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Multiplicity results involving p-biharmonic Kirchhoff-type problems

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Abstract

This paper deals with the existence of multiple solutions for the following Kirchhoff type equations involving p-biharmonic operator:

$$-M\left(\int_\Omega (|\Delta_p u|^2 + |u|^p)\,dx\right)(\Delta_p^2 u - |u|^{p-2}u) = \lambda f(x)|u|^{q-2}u + g(x)|u|^{m-2}u, \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N > 1)$, $\lambda > 0$, $p, q, m > 1$, M is a continuous function, and the weight functions f and g are measurable. We obtain the existence results by combining the variational method with Nehari manifold and fibering maps.

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1 Introduction

The theory of p-Laplacian and p-biharmonic operators has been developed very quickly. The investigation of the existence and multiplicity of solutions has attracted a considerable attention of researchers (see, for instance, [1, 3, 15, 18, 22, 24, 26–28] and the references therein). The motivation of this interest stems from the fact that these nonhomogeneous differential operators are a very productive and rich area of research in recent decades. This theory have relevant applications in various fields; we refer the reader to [17, 20–23].

Kirchhof-type equations, known as nonlocal differential equations, have received specific attention in recent years. An important number of surveys dealing with this type of equations can model phenomenons arising from the study of elastic mechanics, in numerous physical phenomena such as systems of particles in thermodynamical equilibrium, dielectric breakdown, image restoration, biological phenomena, and so on (see [9, 14, 19, 25, 30] and references therein for discussions of various applications).

In recent years, several authors have considered the Nehari manifold to study problems involving sign-changing weight functions [2, 4, 6, 7, 10–13, 15, 16, 26, 28]. More precisely, Ji and Wang [16] proved the existence of two nontrivial solutions for the following perturbed nonlinear p-biharmonic boundary value problem:

$$\begin{cases} \Delta_p^2 u = |u|^{q-2}u + \lambda h(x)|u|^{r-2}u, & x \in \Omega, \\ u = \nabla u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $1 < r < p < q < p^*$ with $p^* = \frac{Np}{N-2p}$ if $p < \frac{N}{2}$ and $p^* = \infty$ if $p \geq \frac{N}{2}$, h is a continuous function in $\overline{\Omega}$, which can change sign, and $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator.

Chen et al. [8] considered the following nonhomogeneous Kirchhof-type problem:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2\,dx)\Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{m-2}u, & x \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < q < 2 < m < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$), $M(s) = a + bs$, and a, b, λ, are positive real numbers. The weight functions f and g are continuous in $\overline{\Omega}$. Based on the Nehari manifold method and the fibering maps, the authors proved that problem (1.1) admits at least two nontrivial solutions.

Inspired by the works mentioned, we study the following Kirchhof-type system:

$$\begin{cases} -M(\int_\Omega(|\Delta_p u|^2 + |u|^p)\,dx)(\Delta_p^2 u - |u|^{p-2}u) \\ \quad = \lambda f(x)|u|^{q-2}u + g(x)|u|^{m-2}u, \quad x \in \Omega, \\ u \in W^{2,p}(\Omega) \setminus \{0\}, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, the functions f, g are measurable in Ω, and the function M is defined on $[0,\infty)$ by $M(s) = a + bs^l$ for some $a, b > 0$ and $0 \leq l < \frac{2p}{N-2p}$.

Before giving our main result, we assume the following hypotheses:

(H_1) g is a measurable function such that $g \in L^{\frac{p^*}{p^*-m}}(\Omega)$ and $g^+ := \max(g, 0) \neq 0$.

(H_2) f is a measurable function such that $f \in L^{\frac{p^*}{p^*-q}}(\Omega)$ and $f^+ := \max(f, 0) \neq 0$.

Our main result of this paper is the following theorem.

Theorem 1.1 *Assume* (H_1)–(H_2). *If* $2p < N$ *and* $1 < m < p \leq p(l+1) < q < p^*$, *then there exists* $\lambda_0 > 0$ *such that for all* $|\lambda| \in (0, \lambda_0)$, *problem* (1.2) *has at least two nontrivial solutions.*

The rest of this paper is organized as follows. In Sect. 2, we give some definitions and basic results that will be used in this paper. Section 3 is devoted to the proof of Theorem 1.1.

2 Definitions and basic results

In this section, we collect some basic preliminary results that will be used in the proof of our main result. To state our main result, let us introduce some definitions and notations. First, we define the Sobolev space

$$W^{2,p}(\Omega) = \{u \in L^p(\Omega), |\Delta u| \in L^p(\Omega)\}$$

equipped with the norm

$$\|u\| = \left(\int_\Omega \left(|\Delta u|^p + |u|^p\right) dx\right)^{\frac{1}{p}}.$$

For $1 < s \leq p^*$, we denote by C_s the best Sobolev constant for the embedding operator $W^{2,p}(\Omega) \hookrightarrow L^s(\Omega)$, which is given by

$$C_s := \inf_{u \in W^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p \, dx}{(\int_\Omega |u|^s \, dx)^{\frac{p}{s}}}.$$

In particular, we have

$$\left(\int_\Omega |u|^s \, dx\right)^{\frac{1}{s}} \leq (C_s)^{-\frac{1}{p}} \|u\|,$$

that is,

$$\|u\|_s \leq (C_s)^{-\frac{1}{p}} \|u\|, \tag{2.1}$$

where $\|\cdot\|_s$ is the usual norm in $L^s(\Omega)$.

Definition 2.1 We say that a function $u \in W^{2,p}(\Omega)$ is a weak solution of (1.2) if for all $v \in W^{2,p}(\Omega)$, we have

$$M\left(\int_\Omega \left(|\Delta_p u|^2 + |u|^p\right) dx\right) \int_\Omega \left(|\Delta|^{p-2} \Delta u \Delta v - |u|^{p-2} uv\right) dx = \lambda \int_\Omega f(x)|u|^{q-2} uv \, dx + \int_\Omega g(x)|u|^{m-2} uv \, dx.$$

Associated with the problem (1.2), we define the functional energy $J_{\lambda,M}(u) : W^{2,p}(\Omega) \longrightarrow \mathbb{R}$ by

$$J_{\lambda,M}(u) = \frac{1}{p}\widehat{M}\left(\|u\|^p\right) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q \, dx - \frac{1}{m}\int_\Omega g(x)|u|^m \, dx, \tag{2.2}$$

where $\widehat{M}(t) = at + \frac{b}{l+1} t^{l+1}$.

Lemma 2.1 *The functional $J_{\lambda,M}$ belongs to $C^1(W^{2,p}(\Omega), \mathbb{R})$. Moreover, for all $u \in W^{2,p}(\Omega)$, we have*

$$\left\langle J'_{\lambda,M}(u), u\right\rangle = a\|u\|^p + b\|u\|^{p(l+1)} - \lambda \int_\Omega f(x)|u|^q \, dx - \int_\Omega g(x)|u|^m \, dx, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between the space $W^{2,p}(\Omega)$ and its dual $W^{-2,p}(\Omega)$.

Proof From the hypotheses (H_1)–(H_2) it is obvious that $J_{\lambda,M} \in C^1(W^{2,p}(\Omega), \mathbb{R})$ and its Gateaux derivative is given by

$$\left\langle J'_{\lambda,M}(u), \varphi\right\rangle = M\left(\int_\Omega \left(|\Delta_p u|^2 + |u|^p\right) dx\right) \int_\Omega \left(|\Delta|^{p-2} \Delta u \Delta \varphi - |u|^{p-2} u\varphi\right) dx - \lambda \int_\Omega f(x)|u|^{q-2} u\varphi \, dx - \int_\Omega g(x)|u|^{m-2} u\varphi \, dx \quad \forall u, \varphi \in W^{2,p}(\Omega).$$

This completes the proof of Lemma 2.1. □

Since the energy functional is not bounded from bellow on $W^{2,p}(\Omega)$, we introduce the following subspace of $W^{2,p}(\Omega)$, which is called Nehari manifold:

$$N_{\lambda,M} = \{u \in W^{2,p}(\Omega) \setminus \{0\}; \langle J'_{\lambda,M}(u), u\rangle = 0\}.$$

Thus $u \in N_{\lambda,M}$ if and only if

$$a\|u\|^p + b\|u\|^{p(l+1)} - \lambda \int_\Omega f(x)|u|^q\,dx - \int_\Omega g(x)|u|^m\,dx = 0. \tag{2.4}$$

Note that the Nehari manifold $N_{\lambda,M}$ contains every nonzero solution of equation (1.2).

Lemma 2.2 *Suppose that* (H_1) *and* (H_2) *hold. Then the energy functional* $J_{\lambda,M}$ *is coercive and bounded below on* $N_{\lambda,M}$.

Proof Let $u \in N_{\lambda,M}$. Then from (2.1), (2.4), and the Hölder inequality we have

$$\begin{aligned}
J_{\lambda,M}(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q\,dx - \frac{1}{m}\int_\Omega g(x)|u|^m\,dx \\
&\geq \frac{q-p}{pq}a\|u\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u\|^{p(l+1)} - \frac{q-m}{mq}\int_\Omega g(x)|u|^m\,dx \\
&\geq \frac{q-p}{pq}a\|u\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u\|^{p(l+1)} \\
&\quad - \frac{q-m}{mq}\left(\int_\Omega |g|^{\frac{p^*}{p^*-m}}\,dx\right)^{\frac{p^*-m}{p^*}}\left(\int_\Omega |u|^{p^*}\,dx\right)^{\frac{m}{p^*}} \\
&\geq \frac{q-p}{pq}a\|u\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u\|^{p(l+1)} - \frac{q-m}{mq}\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}\|u\|^m.
\end{aligned}$$

Since $m < p(l+1) < q$, $J_{\lambda,M}$ is coercive and bounded below on $N_{\lambda,M}$. □

The Nehari manifold $N_{\lambda,M}$ is closely linked to the behavior of the function $h_u : t \longrightarrow J_{\lambda,M}(tu)$ for $t > 0$, defined as follows:

$$h_u(t) = \frac{1}{p}\widehat{M}(t^p\|u\|^p) - \lambda\frac{t^q}{q}\int_\Omega f(x)|u|^q\,dx - \frac{t^m}{m}\int_\Omega g(x)|u|^m\,dx.$$

Such maps, introduced by Drábek and Pohozaev [10], are known as fibering maps. A simple calculation shows that, for each $u \in W^{2,p}(\Omega)$, we have

$$h'_u(t) = at^{p-1}\|u\|^p + bt^{p(l+1)-1}\|u\|^{p(l+1)} - \lambda t^{q-1}\int_\Omega f(x)|u|^q\,dx - t^{m-1}\int_\Omega g(x)|u|^m\,dx$$

and

$$\begin{aligned}
h''_u(t) &= a(p-1)t^{p-2}\|u\|^p + b\big(p(l+1)-1\big)t^{p(l+1)-2}\|u\|^{p(l+1)} \\
&\quad - \lambda(q-1)t^{q-2}\int_\Omega g(x)|u|^q\,dx - (m-1)t^{m-2}\int_\Omega g(x)|u|^m\,dx.
\end{aligned}$$

Clearly,

$$th'_u(t) = \langle J'_{\lambda,M}(tu), tu\rangle = 0.$$

Thus, for all $u \in W^{2,p}(\Omega) \setminus \{0\}$ and $t > 0$, we have

$$h'_u(t) = 0 \quad \text{if and only if} \quad tu \in N_{\lambda,M}.$$

In particular, $h'_u(1) = 0$ if and only if $u \in N_{\lambda,M}$. Also, by equation (2.4) it is easy to see that for $u \in N_{\lambda,M}$,

$$\begin{aligned} h''_u(1) &= a(p-1)\|u\|^p + b\big(p(l+1)-1\big)\|u\|^{p(l+1)} \\ &\quad - \lambda(q-1)\int_\Omega g(x)|u|^q\,dx - (m-1)\int_\Omega g(x)|u|^m\,dx \\ &= a(p-m)\|u\|^p + b\big(p(l+1)-m\big)\|u\|^{p(l+1)} - \lambda(q-m)\int_\Omega f(x)|u|^q\,dx \qquad (2.5) \\ &= a(p-q)\|u\|^p + b\big(p(l+1)-q\big)\|u\|^{p(l+1)} + (q-m)\int_\Omega g(x)|u|^m\,dx \qquad (2.6) \\ &= bpl\|u\|^{p(l+1)} + \lambda(p-q)\int_\Omega f(x)|u|^q\,dx + (p-m)\int_\Omega g(x)|u|^m\,dx. \end{aligned}$$

In order to have multiplicity of solutions, we split $N_{\lambda,M}$ into three parts

$$\begin{aligned} N^+_{\lambda,M} &= \{u \in N_{\lambda,M}; h''_u(1) > 0\}, \\ N^0_{\lambda,M} &= \{u \in N_{\lambda,M}; h''_u(1) = 0\}, \\ N^-_{\lambda,M} &= \{u \in N_{\lambda,M}; h''_u(1) < 0\}. \end{aligned}$$

Furthermore, using arguments similar to those in of Theorem 2.3 in [6], we have the following lemma.

Lemma 2.3 *Let u be a local minimizer for $J_{\lambda,M}$ on $N_{\lambda,M}$ not belonging to $N^0_{\lambda,M}$. Then $J'_{\lambda,M}(u) = 0$.*

Put

$$\lambda_1 = \frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\left(\frac{a(q-p)(C_{p^*})^{\frac{q}{p}}}{(q-m)\|g\|_{\frac{p^*}{p^*-m}}}\right)^{\frac{q-p}{p-m}}.$$

Then we have the following lemma.

Lemma 2.4 *If $0 < |\lambda| < \lambda_1$, then $N^0_{\lambda,M} = \phi$.*

Proof Suppose, otherwise, that $0 < |\lambda| < \lambda_1$ with $N^0_{\lambda,M} \neq \phi$. Let $u \in N^0_{\lambda,M}$. Then we have

$$h''_u(1) = 0.$$

From (2.5) and (2.6) we get

$$(q-m)\int_\Omega g(x)|u|^m\,dx = a(q-p)\|u\|^p + b\big(q-p(l+1)\big)\|u\|^{p(l+1)}$$

and

$$\lambda(q-m)\int_\Omega f(x)|u|^q\,dx = a(p-m)\|u\|^p + b\big(p(l+1)-m\big)\|u\|^{p(l+1)}.$$

Therefore

$$a(q-p)\|u\|^p \le (q-m)\int_\Omega g(x)|u|^m\,dx \tag{2.7}$$

and

$$a(p-m)\|u\|^p \le \lambda(q-m)\int_\Omega f(x)|u|^q\,dx. \tag{2.8}$$

On the other hand, from (2.1) and the Hölder inequality we obtain

$$(q-m)\int_\Omega g(x)|u|^m\,dx \le (q-m)\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}\|u\|^m \tag{2.9}$$

and

$$\lambda(q-m)\int_\Omega f(x)|u|^q\,dx \le |\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}(C_{p^*})^{-\frac{q}{p}}\|u\|^q. \tag{2.10}$$

By combining (2.7) and (2.9) we get

$$\|u\| \le \left(\frac{(q-m)\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}}{a(q-p)}\right)^{\frac{1}{p-m}}. \tag{2.11}$$

Moreover, by combining (2.8) and (2.10) we get

$$\|u\| \ge \left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{1}{q-p}}. \tag{2.12}$$

Finally, by combining (2.11) and (2.12) we obtain $\lambda_1 \le |\lambda|$, which is a contradiction. □

From Lemma 2.4, for $0 < |\lambda| < \lambda_1$, we can write $N_{\lambda,M} = N^+_{\lambda,M} \cup N^-_{\lambda,M}$.
Put

$$\theta_{\lambda,M} = \inf_{u\in N_{\lambda,M}} J_{\lambda,M}(u), \qquad \theta^+_{\lambda,M} = \inf_{u\in N^+_{\lambda,M}} J_{\lambda,M}(u) \quad \text{and} \quad \theta^-_{\lambda,M} = \inf_{u\in N^-_{\lambda,M}} J_{\lambda,M}(u),$$

and

$$\lambda_2 := \frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\left(\frac{ma(q-p)(C_{p^*})^{\frac{m}{p}}}{p(q-m)\|g\|_{\frac{p^*}{p^*-m}}}\right)^{\frac{q-p}{p-m}}.$$

Then we have the following:

Proposition 2.1 *If* $0 < |\lambda| < \lambda_2$, *then*:

(i)

$$\theta_{\lambda,M} \le \theta^+_{\lambda,M} < 0. \tag{2.13}$$

(ii) *There exists* $C > 0$ *such that*

$$\theta^-_{\lambda,M} \ge C > 0. \tag{2.14}$$

Proof (i) Let $u \in N^+_{\lambda,M}$. Then from (2.6) and the fact that $h''_u(1) > 0$ we obtain

$$a(q-p)\|u\|^p + b\big(q - p(l+1)\big)\|u\|^{p(l+1)} < (q-m)\int_\Omega g(x)|u|^m\,dx.$$

So, by (2.4) we obtain

$$\begin{aligned} J_{\lambda,M}(u) &= a\left(\frac{q-p}{pq}\right)\|u\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u\|^{p(l+1)} - \frac{q-m}{mq}\int_\Omega g(x)|u|^m\,dx \\ &< \frac{a(q-p)}{q}\left(\frac{m-p}{pm}\right)\|u\|^p + \frac{b(q-p(l+1))}{q}\frac{m-p}{m-q}\left(\frac{m-p(l+1)}{mp(l+1)}\right)\|u\|^{p(l+1)} < 0. \end{aligned}$$

Thus we can deduce that $\theta_{\lambda,M} \le \theta^+_{\lambda,M} < 0$.

(ii) Let $u \in N^-_{\lambda,M}$. Then from equations (2.5) and the fact that $h''_u(1) < 0$ we get

$$a(p-m)\|u\|^p + b\big(p(l+1) - m\big)\|u\|^{p(l+1)} < \lambda(q-m)\int_\Omega f(x)|u|^q.$$

So,

$$a(p-m)\|u\|^p < \lambda(q-m)\int_\Omega f(x)|u|^q.$$

Therefore equation (2.10) implies that

$$\|u\| \ge \left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{1}{q-p}}.$$

In addition, from equations (2.1), (2.4), and (2.9), using the Hölder inequality, we have

$$\begin{aligned} J_{\lambda,M}(u) &= a\left(\frac{q-p}{pq}\right)\|u\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u\|^{p(l+1)} - \frac{q-m}{mq}\int_\Omega g(x)|u|^m\,dx \\ &\ge a\left(\frac{q-p}{pq}\right)\|u\|^p - \frac{q-m}{mq}\int_\Omega g(x)|u|^m\,dx \\ &\ge a\left(\frac{q-p}{pq}\right)\|u\|^p - \frac{q-m}{mq}\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}\|u\|^m \\ &\ge \|u\|^m\left(a\left(\frac{q-p}{pq}\right)\|u\|^{p-m} - \frac{q-m}{mq}\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{m}{q-p}} \\
&\quad \times \left(a\left(\frac{q-p}{pq}\right)\left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{p-m}{q-p}} - \frac{q-m}{mq}\|g\|_{\frac{p^*}{p^*-m}}(C_{p^*})^{-\frac{m}{p}}\right) \\
&:= C.
\end{aligned}
$$

It is not difficult to see that if $0 < |\lambda| < \lambda_2$, then $C > 0$. This completes the proof of Proposition 2.1. □

Set

$$\lambda_0 = \min(\lambda_1, \lambda_2).$$

Proposition 2.2 *Suppose that* $0 < |\lambda| < \lambda_0$. *Then for each* $u \in W^{2,p}(\Omega)$ *with*

$$\int_\Omega g(x)|u|^m\,dx > 0,$$

there exists $T > 0$ *such that*:

(i) *If* $\lambda \int_\Omega f(x)|u|^q\,dx \leq 0$, *then there exists a unique* $t^+ < T$ *such that* $t^+u \in N^+_{\lambda,M}$ *and*

$$J_{\lambda,M}\big(t^+u\big) = \inf_{0\leq t\leq T} J_{\lambda,M}(tu).$$

(ii) *If* $\lambda \int_\Omega f(x)|u|^q\,dx > 0$, *then there are unique* $0 < t^+ < T < t^-$ *such that* $(t^-u, t^+u) \in N^-_{\lambda,M} \times N^+_{\lambda,M}$ *and*

$$J_{\lambda,M}\big(t^-u\big) = \sup_{t\geq 0} J_{\lambda,M}(tu); \qquad J_{\lambda,M}\big(t^+u\big) = \inf_{0\leq t<T} J_{\lambda,M}(tu).$$

Proof Fix $u \in W^{2,p}(\Omega)$ with $\int_\Omega g(x)|u|^m\,dx > 0$ and define the map Ψ_u on $(0,\infty)$ by

$$\Psi_u(t) = at^{p-q}\|u\|^p + bt^{p(l+1)-q}\|u\|^{p(l+1)} - t^{m-q}\int_\Omega g(x)|u|^m\,dx.$$

A simple calculation shows that

$$h'_u(t) = t^{q-1}\left(\Psi_u(t) - \lambda\int_\Omega f(x)|u|^q\,dx\right).$$

Moreover, for $t > 0$, we have

$$\Psi'_u(t) = t^{m-q-1}\psi_u(t),$$

where

$$\psi_u(t) = a(p-q)t^{p-m}\|u\|^p + b\big(p(l+1)-q\big)t^{p(l+1)-m}\|u\|^{p(l+1)} + (q-m)\int_\Omega g(x)|u|^m\,dx.$$

Since $m < p < p(l+1) < q$, then we have

$$\lim_{t\to 0}\psi_u(t) = (q-m)\int_\Omega g(x)|u|^m\,dx > 0 \quad \text{and} \quad \lim_{t\to\infty}\psi_u(t) = -\infty.$$

Also, ψ_u is decreasing on $(0,\infty)$. So, there is a unique $T > 0$ such that $\psi_u(t) > 0$ for $0 < t < T$, $\psi_u(T) = 0$, and $\psi_u(t) < 0$ for $t > T$. Therefore Ψ_u admits a global maximum at T, ψ_u is increasing on $(0,T)$, decreasing on (T,∞), $\lim_{t\to 0}\Psi_u(t) = -\infty$, and $\lim_{t\to\infty}\Psi_u(t) = 0$.

(i) If $\lambda\int_\Omega f(x)|u|^q\,dx < 0$, then there is a unique $t^+ \in (0,T)$ such that

$$\Psi_u\bigl(t^+\bigr) = \lambda\int_\Omega f(x)|u|^q\,dx \quad \text{and} \quad \Psi_u'\bigl(t^+\bigr) > 0.$$

Therefore $h_u'(t^+) = 0$ and $h_u''(t^+) > 0$, that is, h_u has a global maximum at t^+, and $t^+u \in N^+_{\lambda,M}$.

(ii) Assume that $\lambda\int_\Omega f(x)|u|^q\,dx > 0$, and put

$$T_0 = \left(\frac{(q-m)\int_\Omega g(x)|u|^m\,dx}{a(q-p)\|u\|^p}\right)^{\frac{1}{p-m}}.$$

Then we have

$$\psi_u(T_0) = b\bigl(p(l+1)-q\bigr)T_0^{p(l+1)-m}\|u\|^{p(l+1)} < 0 = \psi_u(T).$$

Since ψ_u is a decreasing function, we get $T_0 > T$. Moreover, since Ψ_u is decreasing on (T,∞), from (2.9) we have

$$\begin{aligned}
\Psi_u(T) &\geq \Psi_u(T_0) \\
&\geq a(T_0)^{p-q}\|u\|^p - (T_0)^{m-q}\int_\Omega g(x)|u|^m\,dx \\
&\geq a\left(\frac{a(q-p)\|u\|^p}{(q-m)\int_\Omega g(x)|u|^m\,dx}\right)^{\frac{q-p}{p-m}}\|u\|^p \\
&\quad - \left(\frac{a(q-p)\|u\|^p}{(q-m)\int_\Omega g(x)|u|^m\,dx}\right)^{\frac{q-m}{p-m}}\int_\Omega g(x)|u|^m\,dx \\
&\geq \frac{a(p-m)}{q-m}\left(\frac{a(q-p)}{q-m}\right)^{\frac{q-p}{q-m}}\frac{\|u\|^{p\frac{q-m}{p-m}}}{(\int_\Omega g(x)|u|^m\,dx)^{\frac{q-p}{p-m}}} \\
&\geq \frac{a(p-m)}{q-m}\left(\frac{a(q-p)}{q-m}\right)^{\frac{q-p}{q-m}}\frac{\|u\|^{p\frac{q-m}{p-m}}}{((C_{p^*})^{-\frac{m}{p}}\|g\|_{\frac{p^*}{p^*-m}}\|u\|^m)^{\frac{q-p}{p-m}}} \\
&\geq \frac{a(p-m)}{q-m}\left(\frac{a(q-p)}{(q-m)(C_{p^*})^{-\frac{m}{p}}\|g\|_{\frac{p^*}{p^*-m}}}\right)^{\frac{q-p}{q-m}}\|u\|^q.
\end{aligned}$$

Therefore by (2.1) we obtain

$$\Psi_u(T)-\lambda\int_\Omega f(x)|u|^q\,dx \geq \frac{a(p-m)}{q-m}\left(\frac{a(q-p)}{(q-m)(C_{p^*})^{-\frac{m}{p}}\|g\|_{\frac{p^*}{p^*-m}}}\right)^{\frac{q-p}{q-m}}\|u\|^q$$
$$-|\lambda|\|f\|_{\frac{p^*}{p^*-q}}(C_{p^*})^{-\frac{q}{p}}\|u\|^q$$
$$\leq \|f\|_{\frac{p^*}{p^*-q}}(C_{p^*})^{-\frac{q}{p}}\|u\|^q\bigl(\lambda_1-|\lambda|\bigr).$$

Since $0<|\lambda|<\lambda_0$, we have

$$0<\lambda\int_\Omega f(x)|u|^q\,dx<\Psi_u(T).$$

Hence there are unique t^- and t^+ such that $0<t^+<T<t^-$,

$$\Psi_u\bigl(t^+\bigr)=\lambda\int_\Omega f(x)|u|^q\,dx=\Psi_u\bigl(t^-\bigr)$$

and

$$\Psi_u'\bigl(t^+\bigr)>0>\Psi_u'\bigl(t^-\bigr).$$

By a similar argument as in case (i) we conclude that $t^-u\in N^-_{\lambda,M}$ and $t^+u\in N^+_{\lambda,M}$. Moreover,

$$J_{\lambda,M}\bigl(t^+u\bigr)\leq J_{\lambda,M}(tu)\leq J_{\lambda,M}\bigl(t^-u\bigr)\quad\text{for each } t\in\bigl[t^+,t^-\bigr],$$

and $J_{\lambda,M}(tu)\leq J_{\lambda,M}(t^-u)$ for each $t\geq 0$. Thus

$$J_{\lambda,M}\bigl(t^+u\bigr)=\inf_{0\leq t\leq T}J_{\lambda,M}(tu)\quad\text{and}\quad J_{\lambda,M}\bigl(t^-u\bigr)=\sup_{T\leq t}J_{\lambda,M}(tu).\qquad\square$$

Proposition 2.3 *For every $u\in W^{2,p}(\Omega)$ with $\lambda\int_\Omega f(x)|u|^m\,dx>0$, there exists $\widetilde{T}$ such that:*

(i) *If $\int_\Omega g(x)|u|^m\,dx\leq 0$, then there exists a unique $t^->\widetilde{T}$ such that $t^-u\in N^-_{\lambda,M}$ and*

$$J_{\lambda,M}\bigl(t^-u\bigr)=\sup_{t\geq\widetilde{T}}J_{\lambda,M}(tu).$$

(ii) *If $\int_\Omega g(x)|u|^m\,dx>0$, then there are unique $0<t^+<\widetilde{T}<t^-$ such that $(t^-u,t^+u)\in N^-_{\lambda,M}\times N^+_{\lambda,M}$ and*

$$J_{\lambda,M}\bigl(t^-u\bigr)=\sup_{t\geq 0}J_{\lambda,M}(tu);\qquad J_{\lambda,M}\bigl(t^+u\bigr)=\inf_{0\leq t<\widetilde{T}}J_{\lambda,M}(tu).$$

Proof Let $u\in W^{2,p}(\Omega)$ be such that $\lambda\int_\Omega f(x)|u|^q\,dx>0$ and define the map Ψ_u by

$$\Psi_u(t)=at^{p-m}\|u\|^p+bt^{p(l+1)-m}\|u\|^{p(l+1)}-\lambda t^{q-m}\int_\Omega f(x)|u|^q\,dx,\quad\text{for } t\geq 0.$$

Put

$$\widetilde{T}_0 = \left(\frac{b(p(l+1)-m)\|u\|^{p(l+1)}}{\lambda(q-m)\int_\Omega f(x)|u|^q\,dx} \right)^{\frac{1}{q-p(l+1)}}.$$

Then by similar arguments as in the proof of Proposition 2.2 we can deduce the results of Proposition 2.3 □

Proposition 2.4 *There exist sequences* $\{u_k^\pm\}$ *in* $N_\lambda^\pm$ *such that*

$$J_{\lambda,M}(u_k^\pm) = \theta_{\lambda,M}^\pm + o(1) \quad \textit{and} \quad J'_{\lambda,M}(u_k^\pm) = o(1).$$

Proof We omit the proof, which is almost the same as that in Wu ([29], Proposition 9). □

3 Proof of our main result

In this section, we apply the method of Nehari manifold combined with the fibering maps to investigate the multiplicity of nontrivial solutions for problem (1.2). To this aim, we assume that $|\lambda| \in (0, \lambda_0)$.

Theorem 3.1 *Assume that* (H_1)*–*(H_2) *hold. Then problem* (1.2) *has a nontrivial solution* $u_{\lambda,M}^+$ *in* $N_{\lambda,M}^+$ *such that*

$$J_{\lambda,M}(u_{\lambda,M}^+) = \theta_{\lambda,M}^+.$$

Proof By Proposition 2.4 there exists a sequence $\{u_k^+\}$ in $N_{\lambda,M}^+$ such that

$$J_{\lambda,M}(u_k^+) = \theta_{\lambda,M}^+ + o(1) \quad \text{and} \quad J'_{\lambda,M}(u_k^+) = o(1) \text{ in } W^{-2,p}. \tag{3.1}$$

Using Lemma 2.2, up to a subsequence, there exists $u_{\lambda,M}^+$ in $W^{2,p}(\Omega)$ such that

$$\begin{cases} u_k^+ \rightharpoonup u_{\lambda,M}^+ & \text{weakly in } W^{2,p}(\Omega), \\ u_k^+ \longrightarrow u_{\lambda,M}^+ & \text{strongly in} L^s(\Omega) \text{ for } 1 < s < p^*, \\ u_k^+ \longrightarrow u_{\lambda,M}^+ & \text{a.e. in } \Omega. \end{cases} \tag{3.2}$$

We will prove that $u_k^+ \longrightarrow u_{\lambda,M}^+$ strongly in $W^{2,p}(\Omega)$ and $J_{\lambda,M}(u_{\lambda,M}^+) = \theta_{\lambda,M}^+$.

Since $u_{\lambda,M}^+ \in N_{\lambda,M}$, by Fatou's lemma and equation (3.1) we get

$$\begin{aligned} \theta_{\lambda,M}^+ \le J_{\lambda,M}(u_{\lambda,M}^+) &= \frac{1}{p}\widehat{M}\left(\|u_{\lambda,M}^+\|^p\right) - \frac{\lambda}{q}\int_\Omega f(x)|u_{\lambda,M}^+|^q\,dx - \frac{1}{m}\int_\Omega g(x)|u_{\lambda,M}^+|^m\,dx \\ &\le \liminf_{k\to\infty}\left(\frac{1}{p}\widehat{M}\left(\|u_k^+\|^p\right) - \frac{\lambda}{q}\int_\Omega f(x)|u_k^+|^q\,dx - \frac{1}{m}\int_\Omega g(x)|u_k^+|^m\,dx\right) \\ &\le \liminf_{k\to\infty} J_{\lambda,M}(u_k^+) = \theta_{\lambda,M} \\ &= \lim_{k\uparrow\infty} J_{\lambda,M}(u_k^+) = \theta_{\lambda,M}^+. \end{aligned}$$

So, it is easy to see that

$$J_{\lambda,M}(u_{\lambda,M}^+) = \theta_{\lambda,M}^+ \quad \text{and} \quad \widehat{M}\left(\|u_k^+\|^p\right) \longrightarrow \widehat{M}\left(\|u_{\lambda,M}^+\|^p\right) \quad \text{as } k \longrightarrow \infty.$$

From the Brezis–Lieb lemma [5] we obtain $\|u_k^+ - u_{\lambda,M}^+\|^p = \|u_k^+\|^p - \|u_{\lambda,M}^+\|^p$. Therefore $u_k \longrightarrow u_{\lambda,M}$ strongly in $W^{2,p}(\Omega)$.

Now we will prove that $u_{\lambda,M}^+ \in N_{\lambda,M}^+$. We proceed by contradiction assuming that $u_{\lambda,M}^+ \in N_{\lambda,M}^-$.

We have

$$\begin{aligned} J_{\lambda,M}(u_k^+) &= \frac{q-p}{pq}a\|u_k^+\|^p + b\left(\frac{q-p(l+1)}{qp(l+1)}\right)\|u_k^+\|^{p(l+1)} - \frac{q-m}{mq}\int_\Omega g(x)|u_k^+|^m\,dx \\ &\geq -\frac{q-m}{mq}\int_\Omega g(x)|u_k^+|^m\,dx. \end{aligned}$$

By letting k tend to infinity we obtain

$$\int_\Omega g(x)|u_{\lambda,M}^+|^m\,dx \geq -\frac{mq}{q-m}\theta_{\lambda,M}^+ > 0.$$

Therefore $u_{\lambda,M}^+$ is nontrivial. Moreover, Propositions 2.2 and 2.3(ii) imply the existence of a unique t^+ such that $t^+u_{\lambda,M}^+ \in N_{\lambda,M}^+$. Since $u_{\lambda,M}^+ \in N_{\lambda,M}^-$, we have

$$\frac{d^2}{dt^2}h_{u_{\lambda,M}^+}(t^+) > 0 \quad \text{and} \quad \frac{d}{dt}h_{u_{\lambda,M}^+}(1) < 0.$$

So, there exists $\widetilde{t} \in (t^+, 1)$ such that

$$h_{u_{\lambda,M}^+}(t^+) = J_{\lambda,M}(t^+u_{\lambda,M}^+) < h_{u_{\lambda,M}^+}(\widetilde{t}) = J_{\lambda,M}(\widetilde{t}u_{\lambda,M}^+).$$

Therefore

$$J_{\lambda,M}(t^+u_{\lambda,M}^+) < J_{\lambda,M}(\widetilde{t}u_{\lambda,M}^+) \leq J_{\lambda,M}(t^-u_{\lambda,M}^+) = J_{\lambda,M}(u_{\lambda,M}^+),$$

which is a contradiction. Therefore $u_{\lambda,M}^+ \in N_{\lambda,M}^+$, Moreover, it is not difficult to see that (3.1) and (3.2) imply that $u_{\lambda,M}^+$ is a weak solution of problem (1.2). The proof is now completed. □

Theorem 3.2 *If* $0 < |\lambda| < \lambda_0$ *and* (H_1)–(H_3) *hold, then problem* (1.2) *admits a nontrivial solution* $u_{\lambda,M}^-$ *in* $N_{\lambda,M}^-$ *satisfying*

$$J_{\lambda,M}(u_{\lambda,M}^-) = \theta_{\lambda,M}^-.$$

Proof By Proposition 2.4 there exists a sequence $\{u_k^-\}$ in $N_{\lambda,M}^-$ such that

$$J_{\lambda,M}(u_k^-) = \theta_{\lambda,M}^- + o(1) \quad \text{and} \quad J'_{\lambda,M}(u_k^-) = o(1) \quad \text{in } W^{-2,p}(\Omega). \tag{3.3}$$

Using Lemma 2.2, up to a subsequence, there exists $u_{\lambda,M}^-$ in $W^{2,p}(\Omega)$ such that

$$\begin{cases} u_k^- \rightharpoonup u_{\lambda,M}^- & \text{weakly in } W^{2,p}(\Omega), \\ u_k^- \longrightarrow u_{\lambda,M}^- & \text{strongly in} L^s(\Omega) \text{ for } 1 < s < p^*, \\ u_k^- \longrightarrow u_{\lambda,M}^- & \text{a.e. in } \Omega. \end{cases} \tag{3.4}$$

We begin by proving that the sequence $\{u_k^-\}$ converges strongly to $u_{\lambda,M}^-$ in $W^{2,p}(\Omega)$. Suppose that, on the contrary,

$$\|u_{\lambda,M}^-\| < \liminf_{k\longrightarrow\infty}\|u_k^-\|.$$

Since $u_k^- \in N_{\lambda,M}^-$, from equations (2.5) and the fact that $h''_{u_k^-}(1) < 0$ we get

$$a(p-m)\|u_k^-\|^p + b\big(p(l+1)-m\big)\|u_k^-\|^{p(l+1)} < \lambda(q-m)\int_\Omega f(x)|u_k^-|^q,$$

which implies that

$$a(p-m)\|u_k^-\|^p < \lambda(q-m)\int_\Omega f(x)|u_k^-|^q. \tag{3.5}$$

Therefore equation (2.10) implies that

$$\|u_k^-\| \geq \left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{1}{q-p}}. \tag{3.6}$$

By combining (3.5) and (3.6) we obtain

$$\lambda\int_\Omega f(x)|u_k^-|^q > \frac{a(p-m)}{q-m}\left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{p}{q-p}}.$$

Passing to the limits as k tends to infinity, we obtain

$$\lambda\int_\Omega f(x)|u_{\lambda,M}^-|^q \geq \frac{a(p-m)}{q-m}\left(\frac{a(p-m)(C_{p^*})^{\frac{q}{p}}}{|\lambda|(q-m)\|f\|_{\frac{p^*}{p^*-q}}}\right)^{\frac{p}{q-p}} > 0.$$

Therefore $u_{\lambda,M}^-$ is nontrivial. Moreover, by Proposition 2.3 there exist a unique $t^- > 0$ such that $t^-u_{\lambda,M}^- \in N_{\lambda,M}^-$. Therefore

$$J_{\lambda,M}\big(t^-u_{\lambda,M}^-\big) < \lim_{k\longrightarrow\infty} J_{\lambda,M}\big(t^-u_k^-\big) \leq \lim_{k\longrightarrow\infty} J_{\lambda,M}\big(u_k^-\big) = \theta_{\lambda,M}^-,$$

a contradiction. Hence $u_k^- \longrightarrow u_{\lambda,M}^-$ strongly in $W^{2,p}(\Omega)$. This implies that

$$J_{\lambda,M}\big(u_k^-\big) \longrightarrow J_{\lambda,M}\big(u_{\lambda,M}^-\big) = \theta_{\lambda,M}^- \quad \text{as } n \longrightarrow \infty.$$

Finally, from (3.3) and (3.4) we obtain that $u_{\lambda,M}^-$ is a weak solution of problem (1.2). This ends the proof of Theorem 3.2 . □

Now Theorems 3.1 and 3.2 and the fact that $N_{\lambda,M}^- \cap N_{\lambda,M}^- = \emptyset$ finishes the proof of Theorem 1.1.

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