Zhou et al. *Boundary Value Problems* (2020) 2020:117
https://doi.org/10.1186/s13661-020-01414-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence–uniqueness and monotone iteration of positive solutions to nonlinear tempered fractional differential equation with p-Laplacian operator

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Abstract

In this paper, without requiring the complete continuity of integral operators and the existence of upper–lower solutions, by means of the sum-type mixed monotone operator fixed point theorem based on the cone P_h, we investigate a kind of p-Laplacian differential equation Riemann–Stieltjes integral boundary value problem involving a tempered fractional derivative. Not only the existence and uniqueness of positive solutions are obtained, but also we can construct successively sequences for approximating the unique positive solution. As an application of our fundamental aims, we offer a realistic example to illustrate the effectiveness and practicability of the main results.

MSC: 34B18; 26A33; 34B27

Keywords: Existence–uniqueness; p-Laplacian operator; Tempered fractional dirivatives; Sum-type mixed monotone operators; Riemann–Stieltjes integral boundary value conditions

1 Introduction

In this paper, we devote our study to the kind of p-Laplacian differential equations Riemann–Stieltjes integral boundary value problems involving tempered fractional derivatives as follows:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha_2,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t))) = f(t,u(t),u(t)) + g(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u)(0) = 0, \\ u(1) = \beta \int_0^1 e^{-\lambda(1-t)}u(t)\,dt, \\ {}_0^R\mathbb{D}_t^{\gamma_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u))(1) = \int_0^\eta a(t){}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t))]\,dA(t), \end{cases} \tag{1.1}$$

where $n-1<\alpha_1\leq n$, $1<\alpha_2\leq 2$, $0<\gamma_2<\gamma_1<\alpha_2-1$, $\beta<\alpha_1$ and $\lambda>0$ is constant, φ_p is a p-Laplacian operator. ${}_0^R\mathbb{D}_t^{\alpha_1,\lambda}$ are tempered fractional derivatives, which are defined by

$$ {}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)=e^{-\lambda t}{}_0^RD_t^{\alpha_1}\bigl(e^{\lambda t}u(t)\bigr). \tag{1.2}$$

Here, ${}_0^RD_t^{\alpha_1}$ denotes the standard Riemann–Liouville fractional derivative

$$ {}_0^RD_t^{\alpha_1}u(t)=\frac{d^n}{dt^n}\bigl({}_0I_t^{n-\alpha_1}u(t)\bigr), \tag{1.3}$$

where ${}_0I_t^{\upsilon}$ for $\upsilon>0$ is the fractional integral operator of order υ defined by

$$ {}_0I_t^{\upsilon}\psi=\frac{1}{\Gamma(\upsilon)}\int_0^t(t-s)^{\upsilon-1}\psi(s)\,ds. \tag{1.4}$$

A is a function of a bounded variation, $\int_0^\eta a(t){}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t))]\,dA(t)$ denotes a Riemann–Stieltjes integral with respect to A. By using the sum-type mixed monotone fixed theorem based on the cone P_h, we show the existence and uniqueness of positive solutions for the p-Laplacian differential system (1.1).

In recent years, many theories and experiments have shown that a large number of abnormal phenomena that occurs in the applied science and engineering can be well described by fractional calculus. Especially, fractional differential equations have been proved to be powerful tools in the modeling of various phenomena in various fields of science and engineering, for example fluid mechanics, physics and heat conduction; see for instance [1–6]. Meanwhile, it is well known that the p-Laplacian operator is also used in analyzing biology, physics, mechanics and the related fields of mathematical modeling; see [7–14]. In [7], for studying the turbulent flow in porous media, Leibenson introduced the p-Laplacian differential equation as follows:

$$ \bigl(\varphi_p\bigl(u'(t)\bigr)\bigr)'=f\bigl(t,u(t),u'(t)\bigr),\quad t\in(0,1), \tag{1.5}$$

where $\varphi_p(s)=|s|^{p-2}s$, $p>1$. Motivated by Leibenson's work, Guo *et al.* [8] studied the existence of a solution for an ordinary differential equation m-point boundary value problem with p-Laplacian operator. Lu *et al.* [9] investigated a fractional differential equation for a two points boundary value problem involving the p-Laplacian operator as follows:

$$ \begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t)))=f(t,u(t)), & 0\leq t\leq 1;\\ u(0)=u'(0)=u'(1)=0;\\ D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0, \end{cases} \tag{1.6}$$

where $2<\alpha\leq 3$, $1<\beta\leq 2$ and $\varphi_p(s)=|s|^{p-2}s$. D_{0+}^{α}, D_{0+}^{β} are standard Riemann–Liouville fractional derivatives. By employing the Guo–Krasnosel'skii fixed-point theorem and upper–lower solutions method, the existence of positive solutions was obtained.

In [10], Ren, Li and Zhang studied the existence of maximum and minimum solutions for the following nonlocal p-Laplacian fractional differential system:

$$\begin{cases} -D_t^{\beta_1}(\varphi_{p_1}(-D_t^{\alpha_1}x_1))(t) = f_1(x_1(t), x_2(t)), \\ -D_t^{\beta_2}(\varphi_{p_2}(-D_t^{\alpha_2}x_2))(t) = f_2(x_1(t), x_2(t)), \\ x_1(0) = 0, \qquad D_t^{\alpha_1}x_1(0) = D_t^{\alpha_1}x_1(1) = 0, \qquad x_1(1) = \int_0^1 x_1(t)\,dA_1(t), \\ x_2(0) = 0, \qquad D_t^{\alpha_2}x_2(0) = D_t^{\alpha_2}x_2(1) = 0, \qquad x_2(1) = \int_0^1 x_2(t)\,dA_2(t), \end{cases} \tag{1.7}$$

where $D_t^{\alpha_i}$, $D_t^{\beta_i}$ are the standard Riemann–Liouville derivatives satisfying $1 < \alpha_i, \beta_i < 2$, $\int_0^1 x_i(t)\,dA_i(t)$ denotes a Riemann–Stieltjes integral and A_i is a function of bounded variation, φ_{p_i} is a p-Laplacian operator. By using the monotone iterative technique, some new results as regards the existence of maximal and minimal solutions were established, and the estimation of the lower and upper bounds of the maximum and minimum solutions was also derived.

Recently, in [15], we investigated the conformable differential equation with p-Laplacian operator as follows:

$$\begin{cases} T_\alpha^{0+}(\varphi_p(T_\alpha^{0+}u(t))) = f(t, u(t), T_\alpha^{0+}u(t)), \\ u^{(i)}(0) = 0, \qquad [\varphi_p(T_\alpha^{0+}u)]^{(i)}(0) = 0, \\ [T_\beta^{0+}u(t)]_{t=1} = 0, \qquad [T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1} = 0, \end{cases} \tag{1.8}$$

where $n-1 \le \alpha < n$ and $T_\alpha^{0^+}$ is a new fractional derivative called "the conformable fractional derivative". By using the Guo–Krasnosel'skii fixed point theorem, some new existence conclusions of positive solutions were obtained to the boundary value problem (1.8).

In [16], we continued to investigate the existence of multiple positive solutions for high order Riemann–Liouville fractional differential equation involving the p-Laplacian operator as follows:

$$\begin{cases} {}_0^RD_t^\alpha(\varphi_p({}_0^RD_t^\alpha u(t))) = f(t, u(t), {}_0^RD_t^\alpha u(t)), \quad 0 \le t \le 1; \\ u^{(i)}(0) = 0, \qquad [\varphi_p({}_0^RD_t^\alpha u)]^{(i)}(0) = 0, \quad i = 0, 1, 2, \dots, n-2; \\ [{}_0^RD_t^\beta u(t)]_{t=1} = 0, \quad 0 < \beta \le \alpha - 1; \\ [{}_0^RD_t^\beta(\varphi_p({}_0^RD_t^\alpha u(t)))]_{t=1} = 0; \end{cases} \tag{1.9}$$

where $n-1 < \alpha \le n$, ${}_0^RD_t^\alpha$ is the standard Riemann–Liouville fractional derivative, φ_p is the p-Laplacian operator. By means of the Leggett–Williams fixed point theorem and a functional-type cone expansion-compression fixed point theorem, not only the existence of two positive solutions was obtained, but also some sufficient conditions for the existence of at least three positive solutions was established.

In addition, Zhang *et al.* [17] investigated the eigenvalue problem for a kind of singular fractional differential equation Riemann–Stieltjes integral boundary value problem involving the p-Laplacian operator as follows:

$$\begin{cases} -D_t^\beta(\varphi_p(D_t^\alpha x(t))) = \lambda f(t, x(t)), \quad 0 \le t \le 1, \\ x(0) = 0, \qquad D_t^\alpha x(0) = 0, \\ x(1) = \int_0^1 x(s)\,dA(s), \end{cases} \tag{1.10}$$

where D_t^β and D_t^α are standard Riemann–Liouville fractional derivatives with $0 < \beta \leq 1$, $1 < \alpha \leq 2$, $\int_0^1 x(s)\,dA(s)$ is the standard Riemann–Stieltjes integral and A is a function of the bounded variation. By using the Schauder fixed point theorem and upper and lower solution methods, some new theorems on existence were obtained.

Inspired by the above work, in this paper, we investigate the existence and uniqueness of positive solutions for a p-Laplacian differential equation Riemann–Stieltjes integral boundary value problem involving a tempered fractional derivative (1.1). To the best of our knowledge, this kind of integral boundary value problem involving a tempered fractional derivative has seldom been researched up to now. Compared with other references, the present article has the following characteristics. Firstly, the tempered fractional derivative ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is more general than the standard fractional derivative ${}_0^R D_t^\alpha$. For example, letting $\lambda = 0$, it is easy to see that ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$ is equivalent to ${}_0^R D_t^\alpha$. Secondly, the Riemann–Stieltjes integral boundary conditions involving a tempered fractional derivative are more general cases, which cover the common integral boundary conditions as special cases. Thirdly, compared with the p-Laplacian differential system (1.8) and (1.9), in this paper, the integral operator need not be completely continuous or compact. Fourthly, in this paper, by employing the sum-type mixed monotone operators fixed points theorem, our conclusions cannot only guarantee the existence of a unique positive solution, but also construct successively sequences for approximating the unique positive solution. Finally, it is worth mentioning that some important properties of two different kernel functions rely on the parameter λ.

The rest of this paper is organized as follows. In Sect. 2, we briefly introduce some necessary basic definitions and preliminary results which will be used to prove our main results. In Sect. 3, we study the existence and uniqueness and monotone iteration of a positive solution to the p-Laplacian differential system (1.1) by means of sum-type mixed monotone fixed points theorems based on the cone P_h. At last, in Sect. 4, we demonstrate the effectiveness and feasibility of the main results by an example.

2 Preliminaries

In the section, we first list some basic notations, concepts in ordered Banach spaces. For convenience, we refer the reader to [18, 19] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. A nonempty closed convex set $P \subset E$ is a cone if it satisfies: (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Definition 2.1 ([18]) P is called normal if there exists $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq \|y\|$; in this case M is the infimum of such a constant, it is called the normality constant of P.

In addition, for a given $h > \theta$, we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$, in which $\sim$ is an equivalence relation, i.e., $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \geq y \geq \mu x$ for all $x, y \in E$.

Definition 2.2 ([20]) An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, i.e., $u_i, v_i (i = 1, 2) \in P$, $u_1 < u_2$, $v_1 > v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. An element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.3 ([9]) Let $p > 1$, the p-Laplacian operator is given by

$$\varphi_p(x) = |x|^{p-2}x \quad \text{and} \quad \varphi_p^{-1} = \varphi_q, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Definition 2.4 ([21]) $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0,1), x \in P.$$

Lemma 2.1 ([16]) *Let* $h(t) \in C[0,1] \cap L^1[0,1]$, $\alpha > 0$, *then*

$${}_0I_t^{\alpha}\,{}_0^RD_t^{\alpha} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, 3, \ldots, n$ $(n = [\alpha] + 1)$.

Lemma 2.2 ([16])

(1) *If* $u \in L^1(0,1)$, $\alpha > \beta > 0$, *then*

$${}_0I_t^{\alpha}\,{}_0I_t^{\beta} u(t) = {}_0I_t^{\alpha+\beta} u(t), \qquad {}_0^RD_t^{\beta}\,{}_0I_t^{\alpha} u(t) = {}_0 I_t^{\alpha-\beta} u(t), \qquad {}_0^RD_t^{\beta}\,{}_0I_t^{\beta} u(t) = u(t).$$

(2) *If* $\rho > 0$, $\mu > 0$, *then*

$${}_0^RD_t^{\rho} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} t^{\mu-\rho-1}.$$

Lemma 2.3 *Let* $g(t) \in C[0,1]$, *then the unique solution of the linear problem*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha_1,\lambda} u(t) + g(t) = 0, \quad n-1 < \alpha \leq n; \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0; \\ u(1) = \beta \int_0^1 e^{-\lambda(1-t)} u(t)\,dt, \quad \beta < \alpha_1; \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 H(t,s) g(s)\,ds, \tag{2.2}$$

where we have the Green function

$$H(t,s) = \begin{cases} \frac{\alpha_1(1-s)^{\alpha_1-1}(\alpha_1-\beta+\beta s)e^{\lambda s}t^{\alpha_1-1} - \alpha_1(\alpha_1-\beta)e^{\lambda s}(t-s)^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)} e^{-\lambda t}, & 0 \leq s \leq t \leq 1; \\ \frac{\alpha_1(1-s)^{\alpha_1-1}(\alpha_1-\beta+\beta s)e^{\lambda s}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)} e^{-\lambda t} t^{\alpha_1-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Proof For the system (2.1), by using Lemma 2.1, we get

$$e^{\lambda t} u(t) = -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} e^{\lambda s} g(s)\,ds + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + \cdots + c_n t^{\alpha_1-n}.$$

Furthermore, the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$ imply that $c_n = c_{n-1} = c_{n-2} = \cdots = c_3 = c_2 = 0$. Thus, we have

$$e^{\lambda t} u(t) = -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} e^{\lambda s} g(s)\,ds + c_1 t^{\alpha_1-1}. \tag{2.4}$$

Integrating both sides of Eq. (2.4) from 0 to 1, we see that

$$\begin{aligned}\int_0^1 e^{\lambda t}u(t)\,dt &= -\int_0^1 \left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}e^{\lambda s}g(s)\,ds\right)dt + c_1\int_0^1 t^{\alpha_1-1}\,dt \\ &= -\int_0^1 \frac{e^{\lambda s}}{\Gamma(\alpha_1)}g(s)\,ds\int_s^1 (t-s)^{\alpha_1-1}\,dt + \frac{c_1}{\alpha_1} \\ &= \frac{c_1}{\alpha_1} - \int_0^1 \frac{(1-s)^{\alpha_1}}{\Gamma(\alpha_1+1)}e^{\lambda s}g(s)\,ds. \end{aligned} \tag{2.5}$$

Letting $t=1$ in (2.4), we obtain

$$e^{\lambda}u(1) = -\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}e^{\lambda s}g(s)\,ds + c_1. \tag{2.6}$$

Combining the integral boundary value condition $u(1)=\beta\int_0^1 e^{-\lambda(1-t)}u(t)\,dt$, (2.6) and (2.5), we can clearly see that

$$c_1 = \int_0^1 \frac{\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{\lambda s}g(s)\,ds. \tag{2.7}$$

Finally, by simply substituting (2.7) into (2.4),

$$\begin{aligned}u(t) &= \int_0^1 \frac{[\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}]t^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s}g(s)\,ds \\ &\quad - \int_0^t \frac{\alpha_1(\alpha_1-\beta)(t-s)^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s}g(s)\,ds \\ &= \int_0^1 H(t,s)g(s),\end{aligned}$$

where the Green function $H(t,s)$ is defined as (2.3). □

Lemma 2.4 *If $\widetilde{g}\in C[0,1]$ is given, then the p-Laplacian tempered fractional differential equation integral boundary value problem*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha_2,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t))) = \widetilde{g}(t), \quad t\in[0,1], \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u)(0)=0, \\ u(1)=\beta\int_0^1 e^{-\lambda(1-t)}u(t)\,dt, \\ {}_0^R\mathbb{D}_t^{\gamma_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u))(1) = \int_0^\eta a(s){}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(s))]\,dA(s), \end{cases} \tag{2.8}$$

has a unique integral formal solution

$$u(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\widetilde{g}(\tau)\,d\tau\right)ds, \tag{2.9}$$

where $H(t,s)$ is given as (2.3), $G(t,s)$ is a Green function and

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha_2-1}e^{-\lambda t}}{\Delta\Gamma(\alpha_2-\gamma_2)}\int_0^\eta a(t)G_2(t,s)\,dA(t), \tag{2.10}$$

in which

$$G_1(t,s)=\frac{e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\begin{cases}(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-1}-(t-s)^{\alpha_2-1}, & 0\le s\le t\le 1,\\(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-1}, & 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s)=\frac{e^{\lambda(s-t)}}{\Gamma(\alpha)}\begin{cases}(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-\gamma_2-1}-(t-s)^{\alpha_2-\gamma_2-1}, & 0\le s\le t\le 1,\\(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-\gamma_2-1}, & 0\le t\le s\le 1,\end{cases}$$

and

$$\Delta=\frac{e^{-\lambda}}{\Gamma(\alpha_2-\gamma_1)}-\frac{\delta}{\Gamma(\alpha_2-\gamma_2)},\qquad \delta=\int_0^\eta e^{-\lambda s}s^{\alpha_2-\gamma_2-1}a(s)\,dA(s).$$

Proof From Lemma 2.1, integrating both sides of the first equation of (2.8), we obtain

$$\begin{aligned}e^{\lambda t}\varphi_p\bigl({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)\bigr)&={}_0I_t^{\alpha_2}\bigl(e^{\lambda t}\widetilde{g}(t)\bigr)+d_1t^{\alpha_2-1}+d_2t^{\alpha_2-2}\\&=\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha)}e^{\lambda s}\widetilde{g}(s)\,ds+d_1t^{\alpha_2-1}+d_2t^{\alpha_2-2}.\end{aligned}$$

Since $\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(0))=0$, we see that $d_2=0$, that is,

$$\varphi_p\bigl({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)\bigr)=e^{-\lambda t}{}_0I_t^{\alpha_2}\bigl(e^{\lambda t}\widetilde{g}(t)\bigr)+d_1e^{-\lambda t}t^{\alpha_2-1}. \tag{2.11}$$

Furthermore, applying the tempered fractional derivative operators ${}_0^R\mathbb{D}_t^{\gamma_i,\lambda}(i=1,2)$ on both sides of Eq. (2.11), we have

$$\begin{aligned}&{}_0^R\mathbb{D}_t^{\gamma_i,\lambda}\bigl(\varphi_p\bigl({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)\bigr)\bigr)\\&\quad={}_0^R\mathbb{D}_t^{\gamma_i,\lambda}\bigl(e^{-\lambda t}{}_0I_t^{\alpha_2}\bigl(e^{\lambda t}\widetilde{g}(t)\bigr)\bigr)+d_1{}_0^R\mathbb{D}_t^{\gamma_i,\lambda}\bigl(e^{-\lambda t}t^{\alpha_2-1}\bigr)\\&\quad=e^{-\lambda t}{}_0I_t^{\alpha_2-\gamma_i}\bigl(e^{\lambda t}\widetilde{g}(t)\bigr)+d_1e^{-\lambda t}{}_0^RD_t^{\gamma_i}\bigl(t^{\alpha_2-1}\bigr)\\&\quad=\int_0^t\frac{(t-s)^{\alpha_2-\gamma_i-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2-\gamma_i)}\widetilde{g}(s)\,ds+d_1\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_i)}e^{-\lambda t}t^{\alpha_2-1-\gamma_i}.\end{aligned} \tag{2.12}$$

From (2.12), we have

$$\begin{cases}{}_0^R\mathbb{D}_t^{\gamma_1,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u)](1)=\int_0^1\frac{(1-s)^{\alpha_2-\gamma_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha_2-\gamma_1)}\widetilde{g}(s)\,ds+d_1\frac{\Gamma(\alpha_2)e^{-\lambda}}{\Gamma(\alpha_2-\gamma_1)},\\{}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u)](t)=\int_0^t\frac{(t-s)^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2-\gamma_2)}\widetilde{g}(s)\,ds+d_1\frac{\Gamma(\alpha_2)e^{-\lambda t}}{\Gamma(\alpha_2-\gamma_2)}t^{\alpha_2-1-\gamma_2}.\end{cases} \tag{2.13}$$

Substituting (2.13) into the integral boundary value condition ${}_0^R\mathbb{D}_t^{\gamma_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u))(1)=\int_0^\eta a(s){}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(s))]\,dA(s)$, we obtain

$$\begin{aligned}d_1=\frac{-1}{\Gamma(\alpha_2)\Delta}\biggl\{&\int_0^1\frac{(1-s)^{\alpha_2-\gamma_1-1}e^{\lambda(s-1)}}{\Gamma(\alpha_2-\gamma_1)}\widetilde{g}(s)\,ds\\&-\int_0^\eta a(t)\,dA(t)\int_0^t\frac{(t-s)^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2-\gamma_2)}\widetilde{g}(s)\,ds\biggr\}.\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.11), we get

$$\begin{aligned}
&\varphi_p\big({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)\big)\\
&= e^{-\lambda t}\int_0^t \frac{(t-s)^{\alpha_2-1}e^{\lambda s}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds - \frac{e^{-\lambda t}t^{\alpha_2-1}}{\Gamma(\alpha_2)\Delta}\bigg\{\int_0^1 \frac{(1-s)^{\alpha_2-\gamma_1-1}e^{-\lambda}}{\Gamma(\alpha_2-\gamma_1)}e^{\lambda s}\widetilde{g}(s)\,ds\\
&\quad - \int_0^\eta a(t)\,dA(t)\int_0^t \frac{(t-s)^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2-\gamma_2)}\widetilde{g}(s)\,ds\bigg\}\\
&= \int_0^t \frac{(t-s)^{\alpha_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds - \frac{e^{-\lambda t}t^{\alpha_2-1}}{\Gamma(\alpha_2)}\int_0^1 (1-s)^{\alpha_2-\gamma_1-1}e^{\lambda s}\widetilde{g}(s)\,ds\\
&\quad - \frac{e^{-\lambda t}t^{\alpha_2-1}\delta}{\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)\Delta}\int_0^1 (1-s)^{\alpha_2-\gamma_1-1}e^{\lambda s}\widetilde{g}(s)\,ds\\
&\quad + \frac{e^{-\lambda t}t^{\alpha_2-1}}{\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)\Delta}\int_0^\eta a(t)\,dA(t)\int_0^t (t-s)^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}\widetilde{g}(s)\,ds\\
&= \int_0^t \frac{(t-s)^{\alpha_2-1}e^{-\lambda t}e^{\lambda s}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds - \int_0^1 \frac{(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds\\
&\quad - \frac{t^{\alpha_2-1}e^{-\lambda t}}{\Gamma(\alpha_2-\gamma_2)\Delta}\int_0^\eta a(t)\,dA(t)\int_0^1 \frac{(1-s)^{\alpha_2-\gamma_1-1}t^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds\\
&\quad + \frac{t^{\alpha_2-1}e^{-\lambda t}}{\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)\Delta}\int_0^\eta a(t)\,dA(t)\int_0^t \frac{(t-s)^{\alpha_2-\gamma_2-1}e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\widetilde{g}(s)\,ds\\
&= -\int_0^1 G_1(t,s)\widetilde{g}(s)\,ds - \frac{t^{\alpha_2-1}e^{-\lambda t}}{\Gamma(\alpha_2-\gamma_2)\Delta}\int_0^1 \widetilde{g}(s)\,ds\int_0^\eta G_2(t,s)a(t)\,dA(t)\\
&= -\int_0^1 G(t,s)\widetilde{g}(s)\,ds.
\end{aligned}$$

By employing the p-Laplacian operator φ_q on both sides of the above equation, we have

$$ {}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t) + \varphi_q\left(\int_0^1 G(t,s)\widetilde{g}(s)\,ds\right) = 0. \tag{2.15}$$

Setting $g(t) :\triangleq \varphi_q(\int_0^1 G(t,s)\widetilde{g}(s)\,ds)$, thus, the p-Laplacian tempered fractional differential system (2.8) is equivalent to the integral boundary value problems as follows:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t) + g(t) = 0, \quad n-1 < \alpha_1 \le n;\\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0;\\ u(1) = \beta\int_0^1 e^{-\lambda(1-t)}u(t)\,dt, \quad \beta < \alpha_1. \end{cases} \tag{2.16}$$

By means of Lemma 2.3, we see that the integral boundary value problem (2.16) has a unique integral solution

$$\begin{aligned} u(t) &= \int_0^1 H(t,s)g(s)\,ds\\ &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\widetilde{g}(\tau)\,d\tau\right)ds, \end{aligned}$$

where the Green function $G(t,s)$ and $H(t,s)$ are given by (2.10) and (2.3), respectively. This constitutes the complete proof. □

Lemma 2.5 *For* $\forall(s,t) \in [0,1] \times [0,1]$, *the Green function* $H(t,s)$ *given by* (2.3) *has the following properties*:

(A_1) *$H(t,s)$ is continuous and $H(t,s) \geq 0$;*

(A_2) $m_1(s)e^{-\lambda t}t^{\alpha_1-1} \leq H(t,s) \leq M_1(s)e^{-\lambda t}t^{\alpha_1-1}$, *where*

$$M_1(s) = \frac{\alpha_1(1-s)^{\alpha_1-1}(\alpha_1-\beta+\beta s)e^{\lambda s}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}, \qquad m_1(s) = \frac{\alpha_1\beta s(1-s)^{\alpha_1-1}e^{\lambda s}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}.$$

Proof Evidently, $H(t,s)$ is continuous and $H(t,s) \leq M_1(s)e^{-\lambda t}t^{\alpha_1-1}$ holds. So, we only need to prove the inequality $H(t,s) \geq m_1(s)e^{-\lambda t}t^{\alpha_1-1}$ and $H(t,s) \geq 0$.

If $0 \leq s \leq t \leq 1$, then we have $0 \leq t-s \leq t-ts = t(1-s)$, and thus $(t-s)^{\alpha_1-1} \leq t^{\alpha_1-1}(1-s)^{\alpha_1-1}$. Hence, we get

$$\begin{aligned} H(t,s) &= \frac{[\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}]t^{\alpha_1-1} - \alpha_1(\alpha_1-\beta)(t-s)^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{[\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}]t^{\alpha_1-1} - \alpha_1(\alpha_1-\beta)t^{\alpha_1-1}(1-s)^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s} \\ &= \frac{\alpha_1\beta s(1-s)^{\alpha_1-1}e^{\lambda s}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda_1 t}t^{\alpha_1-1} \geq 0. \end{aligned}$$

If $0 \leq s \leq t \leq 1$, clearly, we can see that

$$\begin{aligned} H(t,s) &= \frac{[\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}]t^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{[\alpha_1^2(1-s)^{\alpha_1-1} - \alpha_1\beta(1-s)^{\alpha_1}]t^{\alpha_1-1} - \alpha_1(\alpha_1-\beta)t^{\alpha_1-1}(1-s)^{\alpha_1-1}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}e^{\lambda s} \\ &\geq \frac{\alpha_1\beta s(1-s)^{\alpha_1-1}e^{\lambda s}}{(\alpha_1-\beta)\Gamma(\alpha_1+1)}e^{-\lambda t}t^{\alpha_1-1} \geq 0. \end{aligned}$$

Hence, the proof is complete. □

Lemma 2.6 *Suppose that*

$$(H) \quad e^{-\lambda}\Gamma(\alpha_2-\gamma_2) > \Gamma(\alpha_2-\gamma_1)\int_0^\eta e^{-\lambda s}s^{\alpha_2-\gamma_2-1}a(s)\,dA(s),$$

then, for all $(t,s) \in [0,1] \times [0,1]$, *the Green function* $G(t,s)$ *is continuous and satisfies*:

(B_1) $G_1(t,s) \geq 0$, $G_2(t,s) \geq 0$, *and* $G(t,s) \geq 0$;

(B_2) $\frac{e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1}-(1-s)^{\alpha_2-1}]}{\Gamma(\alpha_2)}e^{-\lambda t}t^{\alpha_2-1} \leq G_1(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha_2-\gamma_1-1}}{\Gamma(\alpha_2)}e^{-\lambda t}t^{\alpha_2-1}$;

(B_3) $\frac{e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1}-(1-s)^{\alpha_2-\gamma_2-1}]}{\Gamma(\alpha_2)}e^{-\lambda t}t^{\alpha_2-\gamma_2-1} \leq G_2(t,s) \leq \frac{e^{\lambda s}(1-s)^{\alpha_2-\gamma_1-1}}{\Gamma(\alpha_2)}e^{-\lambda t}t^{\alpha_2-\gamma_2-1}$;

(B_4) $m_2(s)e^{-\lambda t}t^{\alpha_2-1} \leq G(t,s) \leq M_2(s)e^{-\lambda t}t^{\alpha_2-1}$, *where*

$$\begin{cases} M_2(s) = [\frac{1}{\Gamma(\alpha_2)} + \frac{\delta}{\Delta\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)}]e^{\lambda s}(1-s)^{\alpha_2-\gamma_1-1}, \\ m_2(s) = \frac{e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1}-(1-s)^{\alpha_2-1}]}{\Gamma(\alpha_2)} + \frac{\delta e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1}-(1-s)^{\alpha_2-\gamma_2-1}]}{\Delta\Gamma(\alpha_2)\Gamma(\alpha_2-\gamma_2)}. \end{cases}$$

Proof Firstly, for $(t,s) \in [0,1] \times [0,1]$, it is evident that $G(t,s)$ and $G_i(t,s)(i = 1,2)$ are continuous.

Secondly, for (B_2) and (B_3), it is easy to see that the right sides of the inequalities hold, so we only need to prove the left sides of the inequalities. If $0 \le s \le t \le 1$, we have $0 \le t - s \le t - ts = (1 - s)t$, and thus $(t - s)^{\alpha_2 - 1} \le (1 - s)^{\alpha_2 - 1} t^{\alpha_2 - 1}$. Hence, we have

$$\begin{aligned} G_1(t,s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\bigl[t^{\alpha_2-1}(1-s)^{\alpha_2-\gamma_1-1} - (t-s)^{\alpha_2-1}\bigr] \\ &\ge \frac{e^{\lambda(s-t)}}{\Gamma(\alpha_2)}\bigl[t^{\alpha_2-1}(1-s)^{\alpha_2-\gamma_1-1} - (1-s)^{\alpha_2-1}t^{\alpha_2-1}\bigr] \\ &= \frac{e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1} - (1-s)^{\alpha_2-1}]}{\Gamma(\alpha_2)} e^{-\lambda t} t^{\alpha_2-1}. \end{aligned}$$

If $0 \le t \le s \le 1$,

$$\begin{aligned} G_1(t,s) &= \frac{e^{\lambda(s-t)}}{\Gamma(\alpha_2)} t^{\alpha_2-1}(1-s)^{\alpha_2-\gamma_1-1} \\ &\ge \frac{e^{\lambda s}[(1-s)^{\alpha_2-\gamma_1-1} - (1-s)^{\alpha_2-1}]}{\Gamma(\alpha_2)} e^{-\lambda t} t^{\alpha_2-1}. \end{aligned}$$

Furthermore, from $(1-s)^{\alpha_2-\gamma_1-1} > (1-s)^{\alpha_2-1}$, we get $G_1(t,s) \ge 0$ for $\forall (t,s) \in [0,1] \times [0,1]$. In the same way, similar conclusions can be obtained for $G_2(t,s)$.

Finally, from (B_2) and (B_3), we can know that $m_2(s)e^{-\lambda t}t^{\alpha-1} \le G(t,s) \le M_2(s)e^{-\lambda t}t^{\alpha-1}$. Since the condition (H) holds, it is easy to see that $\Delta > 0$. Combining $(1-s)^{\alpha_2-\gamma_1-1} > (1-s)^{\alpha_2-1}$ with $\Delta > 0$, we obtain $m_2(s) \ge 0$. Then $G(s,t) \ge 0$ for $\forall (t,s) \in [0,1] \times [0,1]$. Therefore, our justification for the proof is complete. □

Lemma 2.7 ([20]) *Let $\xi \in (0,1)$, $A : P \times P \to P$ be a mixed monotone operator that satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \ge t^{\xi} A(x,y), \quad t \in (0,1), x, y \in P. \tag{2.17}$$

$B : P \to P$ is an increasing sub-homogeneous operator. Assume that

(I) *there is $h_0 \in P_h$ such that $A(h_0,h_0) \in P_h$ and $Bh_0 \in P_h$;*

(II) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \ge \delta_0 Bx$, $\forall x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;*

(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le A(u_0,v_0) + Bu_0 \le A(v_0,u_0) + B(v_0) \le v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h;*

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1},y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1},x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3 Main results

In this section, we will work in the Banach space $C[0,1]$, the space of all continuous functions on $[0,1]$. It is obvious that this space can be equipped with a partial order

$$x,y\in C[0,1],\quad x\le y\quad\Longleftrightarrow\quad x(t)\le y(t)\quad\text{for } t\in[0,1].$$

Setting $P=\{x\in C[0,1]\mid x(t)\ge 0, t\in[0,1]\}$ and $h(t)=e^{-\lambda t}t^{\alpha_1-1}$, then we see that P is a normal cone in $C[0,1]$.

From Lemma 2.4, we can recognize that the p-Laplacian differential equation integral boundary value problem (1.1) is equivalent to the integral formulation given by

$$u(t)=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\left[f\left(\tau,u(\tau),u(\tau)\right)+g\left(\tau,u(\tau)\right)\right]d\tau\right)ds.$$

For convenience, we define an operator T by

$$T(u,v)(t)=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\left[f\left(\tau,u(\tau),v(\tau)\right)+g\left(\tau,u(\tau)\right)\right]d\tau\right)ds. \tag{3.1}$$

It is evident that u^* is a solution of p-Laplacian differential equation integral boundary value problem (1.1) if and only if $T(u^*,u^*)=u^*$.

Theorem 3.1 *Assume that the condition* (H) *holds, and the following conditions are satisfied*:

(H_1) $f(t,u,v):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *is continuous*, $g(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous with* $g(t,u)\not\equiv 0$ *and* $a(t):[0,1]\to R^+$ *is continuous*; *for fixed* $t\in[0,1]$, $f(t,u,v)$ *is increasing in* $u\in[0,+\infty)$ *and decreasing in* $v\in[0,+\infty)$, $g(t,u)$ *is increasing in* $u\in[0,+\infty)$.

(H_2) *For* $\forall t\in[0,1]$, $\gamma\in(0,1)$, $u,v\in[0,+\infty)$, *there exists a constant* $\xi\in(0,1)$ *such that*

$$f\left(t,\gamma u,\gamma^{-1}v\right)\ge\varphi_p^{\xi}(\gamma)f(t,u,v), \tag{3.2}$$

$$g(t,\gamma u)\ge\varphi_p(\gamma)g(t,u). \tag{3.3}$$

(H_3) *For* $\forall t\in[0,1]$ *and* $u,v\in[0,+\infty)$, *there exists a constant* $\delta_0>0$ *such that*

$$f(t,u,v)\ge\varphi_p(\delta_0)g(t,u). \tag{3.4}$$

Then we have:

(I) *the p-Laplacian differential equation integral boundary value problem involving tempered fractional derivative* (1.1) *has a unique positive solution* $u^*\in P_h$, *where* $h(t)=e^{-\lambda t}t^{\alpha_1-1}$, $t\in[0,1]$;

(II) *for* $\forall t\in[0,1]$, *there exist* $u_0,v_0\in P_h$ *and* $r\in(0,1)$ *such that* $rv_0\le u_0<v_0$ *and*

$$u_0(t)\le\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\left[f\left(\tau,u_0(\tau),v_0(\tau)\right)+g\left(\tau,u_0(\tau)\right)\right]d\tau\right)ds,$$

$$v_0(t)\ge\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\left[f\left(\tau,v_0(\tau),u_0(\tau)\right)+g\left(\tau,v_0(\tau)\right)\right]d\tau\right)ds;$$

(III) *for any initial values* $x_0, y_0 \in P_h$, *making successively the sequences*

$$\begin{aligned} x_n &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,x_{n-1}(\tau),y_{n-1}(\tau)\big)+g\big(\tau,x_{n-1}(\tau)\big)\big]\,d\tau\right)ds,\\ y_n &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)\big[f\big(\tau,y_{n-1}(\tau),x_{n-1}(\tau)\big)+g\big(\tau,y_{n-1}(\tau)\big)\big]\,d\tau\right)ds,\\ n &= 0,1,2,\ldots, \end{aligned}$$

we obtain $x_n \to u^*$ *and* $y_n \to u^*$ *as* $n \to \infty$.

Proof Firstly, we define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$A(u,v)(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds, \tag{3.5}$$

$$B(u)(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\right)ds. \tag{3.6}$$

From (3.1), we have $T(u,v) = A(u,v)+B(u)$ and u is a solution of the p-Laplacian differential system (1.1) if and only if $T(u,u) = u$. We show that the operator A satisfies the condition (2.17) in Lemma 2.7 and the operator B is a sub-homogeneous operator.

From (H_1), Lemma 2.5 and Lemma 2.6, we know that $A : P \times P \to P$ and $B : P \to P$. In addition, it follows from (H_1) and (H_2) that A is a mixed monotone operator and B is an increasing operator. For $\forall \gamma \in (0,1)$ and $u, v \in P$, from (3.2), we obtain

$$\begin{aligned} A\big(\gamma u,\gamma^{-1}v\big)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,\gamma u(\tau),\gamma v(\tau)\big)\,d\tau\right)ds\\ &\geq \int_0^1 H(t,s)\varphi_q\left(\varphi_p^{\xi}(\gamma)\int_0^1 G(s,\tau)f\big(\tau,u(\tau),v(\tau)\big)\,d\tau\right)ds\\ &= \gamma^{\xi}A(u,v)(t). \end{aligned}\tag{3.7}$$

That is, $A(\gamma u,\gamma^{-1}v) \geq \gamma^{\xi}A(u,v)$ for $\forall \gamma \in (0,1)$, $u, v \in P$. Furthermore, for $\forall \gamma \in (0,1)$ and $u \in P$, from (3.3), we have

$$\begin{aligned} B(\gamma u)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\big(\tau,\gamma u(\tau)\big)\,d\tau\right)ds\\ &\geq \varphi_q\big(\varphi_p(\gamma)\big)\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\big(\tau,u(\tau)\big)\,d\tau\right)ds\\ &= \gamma B(u)(t). \end{aligned}\tag{3.8}$$

That is, the operator B is a sub-homogeneous operator.

Secondly, we show that $A(h,h) \in P_h$ and $Bh \in P_h$. From Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned} A(h,h)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau,h(\tau),h(\tau)\big)\,d\tau\right)ds\\ &\leq \int_0^1 H(t,s)\varphi_q\left(\int_0^1 M_2(\tau)e^{-\lambda s}s^{\alpha_2-1}f\big(\tau,h(\tau),h(\tau)\big)\,d\tau\right)ds \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 M_1(s)e^{-\lambda t}t^{\alpha_1-1}\varphi_q\left(\int_0^1 M_2(\tau)e^{-\lambda s}s^{\alpha_2-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\right)ds \\
&\le \left\{\int_0^1 \frac{M_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 M_2(\tau)f(\tau,h_{\max},0)\,d\tau\right)ds\right\}e^{-\lambda t}t^{\alpha_1-1}
\end{aligned}
$$

and

$$
\begin{aligned}
A(h,h)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\right)ds \\
&\ge \int_0^1 H(t,s)\varphi_q\left(\int_0^1 m_2(\tau)e^{-\lambda s}s^{\alpha_2-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\right)ds \\
&\ge \int_0^1 m_1(s)e^{-\lambda t}t^{\alpha_1-1}\varphi_q\left(\int_0^1 m_2(\tau)e^{-\lambda s}s^{\alpha_2-1}f\bigl(\tau,h(\tau),h(\tau)\bigr)\,d\tau\right)ds \\
&\ge \left\{\int_0^1 \frac{m_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m_2(\tau)f(\tau,0,h_{\max})\,d\tau\right)ds\right\}e^{-\lambda t}t^{\alpha_1-1},
\end{aligned}
$$

where $h_{\max} = \max\{h(t) : t \in [0,1]\}$. Setting

$$
\begin{aligned}
L_1 &= \int_0^1 \frac{M_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 M_2(\tau)f(\tau,h_{\max},0)\,d\tau\right)ds, \\
l_1 &= \int_0^1 \frac{m_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m_2(\tau)f(\tau,0,h_{\max})\,d\tau\right)ds,
\end{aligned}
$$

it is easy to see that $L_1 > l_1 > 0$. Hence, we get $l_1 h(t) \le A(h,h) \le L_1 h(t)$. That is, $A(h,h) \in P_h$. Similarly,

$$
\begin{aligned}
B(h)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\bigl(\tau,h(\tau)\bigr)\,d\tau\right)ds \\
&\le \left\{\int_0^1 \frac{M_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 M_2(\tau)g(\tau,h_{\max})\,d\tau\right)ds\right\}e^{-\lambda t}t^{\alpha_1-1}
\end{aligned}
$$

and

$$
\begin{aligned}
B(h)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g\bigl(\tau,h(\tau)\bigr)\,d\tau\right)ds \\
&\ge \left\{\int_0^1 \frac{m_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m_2(\tau)g(\tau,0)\,d\tau\right)ds\right\}e^{-\lambda t}t^{\alpha_1-1}.
\end{aligned}
$$

Set

$$
\begin{aligned}
L_2 &= \int_0^1 \frac{M_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 M_2(\tau)g(\tau,h_{\max})\,d\tau\right)ds, \\
l_2 &= \int_0^1 \frac{m_1(s)s^{(\alpha_2-1)(q-1)}}{e^{\lambda s(q-1)}}\varphi_q\left(\int_0^1 m_2(\tau)g(\tau,0)\,d\tau\right)ds.
\end{aligned}
$$

From $L_2 > l_2 > 0$ and $l_2 h \le B(h) \le L_2 h$, we get $Bh \in P_h$. Since $h \in P_h$, letting $h_0 = h$, we see that the condition (I_1) of Lemma 2.7 is satisfied.

Finally, we show that the condition (II) of Lemma 2.7 is also satisfied. For $u, v \in P$, from (3.4), we get

$$\begin{aligned} A(u,v)(t) &= \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),v(\tau)\bigr)\,d\tau\right)ds \\ &\geq \int_0^1 H(t,s)\varphi_q\left(\int_0^1 \varphi_p(\delta_0)G(s,\tau)g\bigl(\tau,u(\tau)\bigr)\,d\tau\right)ds \\ &= \delta_0 B(u)(t). \end{aligned} \tag{3.9}$$

Now, all conditions of Lemma 2.7 are satisfied. Hence, the conclusions of Theorem 3.1 follow from Lemma 2.7. □

Corollary 3.1 *Assume that the condition* (H) *holds and*

- (H_1') $f(t,u,v):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *is continuous*, $g(t,u)\equiv 0$ *for* $\forall t\in[0,1]$ *and* $u\in[0,+\infty)$, $a(t):[0,1]\to R^+$ *is continuous*;
- (H_2') $f(t,u,v)$ *is increasing in* $u\in[0,+\infty)$ *for fixed* $t\in[0,1]$ *and* $v\in[0,+\infty)$, *decreasing in* $v\in[0,+\infty)$ *for fixed* $t\in[0,1]$ *and* $u\in[0,+\infty)$;
- (H_3') *for* $\forall t\in[0,1]$, $\gamma\in(0,1)$, $u,v\in[0,+\infty)$, *there exists a constant* $\xi\in(0,1)$ *such that*

$$f\bigl(t,\gamma u,\gamma^{-1}v\bigr)\geq\varphi_p^{\xi}(\gamma)f(t,u,v).$$

Then we have:

(I) *The p-Laplacian differential equation integral boundary value problem*

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha_2,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(t)))=f(t,u(t),u(t)), \quad t\in[0,1], \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\ \varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u)(0)=0, \\ u(1)=\beta\int_0^1 e^{-\lambda(1-t)}u(t)\,dt, \\ {}_0^R\mathbb{D}_t^{\gamma_1,\lambda}(\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u))(1)=\int_0^\eta a(s){}_0^R\mathbb{D}_t^{\gamma_2,\lambda}[\varphi_p({}_0^R\mathbb{D}_t^{\alpha_1,\lambda}u(s))]\,dA(s), \end{cases}$$

has a unique positive solution $u^*\in P_h$, *where* $h(t)=e^{-\lambda t}t^{\alpha_1-1}$.

(II) *For* $\forall t\in[0,1]$, *there exist* $u_0,v_0\in P_h$ *and* $r\in(0,1)$ *such that* $rv_0\leq u_0<v_0$ *and*

$$\begin{aligned} u_0(t)&\leq\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,u_0(\tau),v_0(\tau)\bigr)\,d\tau\right)ds, \\ v_0(t)&\geq\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,v_0(\tau),u_0(\tau)\bigr)\,d\tau\right)ds. \end{aligned}$$

(III) *For any initial values* $x_0,y_0\in P_h$, *making successively the sequences*

$$\begin{aligned} x_n&=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,x_{n-1}(\tau),y_{n-1}(\tau)\bigr)\,d\tau\right)ds, \\ y_n&=\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\bigl(\tau,y_{n-1}(\tau),x_{n-1}(\tau)\bigr)\,d\tau\right)ds, \\ &n=0,1,2,\ldots, \end{aligned}$$

we obtain $x_n\to u^*$ *and* $y_n\to u^*$ *as* $n\to\infty$.

Proof Letting $g(t,u(t)) \equiv 0$, from Theorem 3.1, we get the conclusions. □

4 Applications

Example We consider the p-Laplacian differential equation integral boundary value problem involving a tempered fractional derivative as follows:

$$\begin{cases} {}_0^R\mathbb{D}_t^{\frac{3}{2},1}(\varphi_3({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(t))) = f(t,u(t),u(t)) + g(t,u(t)), \quad 0 \le t \le 1; \\ u(0) = u'(0) = 0; \\ \varphi_3({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u)(0) = 0; \\ u(1) = \int_0^1 e^{t-1}u(t)\,dt, \\ {}_0^R\mathbb{D}_t^{\frac{3}{8},1}(\varphi_3({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u))(1) = \int_0^1 {}_0^R\mathbb{D}_t^{\frac{2}{8},1}[\varphi_3({}_0^R\mathbb{D}_t^{\frac{5}{2},1}u(t))]\,d(\frac{t}{2}); \end{cases} \tag{4.1}$$

where $f(t,u,v) = (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}$, $g(t,u) = (1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}$, $p = 3$, $\lambda = 1 > 0$, $\eta = 1$ and $A(t) = \frac{t}{2}$. For any $t \in (0,1)$, $u > 0$ and $v > 0$ and we see that $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{3}{2}\gamma_1 = \frac{3}{8}$, $\gamma_2 = \frac{2}{8}$, $\beta = 1$, $a(t) \equiv 1$ in the systems (4.1).

Let us investigate if all the conditions required in Theorem 3.1 are satisfied.

(1) From $\delta = \int_0^\eta e^{-\lambda s}s^{\alpha_2-\gamma_2-1}a(s)\,dA(s) = 0.2385$, it is easy to see that $\Gamma(\alpha_2-\gamma_1)\int_0^\eta e^{-\lambda s}s^{\alpha_2-\gamma_2-1}a(s)\,dA(s) = 0.2246$ and $e^{-\lambda}\Gamma(\alpha_2-\gamma_2) = 0.3334$, clearly, $e^{-\lambda}\Gamma(\alpha_2-\gamma_2) > \Gamma(\alpha_2-\gamma_1)\int_0^\eta e^{-\lambda s}s^{\alpha_2-\gamma_2-1}a(s)\,dA(s)$. Then the condition (H) is satisfied.

(2) It is obvious that $f(t,u,v):(0,1)\times R^+\times R^+ \to R^+$ and $g(t,u):(0,1)\times R^+ \to R^+$ are continuous. In addition, $f(t,u,v)$ is increasing in u for fixed $t \in (0,1)$ and $v \in R^+$, decreasing in v for fixed $t \in (0,1)$ and $u \in R^+$; furthermore, for fixed $t \in (0,1)$, $g(t,u)$ is increasing in u.

(3) For any $\gamma \in (0,1)$, $t \in (0,1)$, $u,v > 0$, taking $\xi = \frac{1}{2} \in (0,1)$, we have

$$\begin{aligned} f(t,\gamma u,\gamma^{-1}v) &= (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}(\gamma u)^{\frac{1}{3}} + (\gamma^{-1}v)^{-\frac{1}{5}} \\ &\ge \gamma^{\frac{1}{2}}\bigl[(1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}\bigr] \\ &\ge \gamma\bigl[(1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}}\bigr] \\ &= \varphi_p^{\xi}(\gamma)f(t,u,v) \end{aligned}$$

and

$$\begin{aligned} g(t,\gamma u) &= (1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}(\gamma u)^{\frac{1}{3}} \\ &\ge \gamma^2\bigl[(1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}\bigr] \\ &= \varphi_p(\gamma)g(t,u). \end{aligned}$$

(4) Taking $\delta_0 = \frac{1}{2}$, for $\forall t \in (0,1)$ and $u,v \in [0,+\infty)$, we have

$$\begin{aligned} f(t,u,v) &= (1-t)^{-\frac{1}{3}}t^{-\frac{2}{3}}u^{\frac{1}{3}} + v^{-\frac{1}{5}} \\ &\ge \frac{1}{4}\bigl[(1-t)^{-\frac{1}{8}}t^{-\frac{1}{6}}u^{\frac{1}{3}}\bigr] \\ &= \varphi_p(\delta_0)g(t,u). \end{aligned}$$

From the above conditions, we can see that all the assumptions of Theorem 3.1 are satisfied. Hence, Theorem 3.1 implies that the p-Laplacian differential system (4.1) has a unique positive solution $u^* \in P_h$, where $h(t) = e^{-t}t^{\frac{3}{2}}$. Furthermore, for any initial values $x_0, y_0 \in P_h$, making successively the sequences $x_n = T(x_{n-1}, y_{n-1})$, $y_n = T(y_{n-1}, x_{n-1})$, $n = 0, 1, 2, \ldots$, we obtain $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$.

Acknowledgements
We are thankful to the editor and the anonymous reviewers for many valuable suggestions to improve this paper.

Funding
This paper is supported by the opening project of State Key Laboratory of Explosion Science and Technology (Beijing Institute of Technology). The opening project number is KFJJ19-06M. And it is also supported by the Key R&D Program of Shanxi Province (International Cooperation, 201903D421042).

Availability of data and materials
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Publisher's Note

Received: 16 March 2020 Accepted: 14 June 2020 Published online: 23 June 2020

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