
15. Han, X., Ma, X., He, X.M.: Existence of sign-changing solutions for a class of p -Laplacian Kirchhoff-type equations. *Complex Var. Elliptic Equ.* **64**, 181–203 (2019)

16. Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger–Poisson–Slater problem. *Commun. Contemp. Math.* **14**, 1250003 (2012)
17. Ji, C., Fang, F., Zhang, B.: Least energy sign-changing solutions for the nonlinear Schrödinger–Poisson system. *Electron. J. Differ. Equ.* **2017**, 282 (2017)
18. Kim, S., Seok, J.: On nodal solutions of the nonlinear Schrödinger–Poisson equations. *Commun. Contemp. Math.* **14**, 1250041 (2012)
19. Li, F., Song, Z., Zhang, Q.: Existence and uniqueness results for Kirchhoff–Schrödinger–Poisson system with general singularity. *Appl. Anal.* **96**, 2906–2916 (2017)
20. Li, Q., Du, X., Zhao, Z.: Existence of sign-changing solutions for nonlocal Kirchhoff–Schrödinger-type equations in $\mathbb{R}^3$. *J. Math. Anal. Appl.* **477**, 174–186 (2019)
21. Li, Y., Wang, D.B., Zhang, J.: Sign-changing solutions for a class of p -Laplacian Kirchhoff-type problem with logarithmic nonlinearity. *AIMS Math.* **5**, 2100–2112 (2020)
22. Liang, Z., Xu, J., Zhu, X.: Revisit to sign-changing solutions for the nonlinear Schrödinger–Poisson system in $\mathbb{R}^3$. *J. Math. Anal. Appl.* **435**, 783–799 (2016)
23. Liu, Z., Wang, Z., Zhang, J.: Infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system. *Ann. Mat. Pura Appl.* **4**, 775–794 (2016)
24. Lü, D.: Positive solutions for Kirchhoff–Schrödinger–Poisson systems with general nonlinearity. *Commun. Pure Appl. Anal.* **17**, 605–626 (2018)
25. Lu, S.: Signed and sign-changing solutions for a Kirchhoff-type equation in bounded domains. *J. Math. Anal. Appl.* **432**, 965–982 (2015)
26. Mao, A., Luan, S.: Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems. *J. Math. Anal. Appl.* **383**, 239–243 (2011)
27. Mingqi, X., Rădulescu, V.D., Zhang, B.L.: Combined effects for fractional Schrödinger–Kirchhoff systems with critical nonlinearities. *ESAIM Control Optim. Calc. Var.* **24**, 1249–1273 (2018)
28. Ruiz, D.: On the Schrödinger–Poisson system: behavior of minimizers, radial and nonradial cases. *Arch. Ration. Mech. Anal.* **198**, 349–368 (2010)
29. Shuai, W.: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. *J. Differ. Equ.* **259**, 1256–1274 (2015)
30. Shuai, W., Wang, Q.: Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger–Poisson system in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* **66**, 3267–3282 (2015)
31. Sun, J., Li, L., Cencelj, M., Gabrovšek, B.: Infinitely many sign-changing solutions for Kirchhoff type problems in $\mathbb{R}^3$. *Nonlinear Anal.* **186**, 33–54 (2019)
32. Szulkin, A., Weth, T.: The method of Nehari manifold. In: Gao, D.Y., Motreanu, D. (eds.) *Handbook of Nonconvex Analysis and Applications*, pp. 597–632. International Press, Boston (2010)
33. Tang, X.H., Cheng, B.: Ground state sign-changing solutions for Kirchhoff type problems in bounded domains. *J. Differ. Equ.* **261**, 2384–2402 (2016)
34. Wang, D.B.: Least energy sign-changing solutions of Kirchhoff-type equation with critical growth. *J. Math. Phys.* **61**, 011501 (2020). <https://doi.org/10.1063/1.5074163>
35. Wang, D.B., Li, T., Hao, X.: Least-energy sign-changing solutions for Kirchhoff–Schrödinger–Poisson systems in $\mathbb{R}^3$. *Bound. Value Probl.* **2019**, 75 (2019). <https://doi.org/10.1186/s13661-019-1183-3>
36. Wang, D.B., Ma, Y., Guan, W.: Least energy sign-changing solutions for the fractional Schrödinger–Poisson systems in $\mathbb{R}^3$. *Bound. Value Probl.* **2019**, 25 (2019). <https://doi.org/10.1186/s13661-019-1128-x>
37. Wang, D.B., Zhang, H., Guan, W.: Existence of least-energy sign-changing solutions for Schrödinger–Poisson system with critical growth. *J. Math. Anal. Appl.* **479**, 2284–2301 (2019)
38. Wang, D.B., Zhang, H., Ma, Y., Guan, W.: Ground state sign-changing solutions for a class of nonlinear fractional Schrödinger–Poisson system with potential vanishing at infinity. *J. Appl. Math. Comput.* **61**, 611–634 (2019)
39. Wang, D.B., Zhang, J.: Least energy sign-changing solutions of fractional Kirchhoff–Schrödinger–Poisson system with critical growth. *Appl. Math. Lett.* **106**, 106372 (2020)
40. Wang, L., Zhang, B.L., Cheng, K.: Ground state sign-changing solutions for the Schrödinger–Kirchhoff equation in $\mathbb{R}^3$. *J. Math. Anal. Appl.* **466**, 1545–1569 (2018)
41. Wang, Z., Zhou, H.: Sign-changing solutions for the nonlinear Schrödinger–Poisson system in $\mathbb{R}^3$. *Calc. Var. Partial Differ. Equ.* **52**, 927–943 (2015)
42. Willem, M.: *Minimax Theorems*. Birkhäuser, Basel (1996)
43. Xiang, M., Pucci, P., Squassina, M., Zhang, B.L.: Nonlocal Schrödinger–Kirchhoff equations with external magnetic field. *Discrete Contin. Dyn. Syst.* **37**, 1631–1649 (2017)
44. Xiang, M., Zhang, B.L., Rădulescu, V.D.: Superlinear Schrödinger–Kirchhoff type problems involving the fractional p -Laplacian and critical exponent. *Adv. Nonlinear Anal.* **9**, 690–709 (2020)
45. Yang, M., Santos, C.A., Zhou, J.: Least action nodal solutions for a quasilinear defocusing Schrödinger equation with supercritical nonlinearity. *Commun. Contemp. Math.* **21**, 1850026 (2019)
46. Zhang, H., Guan, W.: Least energy sign-changing solutions for fourth-order Kirchhoff-type equation with potential vanishing at infinity. *J. Appl. Math. Comput.* <https://doi.org/10.1007/s12190-020-01349-0>
47. Zhang, J., Wang, D.B.: Existence of least energy nodal solution for Kirchhoff-type system with Hartree-type nonlinearity. *AIMS Math.* **5**, 4494–4511 (2020)
48. Zhang, Z., Perera, K.: Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. *J. Math. Anal. Appl.* **317**, 456–463 (2006)
49. Zhao, G., Zhu, X., Li, Y.: Existence of infinitely many solutions to a class of Kirchhoff–Schrödinger–Poisson system. *Appl. Math. Comput.* **256**, 572–581 (2015)
50. Zhong, X., Tang, C.: Ground state sign-changing solutions for a Schrödinger–Poisson system with a critical nonlinearity in $\mathbb{R}^3$. *Nonlinear Anal., Real World Appl.* **39**, 166–184 (2018)