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Controllability of fractional evolution systems of Sobolev type via resolvent operators

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Abstract

In this paper, we consider the nonlocal controllability of $\alpha \in (1,2)$-order fractional evolution systems of Sobolev type in abstract spaces. By utilizing fixed point theorems and the theory of resolvent operators we establish some sufficient conditions for the nonlocal controllability of Sobolev-type fractional evolution systems.

MSC: 34K35; 47A10

Keywords: Fractional evolution systems; Nonlocal controllability; Fractional resolvent family; The measure of noncompactness; Relative compactness

1 Introduction

The theory of fractional differential equations admits wide applications in the fields of biology, physics, chemistry, control theory, and so on, and hence it has been regarded as an active aspect of mathematics in recent years. Controllability of fractional differential systems of order $0 < \alpha < 1$ has been investigated by many authors; we refer the readers to [6, 13, 15–17] for more detail. However, as far as we know, the works on the fractional order $1 < \alpha < 2$ are limited. In 2013, using the Sadovskii fixed point theorem and vector-valued operator theory, Li et al. [10] proved the controllability of fractional differential systems of order $\alpha \in (1,2]$ of the form

$$\begin{cases} {}^C D_t^{\alpha} x(t) = Ax(t) + f(t,x(t)) + Bu(t), & t \in J, \\ x(0) + g(x) = x_0, \qquad x'(0) = y_0, \end{cases}$$

where $J = [0,b]$, $b > 0$ is a constant, ${}^C D_t^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha \in (1,2]$. Recently, Lian et al. [12], by using Schauder's fixed point theorem and approximate techniques, studied the approximate controllability of fractional evolution equations of order $\alpha \in (1,2)$. However, investigation of the controllability for fractional evolution systems of order $\alpha \in (1,2)$ of Sobolev type is seldom.

In the present work, we consider the controllability of the fractional control system of Sobolev type with nonlocal conditions in a Hilbert space X of the form

$$\begin{cases} {}^CD_t^\alpha(Ex(t)) = Ax(t) + f(t,x(t)) + Bu(t), & t \in J, \\ Ex(0) = x_0 - g(x), \qquad Ex'(0) = y_0 - h(x), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, ${}^CD_t^\alpha$ denotes the Caputo fractional derivative operator of order α, A and E are two closed linear operators defined in X with domains $D(A)$ and $D(E)$, respectively, the control function u is given in $L^2(J,U)$, U is a Hilbert space, B is a bounded linear operator from U to X, and f, g, and h are appropriate functions to be specified later.

To deal with the Sobolev-type differential equations, the common assumptions are:

(1) E, A are linear operators, and A is closed;

(2) $D(E) \subset D(A)$, and E is bijective;

(3) E^{-1} is a compact operator.

In this case, $-AE^{-1}$ is a bounded operator, which generates a uniformly continuous semigroup; see [2, 7] for more detail. In this paper, without assuming the existence and compactness of E^{-1}, we define the solution operator of (1.1) by fractional resolvent family generated by the pair (A,E). More precisely, we assume that the pair (A,E) generates an $(\alpha,1)$-resolvent family $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$. Then we prove some properties of $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$. Applying these properties and the Laplace transform, we define the solution operator of the fractional control system (1.1). By utilizing fixed point theorems and resolvent operator theory we obtain some controllability results without any compactness conditions on the $(\alpha,1)$-resolvent family $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$.

2 Preliminaries

Let X be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle_X$. We denote by $C(J,X)$ the set of all X-valued continuous functions on J. Then $C(J,X)$ is a Banach space with norm $\|x\|_C = \sup_{t\in J}\|x(t)\|$. For $1 \leq p < +\infty$, $L^p(J,X)$ denotes the Banach space of all Bochner-measurable functions $F: J \to X$ normed by $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p\,dt)^{\frac{1}{p}}$. Let $\mathcal{B}(X) := \mathcal{B}(X,X)$ be the Banach space of all bounded linear operators from X to X with operator norm $\|\cdot\|$.

We recall some definitions of fractional calculus,; see [1, 3, 5] and the reference therein for more detail. For simplicity, for every $\nu \geq 0$, let

$$g_\nu(t) = \begin{cases} \frac{t^{\nu-1}}{\Gamma(\nu)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where Γ is the gamma function. As usual, we define

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

Denote by $n = \lceil\alpha\rceil$ the smallest integer greater than or equal to α.

Definition 1 Let $u \in L^1(J)$. The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by

$$J_t^\alpha u(t) = (g_\alpha * u)(t), \quad t > 0.$$

Definition 2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined for all $u \in L^1(J)$ satisfying $g_{n-\alpha} * u \in W^{n,1}(J)$ by

$$^{L}D_t^{\alpha} u(t) = D_t^n (g_{n-\alpha} * u)(t), \quad t > 0,$$

where $D_t^n = \frac{d^n}{dt^n}$.

Definition 3 The Caputo fractional derivative of order $\alpha > 0$ is defined for all $u \in L^1(J)$ by

$$^{C}D_t^{\alpha} u(t) = J_t^{n-\alpha\,L} D_t^n u(t), \quad t > 0,$$

If $u \in C^n[0,\infty)$, then the Caputo fractional derivative of order $\alpha \in (n-1, n)$ is

$$^{C}D_t^{\alpha} u = \bigl(g_{n-\alpha} * u^{(n)}\bigr)(t), \quad t > 0.$$

By (1.23) of [1] the Laplace transform of Caputo fractional derivative is given by

$$\widehat{^{C}D_t^{\alpha}} u(\lambda) = \lambda^{\alpha} \widehat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0) \lambda^{\alpha-1-k}, \tag{2.1}$$

where $n = \lceil \alpha \rceil$.

We further introduce some results on fractional resolvent family; see [4, 14] for more detail. We assume that A is a closed linear densely defined operator in X. Denote

$$\begin{aligned} \rho_E(A) := \bigl\{ &\lambda \in \mathbb{C} \mid (\lambda E - A) : D(A) \cap D(E) \to X \text{ is invertible, and} \\ &(\lambda E - A)^{-1} \in \mathcal{B}\bigl(X, D(A) \cap D(E)\bigr) \bigr\}. \end{aligned}$$

We call $R(\lambda E, A) := (\lambda E - A)^{-1}$ the E-modified resolvent operator of A.

Definition 4 ([1], Definition 2.4) A strongly continuous family $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be exponentially bounded if there are constants $\overline{M} \geq 1$ and $\omega \geq 0$ such that

$$\bigl\| T(t) \bigr\| \leq \overline{M} e^{\omega t}, \quad t \geq 0.$$

Definition 5 Let $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ be closed linear operators on the Hilbert space X satisfying $D(A) \cap D(E) \neq \{0\}$. Let $\alpha, \beta > 0$. The pair (A, E) is said to be the generator of an (α,β)-resolvent family if there exist a constant $\omega \geq 0$ and a strongly continuous function $C_{\alpha,\beta}^E : [0,\infty) \to \mathcal{B}(X)$ such that $C_{\alpha,\beta}^E(t)$ is exponentially bounded, $\{\lambda^{\alpha} : \operatorname{Re}\lambda > \omega\} \subset \rho_E(A)$, and for all $x \in X$,

$$\lambda^{\alpha-\beta} R\bigl(\lambda^{\alpha} E, A\bigr) x = \int_0^{\infty} e^{-\lambda t} C_{\alpha,\beta}^E(t) x \, dt, \quad \operatorname{Re}\lambda > \omega. \tag{2.2}$$

In this case, $\{C_{\alpha,\beta}^E(t)\}_{t\geq 0}$ is called the (α,β)-resolvent family generated by the pair (A, E).

Let $\{C_{\alpha,1}^E(t)\}_{t\geq 0}$ be the $(\alpha, 1)$-resolvent family generated by the pair (A, E). Then $\{C_{\alpha,1}^E(t)\}_{t\geq 0}$ is exponentially bounded if $M := \sup_{t\in J} \|C_{\alpha,1}^E(t)\| < +\infty$. From [4, 14], using the properties of Laplace transform, (2.1), and (2.2), we obtain the definition of a mild solution of (1.1).

Definition 6 A function $x \in C(J,X)$ is called a mild solution of (1.1) if for each $t \in J$, x satisfies the integral equation

$$\begin{aligned} x(t) &= C^E_{\alpha,1}(t)\bigl[x_0 - g(x)\bigr] + S^E_{\alpha,1}(t)\bigl[y_0 - h(x)\bigr] \\ &\quad + \int_0^t P^E_{\alpha,1}(t-s)\bigl[f\bigl(s,x(s)\bigr) + Bu(s)\bigr]\,ds, \quad t \in J, \end{aligned} \tag{2.3}$$

where

$$S^E_{\alpha,1}(t) = \int_0^t C^E_{\alpha,1}(s)\,ds, \quad t \geq 0, \tag{2.4}$$

$$P^E_{\alpha,1}(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} C^E_{\alpha,1}(s)\,ds, \quad t \geq 0. \tag{2.5}$$

Lemma 1 *The operator families* $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ *and* $\{P^E_{\alpha,1}(t)\}_{t\geq 0}$ *are bounded, that is,*

$$\begin{aligned} &\bigl\|S^E_{\alpha,1}(t)\bigr\| \leq Mb, \quad t \geq 0, \\ &\bigl\|P^E_{\alpha,1}(t)\bigr\| \leq \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}, \quad t \geq 0. \end{aligned}$$

Proof Since $M := \sup_{t\in J}\|C^E_{\alpha,1}(t)\| < +\infty$, by (2.4) and (2.5), for any $t \geq 0$, we have

$$\bigl\|S^E_{\alpha,1}(t)\bigr\| \leq \int_0^t \bigl\|C^E_{\alpha,1}(s)\bigr\|\,ds \leq Mb$$

and

$$\bigl\|P^E_{\alpha,1}(t)\bigr\| \leq \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\bigl\|C^E_{\alpha,1}(s)\bigr\|\,ds \leq \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}.$$

Thus the conclusion is proved. □

Lemma 2 *The operator* $P^E_{\alpha,1}(t)$ *is equicontinuous for* $t \in J$.

Proof For $0 \leq t_1 < t_2 \leq b$, by the definition of $P^E_{\alpha,1}(t)$ we have

$$\begin{aligned} &\bigl\|P^E_{\alpha,1}(t_2) - P^E_{\alpha,1}(t_1)\bigr\| \\ &\quad\leq \frac{1}{\Gamma(\alpha-1)}\left\|\int_0^{t_1}\bigl[(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\bigr]C^E_{\alpha,1}(s)\,ds\right\| \\ &\qquad + \frac{1}{\Gamma(\alpha-1)}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}C^E_{\alpha,1}(s)\,ds\right\| \\ &\quad\leq \frac{M}{\Gamma(\alpha-1)}\int_0^{t_1}\bigl|(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\bigr|\,ds \\ &\qquad + \frac{M}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\,ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{M}{\Gamma(\alpha)}\bigl(t_1^{\alpha-1} - t_2^{\alpha-1} + 2(t_2 - t_1)^{\alpha-1}\bigr) \\
&\to 0
\end{aligned}
$$

as $t_2 \to t_1$. Hence $P^E_{\alpha,1}(t)$ is equicontinuous for $t \in [0, b]$. □

Now we recall some definitions and lemmas on the Hausdorff measure of noncompactness (H-MNC). Let $D \subset X$ be a nonempty bounded subset of X. Denote by $\gamma(D)$ the H-MNC of D with respect to X, that is, $\gamma(D) := \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net in } X\}$.

We denote by $\gamma(\cdot)$ and $\gamma_C(\cdot)$ the H-MNCs of a bounded subset of X and $C(J, X)$, respectively. Let $B \subset C(J, X)$ be a bounded subset, and let $t \in J$. Then $B(t) := \{u(t) : u \in B\}$ is a bounded subset of X, and $\gamma(B(t)) \le \gamma_C(B)$. It is well known (see [9]) that the H-MNC $\gamma(\cdot)$ has the following properties: For any bounded subsets D_1, D_2, and D of X, we have

(i) if $D_1 \subset D_2$, then $\gamma(D_1) \le \gamma(D_2)$;
(ii) $\gamma(D_1 + D_2) \le \gamma(D_1) + \gamma(D_2)$, where $D_1 + D_2 = \{x + y : x \in D_1, y \in D_2\}$;
(iii) $\gamma(D_1 \cup D_2) \le \max\{\gamma(D_1), \gamma(D_2)\}$;
(iv) $\gamma(\rho D) \le |\rho|\gamma(D)$ for $\rho \in \mathbb{R}$;
(v) $\gamma(D) = 0 \Leftrightarrow D$ is relatively compact in X.

By Example 2.1.1 of [9], if a mapping $F : X \to X$ satisfies the Lipschitz condition

$$
\bigl\|F(x) - F(y)\bigr\| \le L\|x - y\|, \quad \forall x, y \in D \subset X,
$$

then

$$
\gamma\bigl(F(D)\bigr) \le L\gamma(D)
$$

for all nonempty bounded subsets D of X.

Definition 7 Let $0 \le k < 1$. An operator $Q : X \to X$ is said to be condensing if $\gamma(Q(D)) < k\gamma(D)$ for every subset $D \subset X$.

Furthermore, for the H-MNC, we have the following lemmas; see [8, 9, 11, 17] for more detail.

Lemma 3 *Let $B \subset C(J, X)$ be a bounded and equicontinuous subset. Then $\gamma(B(t))$ is continuous on J, and $\gamma_C(B) = \max_{t\in J} \gamma(B(t))$.*

Lemma 4 *Let $B \subset C(J, X)$ be a bounded subset. Then there exists a countable subset $B_0 \subset B$ such that $\gamma_C(B) \le 2\gamma_C(B_0)$.*

Lemma 5 *Let X be a separable Hilbert space, and let $B_0 := \{u_n : n \ge 1\} \subset C(J, X)$ be countable. If there exists $\phi \in L^1(J, \mathbb{R}^+)$ such that $\|u_n(t)\| \le \phi(t)$ for a.e. $t \in J$, $n = 1, 2, \ldots$, then $\gamma(B_0(t))$ is Lebesgue integrable on J, and*

$$
\gamma\left(\left\{\int_J u_n(t)\,ds : n \ge 1\right\}\right) \le \int_J \gamma\bigl(B_0(t)\bigr)\,dt.
$$

To investigate the nonlocal controllability of system (1.1), we introduce the definition of controllability.

Definition 8 (Nonlocal controllability) System (1.1) is said to be nonlocally controllable on $[0,b]$ if for all $x_0, y_0, x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution x of system (1.1) satisfies $x(b) + g(x) = x_1$.

At the end of this section, we present a fixed point theorem, on which the proof of our main results are based.

Lemma 6 ([10], Lemma 2.1 (Sadovskii's Fixed Point Theorem)) *Let Q be a condensing operator on a Banach space X. If $Q(S) \subset S$ for a convex closed bounded subset S of X, then Q has at least one fixed point in S.*

3 Nonlocal controllability

In this section, we state and prove some results on the nonlocal controllability of system (1.1). The discussion is based on the theory of resolvent operators and fixed point theorems. For this purpose, we first make the following assumptions.

(H_{AE}) The pair (A,E) generates an $(\alpha,1)$-resolvent family $\{C^E_{\alpha,1}(t)\}_{t\geq 0}$ in X, and

$$M := \sup_{t\in J} \left\| C^E_{\alpha,1}(t) \right\| < +\infty.$$

(H_W) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu := \int_0^b P^E_{\alpha,1}(b-s)Bu(s)\,ds$$

has a linear bounded inverse operator W^{-1} taking values in $L^2(J,U) \setminus \mathrm{Ker}(W)$, and let $M_1 := \|W^{-1}\|$.

(H_{f1}) $f : J \times X \to X$ satisfies the Carathéodory condition, that is, for each $x \in X$, $f(\cdot,x) : J \to X$ is strongly measurable; for each $t \in J$, $f(t,\cdot) : X \to X$ is continuous.

(H_{f2}) There is a function $L_f \in L^1(J,\mathbb{R}^+)$ such that

$$\left\| f(t,x) - f(t,y) \right\| \leq L_f(t)\|x-y\|, \quad \forall t \in J, x,y \in X.$$

(H_g) $g : C(J,X) \to X$, and there exists a constant $L_g > 0$ such that

$$\left\| g(x) - g(y) \right\| \leq L_g\|x-y\|_C, \quad \forall x,y \in C(J,X).$$

(H_h) $h : C(J,X) \to X$, and there exists a constant $L_h > 0$ such that

$$\left\| h(x) - h(y) \right\| \leq L_h\|x-y\|_C, \quad \forall x,y \in C(J,X).$$

(H_B) $B : U \to X$ is a linear bounded operator, and let $M_B := \|B\|$.

By assumption (H_W), for any $x_1 \in X$ and $x \in C(J,X)$, we define the control $u_x \in L^2(J,U)$ as

$$\begin{aligned} u_x(t) = W^{-1}\bigg\{ & x_1 - g(x) - C^E_{\alpha,1}(b)\big[x_0 - g(x)\big] - S^E_{\alpha,1}(b)\big[y_0 - h(x)\big] \\ & - \int_0^b P^E_{\alpha,1}(b-s)f\big(s,x(s)\big)\,ds\bigg\}(t), \quad t \in J. \end{aligned}$$

If $x \in C(J,X)$ is a mild solution of system (1.1) corresponding to the control u_x, then by (H_W) and (2.3) we have

$$\begin{aligned} x(b) &= C^E_{\alpha,1}(b)\big[x_0 - g(x)\big] + S^E_{\alpha,1}(b)\big[y_0 - h(x)\big] \\ &\quad + \int_0^b P^E_{\alpha,1}(b-s)f\big(s,x(s)\big)\,ds + \int_0^b P^E_{\alpha,1}(b-s)Bu_x(s)\,ds \\ &= x_1 - g(x), \end{aligned}$$

which implies $x(b) + g(x) = x_1$, and system (1.1) is nonlocally controllable on J. Hence we will now prove that system (1.1) has mild solutions by using resolvent operator theory and fixed point theorems. Define the operator $Q : C(J,X) \to C(J,X)$ by

$$\begin{aligned} (Qx)(t) &= C^E_{\alpha,1}(t)\big[x_0 - g(x)\big] + S^E_{\alpha,1}(t)\big[y_0 - h(x)\big] \\ &\quad + \int_0^t P^E_{\alpha,1}(t-s)\big[f\big(s,x(s)\big) + Bu_x(s)\big]\,ds, \quad t \in J. \end{aligned} \tag{3.1}$$

By Definition 6 the mild solution of system (1.1) is equivalent to the fixed point of Q. We first apply the contraction mapping principle to prove that Q has a fixed point in $C(J,X)$.

Lemma 7 *Assume that conditions (H_{AE}), (H_W), (H_{f2}), (H_g), and (H_h) are satisfied. Then for all $x, y \in C(J,X)$ and $t \in J$, we have*

$$\big\|u_x(t) - u_y(t)\big\| \le M_1\bigg[(1+M)L_g + MbL_h + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1}\bigg]\|x-y\|_C.$$

Proof For any $x, y \in C(J,X)$ and $t \in J$, by the definition of u_x and u_y we have

$$\begin{aligned} &\big\|u_x(t) - u_y(t)\big\| \\ &\quad \le M_1\bigg[(1+M)\big\|g(x) - g(y)\big\| + Mb\big\|h(x) - h(y)\big\| \\ &\qquad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b \big\|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big\|\,ds\bigg] \\ &\quad \le M_1\bigg[(1+M)L_g\|x-y\|_C + MbL_h\|x-y\|_C \\ &\qquad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^b L_f(s)\big\|x(s) - y(s)\big\|\,ds\bigg] \\ &\quad \le M_1\bigg[(1+M)L_g + MbL_h + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1}\bigg]\|x-y\|_C. \end{aligned}$$

This completes the proof. □

Theorem 1 *Let assumptions (H_{AE}), (H_W), (H_{f1}), (H_{f2}), (H_g), (H_h), and (H_B) hold. Then system (1.1) is nonlocally controllable on J, provided that*

$$M^* := \bigg\{\frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}L_g + M\bigg(1 + \frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}\bigg)\bigg[L_g + bL_h + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1}\bigg]\bigg\} < 1. \tag{3.2}$$

Proof For any $x, y \in C(J, X)$ and $t \in J$, by (3.1) and Lemma 7 we have

$$\begin{aligned}
&\|(Qx)(t) - (Qy)(t)\| \\
&\leq M\|g(x) - g(y)\| + Mb\|h(x) - h(y)\| \\
&\quad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\left[\int_0^t \|f(s, x(s)) - f(s, y(s))\| \, ds + M_B \int_0^t \|u_x(s) - u_y(s)\| \, ds\right] \\
&\leq \left(ML_g + MbL_h + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1}\right)\|x - y\|_C \\
&\quad + \frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}\left[(1 + M)L_g + MbL_h + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|L_f\|_{L^1}\right]\|x - y\|_C \\
&= M^*\|x - y\|_C.
\end{aligned}$$

From (3.2) it follows that $M^* < 1$. Hence by the contraction mapping principle, Q has a unique fixed point x in $C(J, X)$ satisfying $x(b) + g(x) = x_1$. In other words, system (1.1) is nonlocally controllable on J. □

Remark 1 If $E = I$, where I denotes the identity operator in X, and $h(x) \equiv 0$ for all $x \in C(J, X)$, then Theorem 1 is a natural extension of Theorem 3.1 in [10], because we delete the compactness condition (H_4) in [10].

The Lipschitz condition (H_{f2}) of the nonlinear term f is difficult to verify in applications. If we apply more weaker conditions on f, we can also prove the controllability results for system (1.1). For $r > 0$, set $\Omega_r := \{x \in C(J, X) : \|x\|_C \leq r\}$.

Lemma 8 *Assume that conditions* (H_{AE}), (H_W), (H_g), (H_h), *and* $(H_{f2})'$ *are satisfied, where*

$(H_{f2})'$ *For each* $r > 0$, *there is a function* $\varphi_r \in L^1(J, \mathbb{R}^+)$ *satisfying* $\lim_{r\to\infty} \frac{\|\varphi_r\|_{L^1}}{r} = \sigma < \infty$ *such that*

$$\sup_{\|x\|\leq r} \|f(t, x)\| \leq \varphi_r(t), \quad \forall t \in J.$$

Then for any $x \in \Omega_r$ *and* $t \in J$, *we have*

$$\|u_x(t)\| \leq \mathfrak{C} + M_1\left[(1 + M)L_g r + MbL_h r + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}\right], \tag{3.3}$$

where $\mathfrak{C} := M_1[\|x_1\| + M\|x_0\| + Mb\|y_0\| + (1 + M)\|g(0)\| + Mb\|h(0)\|]$.

Proof Applying assumptions (H_{AE}), (H_W), (H_g), (H_h), and $(H_{f2})'$, by direct calculation we can easily prove that u_x satisfies inequality (3.3). So we omit the details. □

Remark 2 If f satisfies the linear growth conditions, for example, $f(t, x) \leq a_1(t)x + a_2(t)$, $t \in J$, $x \in X$, where $a_1, a_2 \in L^1(J, \mathbb{R})$, then condition $(H_{f2})'$ holds when we choose $\varphi_r(t) = \|a_1(t)\|r + \|a_2(t)\|$.

Lemma 9 *Assume that the conditions* (H_{AE}), (H_W), (H_{f1}), $(H_{f2})'$, (H_g), (H_h) *and* (H_B) *are satisfied. Then the operator* Q, *defined as in* (3.1), *maps* Ω_r *into itself for some* $r > 0$ *provided* *that*

$$\frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}L_g + M\left(1 + \frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}\right)\left(L_g + bL_h + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\sigma\right) < 1. \tag{3.4}$$

Moreover, $Q : \Omega_r \to \Omega_r$ *is continuous.*

Proof It is obvious that the operator $Q : C(J,X) \to C(J,X)$ is continuous under these assumptions. Hence we just prove $Q(\Omega_r) \subset \Omega_r$ for some $r > 0$. If this were not true, then for any $r > 0$, there would be $x \in \Omega_r$ such that $r < \|(Qx)(t)\|$ for all $t \in J$. By Lemma 7 and (3.1) we have

$$\begin{aligned}
r &< \|(Qx)(t)\| \\
&\le M\left(\|x_0\| + \|g(x)\|\right) + Mb\left(\|y_0\| + \|h(x)\|\right) \\
&\quad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|f(s,x(s))\|\,ds + M_B\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|u_x(s)\|\,ds \\
&\le M\left(\|x_0\| + L_g r + \|g(0)\|\right) + Mb\left(\|y_0\| + L_h r + \|h(0)\|\right) \\
&\quad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1} + M_B\frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|u_x(s)\|\,ds \\
&\le M\left(\|x_0\| + \|g(0)\|\right) + Mb\left(\|y_0\| + \|h(0)\|\right) + \frac{MM_Bb^\alpha}{\Gamma(\alpha)}\mathfrak{C} \\
&\quad + ML_g r + MbL_h r \\
&\quad + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}\frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}\left[(1+M)L_g r + MbL_h r + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}\right].
\end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \to \infty$, we obtain

$$1 \le \frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}L_g + \left(1 + \frac{MM_1M_Bb^\alpha}{\Gamma(\alpha)}\right)\left(ML_g + MbL_h + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\sigma\right),$$

which is a contradiction to (3.4). Thus there is $r > 0$ such that $Q(\Omega_r) \subset \Omega_r$. □

Theorem 2 *Let assumptions* (H_{AE}), (H_W), (H_{f1}), $(H_{f2})'$, (H_g), (H_h), (H_B), *and* (H_{f3}) *hold, where*

(H_{f3}) *For* $t \in [0,b]$, *the set* $V_\varepsilon := \{P^E_{\alpha,1}(t-s)[f(s,x(s)) + Bu(s)] : x \in \Omega_r, s \in [0,t-\varepsilon], \varepsilon \in (0,t)\}$ *is compact.*

Then system (1.1) *is nonlocally controllable on* J *when* (3.4) *is satisfied.*

Proof We define two operators $Q_1, Q_2 : C(J,X) \to C(J,X)$ by

$$(Q_1x)(t) = C^E_{\alpha,1}(t)\left[x_0 - g(x)\right] + S^E_{\alpha,1}(t)\left[y_0 - h(x)\right], \quad t \in J, \tag{3.5}$$

and

$$(Q_2x)(t) = \int_0^t P^E_{\alpha,1}(t-s)\left[f(s,x(s)) + Bu(s)\right]ds, \quad t \in J. \tag{3.6}$$

Then by (3.1) we know that $Q = Q_1 + Q_2$. By (H_{AE}), (H_g), and (H_h) it is clear that

$$\left\|(Q_1x)(t) - (Q_1y)(t)\right\| \le M(L_g + bL_h)\|x - y\|_C, \quad \forall x, y \in C(J, X). \tag{3.7}$$

Next, we prove that the set $V := \{Q_2x : x \in \Omega_r\}$ is relatively compact in $C(J, X)$. To apply the Ascoli–Arzelà theorem, we prove that $V := \{Q_2x : x \in \Omega_r\}$ is equicontinuous in $C(J, X)$ and $V(t) := \{(Q_2x)(t) : x \in \Omega_r\}$ is relatively compact in X. For any $0 \le t_1 < t_2 \le b$ and $x \in \Omega_r$, by Lemmas 2 and 8 we have

$$\begin{aligned}
\left\|(Q_2x)(t_2) - (Q_2x)(t_1)\right\| \le{} & \left\|\int_0^{t_1} \left[P^E_{\alpha,1}(t_2 - s) - P^E_{\alpha,1}(t_1 - s)\right]\left[f\left(s, x(s)\right) + Bu(s)\right] ds\right\| \\
& + \left\|\int_{t_1}^{t_2} P^E_{\alpha,1}(t_2 - s)\left[f\left(s, x(s)\right) + Bu(s)\right] ds\right\| \\
\le{} & \int_0^{t_1} \left\|f\left(s, x(s)\right) + Bu(s)\right\| ds \sup_{s \in [0,t_1]} \left\|P^E_{\alpha,1}(t_2 - s) - P^E_{\alpha,1}(t_1 - s)\right\| \\
& + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left\|f\left(s, x(s)\right) + Bu(s)\right\| ds \\
\to{} & 0
\end{aligned}$$

as $t_2 - t_1 \to 0$, which implies that the set V is equicontinuous in $C(J, X)$.

Let

$$\left(Q^\varepsilon_2 x\right)(t) = \int_0^{t-\varepsilon} P^E_{\alpha,1}(t - s)\left[f\left(s, x(s)\right) + Bu(s)\right] ds, \quad t \in J.$$

By the assumption (H_{f3}), $\overline{\operatorname{conv}(V_\varepsilon)}$ is also a compact set, where $\overline{\operatorname{conv}(V_\varepsilon)}$ means the convex closure of V_ε. By the mean value theorem for Bochner integrals, we deduce that $(Q^\varepsilon_2x)(t) \in (t - \varepsilon)\overline{\operatorname{conv}(V_\varepsilon)}$ for $t \in J$. So the set $V^\varepsilon_2(t) := \{(Q^\varepsilon_2x)(t) : x \in \Omega_r\}$ is relatively compact in X. Moreover, for any $x \in \Omega_r$, we have

$$\begin{aligned}
\left\|(Q_2x)(t) - \left(Q^\varepsilon_2x\right)(t)\right\| \le{} & \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} \left\|f\left(s, x(s)\right) + Bu(s)\right\| ds \\
\le{} & \frac{Mb^{\alpha-1}}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} \varphi_r(s)\, ds + \frac{MM_Bb^{\alpha-1}M^{**}}{\Gamma(\alpha)}\varepsilon \\
\to{} & 0
\end{aligned}$$

as $\varepsilon \to 0^+$, where $M^{**} := \mathfrak{C} + M_1[(1 + M)L_gr + MbL_hr + \frac{Mb^{\alpha-1}}{\Gamma(\alpha)}\|\varphi_r\|_{L^1}]$. Thus the set $V(t) := \{(Q_2x)(t) : x \in \Omega_r\}$ is relatively compact in X. By the Ascoli–Arzelà theorem the set V is relatively compact. Hence $\gamma_C(V) = \gamma_C(Q_2(\Omega_r)) = 0$.

At last, by the properties of H-MNC and because of $M(L_g + bL_h) < 1$, we obtain that

$$\begin{aligned}
\gamma_C\left(Q(\Omega_r)\right) &\le \gamma_C\left(Q_1(\Omega_r)\right) + \gamma_C\left(Q_2(\Omega_r)\right) \\
&\le M(L_g + bL_h)\gamma_C(\Omega_r) \\
&< \gamma_C(\Omega_r),
\end{aligned}$$

which implies that $Q : \Omega_r \to \Omega$ is a condensing mapping. By Sadovskii's fixed point theorem (see Lemma 6) Q has at least one fixed point x in Ω_r, which is the mild solution of system (1.1) satisfying $x(b) + g(x) = x_1$. Therefore system (1.1) is nonlocally controllable. □

H-MNC condition is another important tool guaranteeing the compactness of the solution operator. In what follows, we assume that f satisfies the following H-MNC condition:

(H_{f4}) There exists a constant $L_1 > 0$ such that

$$\gamma\big(f(t,D_0)\big) \le L_1\gamma(D_0), \quad t \in J,$$

for every countable subset $D_0 \subset X$.

Lemma 10 *Let X be a separable Hilbert space. Assume that conditions (H_{AE}), (H_W), (H_{f2})′, (H_{f4}), (H_g), and (H_h) hold. Then*

$$\gamma\big(\{u_x(s) : x \in D_0\}\big) \le M_1\left[(1+M)L_g + MbL_h + \frac{2Mb^\alpha L_1}{\Gamma(\alpha)}\right]\gamma_C(D_0), \quad s \in J,$$

where $D_0 \subset \Omega_r$ is a countable subset of Ω_r.

Proof By Lemma 5 we obtain that

$$\begin{aligned}\gamma\big(\{u_x(s) : x \in D_0\}\big) &\le M_1\big((1+M)L_g\gamma(D_0) + MbL_h\gamma(D_0)\big)\\ &\quad + \frac{MM_1b^{\alpha-1}}{\Gamma(\alpha)}\int_0^b L_1\gamma\big(D_0(s)\big)\,ds\\ &\le M_1\left[(1+M)L_g + MbL_h + \frac{Mb^\alpha L_1}{\Gamma(\alpha)}\right]\gamma_C(D_0).\end{aligned}$$

The proof is completed. □

Theorem 3 *Let X be a separable Hilbert space. Assume that assumptions (H_{AE}), (H_W), (H_{f1}), (H_{f2})′, (H_{f4}), (H_g), (H_h), and (H_B) are satisfied. If the inequality conditions (3.4) and*

$$\frac{2MM_Bb^\alpha M_1}{\Gamma(\alpha)}L_g + \left(1 + \frac{2MM_Bb^\alpha M_1}{\Gamma(\alpha)}\right)(ML_g + MbL_h) + \frac{2Mb^\alpha L_1}{\Gamma(\alpha)}\left(1 + \frac{MM_Bb^\alpha M_1}{\Gamma(\alpha)}\right) < 1$$

hold, then system (1.1) is nonlocally controllable on J.

Proof Define two operators Q_1 and Q_2 as in (3.5) and (3.6), respectively. By the properties of H-MNC and (3.7) we easily obtain that

$$\gamma_C\big(Q_1(\Omega_r)\big) \le M(L_g + bL_h)\gamma_C(\Omega_r). \tag{3.8}$$

On the other hand, since $Q_2(\Omega_r) \subset \Omega_r$ and the set $Q_2(\Omega_r)$ is equicontinuous in $C(J,X)$, by Lemmas 3 and 4 there is a countable set $D_0 \subset \Omega_r$ such that

$$\gamma_C\big(Q_2(\Omega_r)\big) \le 2\gamma_C\big(Q_2(D_0)\big) = 2\max_{t\in J}\gamma\big(Q_2(D_0)(t)\big). \tag{3.9}$$

Applying assumption (H_{f4}) and Lemma 10, we have

$$\begin{aligned}\gamma\big(Q_2(D_0)(t)\big) &= \gamma\left(\left\{\int_0^t P_{\alpha,1}^E(t-s)\big[f\big(s,x(s)\big)+Bu_x(s)\big]\,ds : x\in D_0\right\}\right)\\ &\le \frac{Mb^{\alpha-1}L_1}{\Gamma(\alpha)}\int_0^t \gamma\big(D_0(s)\big)\,ds + \frac{MM_Bb^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \gamma\big(\{u_x(s):x\in D_0\}\big)\,ds\\ &\le \frac{Mb^{\alpha}L_1}{\Gamma(\alpha)}\gamma_C(D_0) + \frac{MM_Bb^{\alpha}M_1}{\Gamma(\alpha)}\left[(1+M)L_g + MbL_h + \frac{Mb^{\alpha}L_1}{\Gamma(\alpha)}\right]\gamma_C(D_0).\end{aligned}$$

This, together with (3.9), gives

$$\begin{aligned}\gamma_C\big(Q_2(\Omega_r)\big) \le{}& \frac{2Mb^{\alpha}L_1}{\Gamma(\alpha)}\gamma_C(D_0)\\ &+ \frac{2MM_Bb^{\alpha}M_1}{\Gamma(\alpha)}\left[(1+M)L_g + MbL_h + \frac{Mb^{\alpha}L_1}{\Gamma(\alpha)}\right]\gamma_C(D_0).\end{aligned}\tag{3.10}$$

Combining (3.8) and (3.10), because of $\gamma_C(D_0)\le\gamma_C(\Omega_r)$, we obtain that

$$\begin{aligned}\gamma_C\big(Q(\Omega_r)\big) &= \gamma_C\big(Q_1(\Omega_r)\big) + \gamma_C\big(Q_2(\Omega_r)\big)\\ &\le \left[\frac{2MM_Bb^{\alpha}M_1}{\Gamma(\alpha)}L_g + \left(1+\frac{2MM_Bb^{\alpha}M_1}{\Gamma(\alpha)}\right)(ML_g+MbL_h)\right.\\ &\quad \left. + \frac{2Mb^{\alpha}L_1}{\Gamma(\alpha)}\left(1+\frac{MM_Bb^{\alpha}M_1}{\Gamma(\alpha)}\right)\right]\gamma_C(\Omega_r)\\ &< \gamma_C(\Omega_r).\end{aligned}$$

Thus we conclude that $Q:\Omega_r\to\Omega_r$ is a condensing mapping. By Sadovskii's fixed point theorem, Q has at least one fixed point x in Ω_r, which is the mild solution of system (1.1) satisfying $x(b)+g(x)=x_1$. Therefore system (1.1) is nonlocally controllable. □

4 Conclusion

In this paper, we investigated the nonlocal controllability of $\alpha\in(1,2)$-ordered fractional evolution systems of Sobolev type of the form (1.1) in a Hilbert space X. We first define the $(\alpha,1)$-resolvent family $\{C_{\alpha,1}^E(t)\}_{t\ge0}$ generated by the pair (A,E). Without assuming the compactness of $\{C_{\alpha,1}^E(t)\}_{t\ge0}$, we prove some nonlocal controllability results for the fractional evolution system (1.1) by using Banach's contraction mapping principle and Sadovskii's fixed point theorem. The discussion is based on fractional resolvent operator theory. Our results improve and extend some existing results.

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