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Existence of ground state for fractional Kirchhoff equation with L^2 critical exponents

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Abstract

In this paper, we consider a class of fractional Kirchhoff equations with L^2 critical exponents. By using the scaling technique and concentration-compactness principle we obtain the existence and nonexistence of ground state for fractional Kirchhoff equation with L^2 critical exponent.

Keywords: L^2 critical exponent; Besov space; Fractional Kirchhoff equation; Ground state

1 Introduction

In this paper, we consider the existence of ground state for the following fractional Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)(-\Delta)^s u+V(x)|u|^{\gamma}u=|u|^{\frac{8s}{N}}u+\mu u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $a,b>0$, $N>2s>\frac{N}{2}$ with $s\in(0,1)$, $0\leq\gamma\leq\frac{8s}{N}$, $2^*(s)=\frac{2N}{N-2s}$, and $V(x)$ is a bounded function in $\mathbb{R}^N$.

If $s=1$, then equation (1) is related to the stationary solutions of

$$u_{tt}-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=f(x,u), \tag{2}$$

where $f(x,u)$ is a general nonlinear function. Equation (2) comes from free vibrations of elastic strings by taking into account the changes in length of the string produced by transverse vibrations [13]. After the pioneering works [17] and [15], equation (1) has attracted considerable attention. The existence and asymptotic behavior of nodal solutions of equation (1) were considered by Deng, Peng, and Shuai [5]. The existence and concentration behavior of positive solutions were studied in [8, 9]. The uniqueness and nondegeneracy of positive solutions were obtained by Li et al. [14] and the references therein. The existence of multipeak solutions was considered in [23].

Equation (1) can be viewed as an eigenvalue problem by taking μ as an unknown Lagrange multiplier. Hence some mathematicians considered equation (1) by studying some constrained variational problems and obtained the existence of ground state of equation (1). This technique was generally used for other types of equations, for example, semilinear Schrödinger equation [11, 24], Schrödinger–Poisson equation [4, 12], quasilinear Schrödinger equation [29, 30]; see also [1, 2, 18, 21, 22]. For $s = 1$, as far as we know, the first work comes from Ye [25], who considered the following minimization problem:

$$I_{c^2} := \inf_{u\in S_c} I(u), \tag{3}$$

where

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx$$

and

$$S_c := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = c^2\right\}.$$

Using the scaling technique and concentration-compactness principle, Ye obtained the sharp existence of global constraint minimizers of problem (3). Then Zeng and Zhang [28] improved the results of [25] and obtained the sharp existence and uniqueness of global constraint minimizers of problem (3). From [25, 28] we know that there is an L^2 critical exponent $p^* = 2 + \frac{8}{N}$ such that problem (3) has global constraint minimizers for $p < p^*$ and no global constraint minimizers for $p \ge p^*$. Then, for the L^2 critical exponent, Ye [26] and Zeng and Chen [31] added a perturbation function and obtained the existence of minimizers on S_c. Moreover, for the L^2 critical exponent, Ye [27] gave some mass concentration behavior. Recently, Guo, Zhang, and Zhou [7] considered the following minimization problem:

$$d_\beta(p) := \inf_{u\in S_1} E_p^\beta(u), \tag{4}$$

where

$$E_p^\beta = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2\,dx - \frac{\beta}{p}\int_{\mathbb{R}^N} |u|^p\,dx,$$

and $S_1 := \{u \in H(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = 1\}$ with $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\,dx < \infty\}$. They first proved the sharp existence and nonexistence of global minimizer of problem (4) with $V(x) = 0$. Then, for the trapping potential $V(x)$, they considered the existence of minimizers for problem (4). Especially, for the L^2 critical exponent, they proved that there is $\beta_{p^*} > 0$ such that problem (4) has at least one minimizer for $\beta \le \beta_{p^*}$ and has no minimizers for $\beta > \beta_{p^*}$. Furthermore, for minimizers of problem (4) with $p < p^*$ and $\beta > \beta_{p^*}$, they obtained the blowup behavior of minimizers as p tends to p^*.

For $s \in (0, 1)$, Autuori, Fiscella, and Pucci [3] obtained the existence of solutions for equation (1) with critical nonlinearity. The existence of solutions of (1) with critical exponents was also considered in [19]. The multiplicity of solutions was obtained by Pucci, Xiang, and Zhang [20] and so on. Recently, Huang and Zhang [10] considered the existence and uniqueness of minimizers for the following problem:

$$e(c) := \inf_{u \in S_c} E_p(u), \tag{5}$$

where

$$E_p(u) = \frac{a}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 - \frac{1}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} \, dx,$$

and $S_c := \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = c^2\}$. Using the scaling technique and some energy estimates, they obtained the existence and uniqueness of minimizers for problem (5) if $p < \frac{8s}{N}$ and proved that there are no minimizers for problem (5) when $p \geq \frac{8s}{N}$.

For the existence of ground state of equation (1), we consider the following minimization problem:

$$e(c) := \inf_{u \in S_c} I_p(u), \tag{6}$$

where

$$\begin{aligned} I_p(u) = {} & \frac{a}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 \\ & + \frac{1}{\gamma+2} \int_{\mathbb{R}^N} V(x) |u|^{\gamma+2} \, dx - \frac{1}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} \, dx, \end{aligned}$$

and

$$S_c := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = c^2 \right\}.$$

Here $H^s(\mathbb{R}^N)$ is the Besov space defined by

$$H^s = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + |u|^2 \right) dx \right)^{\frac{1}{2}},$$

where

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx = \int \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

It is easy to see that there are no minimizers for problem (6) if $p > \frac{8s}{N}$. Indeed, for any $u \in S_c$ and constant $\lambda > 0$, let $u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$. Then

$$\int_{\mathbb{R}^N} u_\lambda^2(x) \, dx = \int_{\mathbb{R}^N} u^2(x) \, dx = c^2,$$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\lambda^2(x)-u_\lambda^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy &= \lambda^{2s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u^2(x)-u^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\ &= \lambda^{2s}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx,\\ \int_{\mathbb{R}^N}\left|u_\lambda(x)\right|^{2+p}dx &= \lambda^{\frac{Np}{2}}\int_{\mathbb{R}^N}\left|u(x)\right|^{2+p}dx,\\ \int_{\mathbb{R}^N}V(x)\left|u_\lambda(x)\right|^{2+\gamma}dx &= \lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N}V\left(\frac{x}{\lambda}\right)\left|u(x)\right|^{2+\gamma}dx.\end{aligned}$$

Hence we can deduce that

$$\begin{aligned}I_p(u_\lambda) = {}&\frac{a}{2}\lambda^{2s}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{b}{4}\lambda^{4s}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2\\ &+\frac{1}{2+\gamma}\lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N}V\left(\frac{x}{\lambda}\right)\left|u(x)\right|^{2+\gamma}dx-\frac{1}{p+2}\lambda^{\frac{Np}{2}}\int_{\mathbb{R}^N}\left|u(x)\right|^{2+p}dx.\end{aligned}\tag{7}$$

Since $\gamma < \frac{8s}{N}$, it is easy to see that $\frac{N\gamma}{2} < 4s$. If $p > \frac{8s}{N}$, then for λ large enough, the dominant term in (7) is $\frac{1}{p+2}\lambda^{\frac{Np}{2}}\int_{\mathbb{R}^N}|u(x)|^{2+p}\,dx$. Then $I_p(u_\lambda)\to-\infty$ as $\lambda\to\infty$. This means that there are no minimizers for problem (6) if $p > \frac{8s}{N}$. Therefore it seems that $p = \frac{8s}{N}$ is the L^2 critical exponent for problem (6). Moreover, from (7) with $V(x) = 0$ we have $I_p(u_\lambda)\to 0$ as $\lambda\to 0$. Hence $e(c)\le 0$ for any $c > 0$, and $0 < p < 2^*(s)-2$. For $p = \frac{8s}{N}$, similarly to the proof of [10, 28], using the Gagliardo–Nirenberg inequality (12), we have

$$\begin{aligned}I_p(u) &= \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2-\frac{N}{2N+8s}\int_{\mathbb{R}^N}|u|^{2+\frac{8s}{N}}\,dx\\ &\ge \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx+\frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2,\end{aligned}\tag{8}$$

where the definition of c^* is given further. If $c\le c^*$, then (8) means that $e(c) > 0$, a contradiction to $e(c)\le 0$, which indicates that for $p = \frac{8s}{N}$ and $c\le c^*$, problem (6) with $V(x) = 0$ has no minimizers. If $c > c^*$, then in view of Lemma 2.3, let $u_\lambda(x) = \frac{c\lambda^{\frac{N}{2}}}{|U|_2}U(\lambda x)$. Then we have $e(c)\le-\infty$, which means that for $p = \frac{8s}{N}$ and $c > c^*$, there are no minimizers for problem (6) with $V(x) = 0$. In other words, for $V(x) = 0$, there is no minimizer for problem (6) with $p = \frac{8s}{N}$. Hence, in this paper, when the potential function $V(x)$ satisfies some conditions, we consider the existence and nonexistence of minimizers for problem (6) with $p = \frac{8s}{N}$. In addition, we consider the existence and nonexistence of ground states for equation (1) under some conditions on the function $V(x)$. Moreover, in this paper, the energy estimate method used in [10, 28] is invalid because of the existence of a potential function $V(x)$. Hence we use the concentration-compactness principle to overcome the compactness of a minimizing sequence. Using this technique, it is natural that $\gamma\ge 2$ is necessary by Lemma 2.6.

In this paper, we assume that

$$V(x)\in L^\infty\left(\mathbb{R}^N\right).\tag{9}$$

Let

$$c^* = \left(b\left|U(x)\right|_2^{\frac{8s}{N}}\right)^{\frac{N}{8s-2N}},$$

where the function $U(x)$ is defined in Sect. 2. We first give a nonexistence result.

Theorem 1.1 *Let $p = \frac{8s}{N}$, and let $V(x)$ satisfy (9). Then problem (6) has no minimizers if one of the following conditions holds*:

(1) $c > c^*$ *for any* $\gamma \in [0, \frac{8s}{N})$.
(2) $V(x) \geq 0$ *for any* $c \in (0, c^*)$ *and* $\gamma \in [0, \frac{8s}{N})$.
(3) *For* $\gamma \in (\frac{4s}{N}, \frac{8s}{N})$ *and* $|V|_\infty c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}$ *small enough, we have*

$$\begin{aligned}&\frac{|V|_\infty}{\gamma+2}\left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}}\right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}\\ &\quad\leq \left(\frac{2as}{8s-\gamma N}\right)^{\frac{8s-\gamma N}{4s}}\left(\frac{bs}{\gamma N-4s}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\right)^{\frac{\gamma N-4s}{4s}}.\end{aligned}$$

From (2) of Theorem 1.1 we know that problem (6) has minimizers if and only if the function $V(x)$ has a negative part. Hence, in this paper, we first give a certain condition for $V(x)$ at infinity and get the following existence result.

Theorem 1.2 *Let $p = \frac{8s}{N}$, $c \in (0, c^*)$, $\gamma \in [2, \frac{8s}{N})$, $\frac{N\gamma}{2} + \alpha < 4s$ for some $\alpha > 0$, and let a be small enough. Suppose that the function $V(x)$ satisfies (9) and*

$$V(x) \sim -|x|^{-\beta} \quad \text{as } |x| \to \infty. \tag{10}$$

Then problem (6) has at least a minimizer.

According Theorem 1.2, we get the existence of minimizers of problem (6) for $V(x)$ tending to 0 at infinity with some rates as $|x| \to \infty$. Next, if we assume a general condition for $V(x)$ at infinity, then we have the following:

Theorem 1.3 *Let $p = \frac{8s}{N}$, $c \in (0, c^*)$, and $\gamma \in [2, \frac{8s}{N})$, and suppose that the function $V(x)$ satisfies (9) and*

$$\lim_{|x|\to\infty} V(x) = 0. \tag{11}$$

Then if $e(c) < 0$, the problem (6) has at least one minimizer.

Throughout the paper, C denotes some constant, and $|u|_p$ denotes the L^p-norm of a function u.

2 Preliminary results

Since we want to consider the existence of minimizers for problem (6) with $p = \frac{8s}{N}$, we first introduce the following Gagliardo–Nirenber inequality [6]:

$$\int_{\mathbb{R}^N} |u|^{2+\frac{8s}{N}}\,dx \leq \frac{N+4s}{2N|U(x)|_2^{\frac{8s}{N}}}\left(\int_{\mathbb{R}^N}|u|^2\,dx\right)^{\frac{4s-N}{N}}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx\right)^2. \tag{12}$$

Here the function $U(x)$ is the unique ground state of the equation

$$(-\Delta)^s u + \frac{4s-N}{2N} u = |u|^{\frac{8s}{N}} u, \quad x \in \mathbb{R}^N. \tag{13}$$

Using the Pohozaev identity and equation (13) [6, 10], we can get that

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx = \int_{\mathbb{R}^N} |u|^2 \, dx = \frac{2N}{N+4s} \int_{\mathbb{R}^N} |u|^{2+\frac{8s}{N}} \, dx. \tag{14}$$

Lemma 2.1 *Assume that $V(x) \geq 0$. Then, for any $c \in (0, c^*)$, we have $e(c) \geq 0$.*

Proof For any $u \in S_c$, using the Gagliardo–Nirenberg inequality (12), we get that

$$\begin{aligned}
I_p(u) &= \frac{a}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 + \frac{1}{\gamma+2} \int_{\mathbb{R}^N} V(x) |u|^{\gamma+2} \, dx \\
&\quad - \frac{N}{2N+8s} \int_{\mathbb{R}^N} |u|^{2+\frac{8s}{N}} \, dx \\
&\geq \frac{b}{4} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 - \frac{c^{\frac{8s-2N}{N}}}{4|U(x)|_2^{\frac{8s}{N}}} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 \\
&\quad + \frac{1}{\gamma+2} \int_{\mathbb{R}^N} V(x) |u|^{\gamma+2} \, dx \\
&\geq \frac{b}{4} \left(1 - \left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}} \right) \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^2 + \frac{1}{\gamma+2} \int_{\mathbb{R}^N} V(x) |u|^{\gamma+2} \, dx,
\end{aligned} \tag{15}$$

which, together with $V(x) \geq 0$, implies that $I_p(u) > 0$. Hence we have

$$e(c) \geq 0.$$

□

Lemma 2.2 *Let $\gamma \in (\frac{4s}{N}, \frac{8s}{N})$, and let $|V|_\infty c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}$ be small enough such that*

$$\begin{aligned}
&\frac{|V|_\infty}{\gamma+2} \left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}} \right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}} \\
&\quad \leq \left(\frac{2as}{8s-\gamma N} \right)^{\frac{8s-\gamma N}{4s}} \left(\frac{bs}{\gamma N-4s} \left(1 - \left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}} \right) \right)^{\frac{\gamma N-4s}{4s}}.
\end{aligned}$$

Then $e(c) \geq 0$.

Proof For any $u \in S_c$, using the Hölder and Gagliardo–Nirenberg inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^{\gamma+2} \, dx &\leq \left(\int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{8s-\gamma N}{8s}} \left(\int_{\mathbb{R}^N} |u|^{2+\frac{8s}{N}} \, dx \right)^{\frac{\gamma N}{8s}} \\
&\leq \left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}} \right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}} \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx \right)^{\frac{\gamma N}{4s}},
\end{aligned}$$

which, combined with (15), indicates that

$$\begin{aligned}
I_p(u) &= \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 + \frac{1}{\gamma+2}\int_{\mathbb{R}^N}V(x)|u|^{\gamma+2}\,dx \\
&\quad - \frac{N}{2N+8s}\int_{\mathbb{R}^N}|u|^{2+\frac{8s}{N}}\,dx \\
&\ge \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^2 \\
&\quad - \frac{|V|_\infty}{\gamma+2}\left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}}\right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^{\frac{\gamma N}{4s}}.
\end{aligned} \tag{16}$$

Let $\delta = \frac{8s-\gamma N}{4s}$ and $\beta = 1-\delta = \frac{\gamma N-4s}{4s}$. Using the Young inequality, we have

$$\begin{aligned}
&\frac{a}{2}t + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)t^2 \\
&\quad \ge \left(\frac{a}{2\delta}\right)^{\delta}\left(\frac{b(1-(\frac{c}{c^*})^{\frac{8s-2N}{N}})}{4\beta}\right)^{\beta} t^{\delta+2\beta} \\
&\quad = \left(\frac{2as}{8s-\gamma N}\right)^{\frac{8s-\gamma N}{4s}}\left(\frac{bs}{\gamma N-4s}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\right)^{\frac{\gamma N-4s}{4s}} t^{\frac{\gamma N}{4s}}.
\end{aligned}$$

Thus from (16) it follows that

$$\begin{aligned}
I_p(u) &\ge \left[\left(\frac{2as}{8s-\gamma N}\right)^{\frac{8s-\gamma N}{4s}}\left(\frac{bs}{\gamma N-4s}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\right)^{\frac{\gamma N-4s}{4s}}\right. \\
&\quad \left. - \frac{|V|_\infty}{\gamma+2}\left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}}\right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}\right]\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2dx\right)^{\frac{\gamma N}{4s}}.
\end{aligned} \tag{17}$$

If we choose $|V|_\infty c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}}$ small enough such that

$$\begin{aligned}
&\frac{|V|_\infty}{\gamma+2}\left(\frac{N+4s}{2N|U|_2^{\frac{8s}{N}}}\right)^{\frac{\gamma N}{8s}} c^{\frac{8s-\gamma N+\gamma(4s-N)}{4s}} \\
&\quad \le \left(\frac{2as}{8s-\gamma N}\right)^{\frac{8s-\gamma N}{4s}}\left(\frac{bs}{\gamma N-4s}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\right)^{\frac{\gamma N-4s}{4s}},
\end{aligned}$$

then (17) indicates that

$$e(c) \ge 0. \qquad \square$$

Lemma 2.3 *If $c > c^*$, then $e(c) < -\infty$.*

Proof Set

$$u_\lambda(x) = \frac{c\lambda^{\frac{N}{2}}}{|U|_2}U(\lambda x).$$

Then using (14), we have

$$\int_{\mathbb{R}^N} u_\lambda^2(x)\,dx = c^2,$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\lambda^2(x)-u_\lambda^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \frac{c^2\lambda^{2s}}{|U|_2^2}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}U\bigr|^2\,dx = c^2\lambda^{2s},$$
$$\int_{\mathbb{R}^N}\bigl|u_\lambda(x)\bigr|^{2+\frac{8s}{N}}\,dx = \frac{(N+4s)c^{2+\frac{8s}{N}}\lambda^{4s}}{2N|U|_2^{\frac{8s}{N}}},$$
$$\int_{\mathbb{R}^N} V(x)\bigl|u_\lambda(x)\bigr|^{2+\gamma}\,dx = \frac{c^{2+\gamma}\lambda^{\frac{N\gamma}{2}}}{|U|_2^{2+\gamma}}\int_{\mathbb{R}^N} V\Bigl(\frac{x}{\lambda}\Bigr)\bigl|U(x)\bigr|^{2+\gamma}\,dx.$$

Hence we can deduce that $u_\lambda \in S_c$ and

$$\begin{aligned} I_p(u_\lambda) &= \frac{a}{2}c^2\lambda^{2s} + \frac{b}{4}c^4\lambda^{4s} + \frac{c^{\gamma+2}}{(2+\gamma)|U|_2^{2+\gamma}}\lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N} V\Bigl(\frac{x}{\lambda}\Bigr)|U|^{2+\gamma}\,dx - \frac{c^{2+\frac{8s}{N}}}{4|U|_2^{\frac{8s}{N}}}\lambda^{4s} \\ &= \frac{a}{2}c^2\lambda^{2s} + \frac{b}{4}c^4\lambda^{4s}\biggl(1-\Bigl(\frac{c}{c^*}\Bigr)^{\frac{8s-2N}{N}}\biggr) \\ &\quad + \frac{c^{\gamma+2}}{(2+\gamma)|U|_2^{2+\gamma}}\lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N} V\Bigl(\frac{x}{\lambda}\Bigr)|U|^{2+\gamma}\,dx. \end{aligned} \tag{18}$$

From $\gamma < \frac{8s}{N}$ we get that $\frac{N\gamma}{2} < 4s$. Then (18) indicates that $I_p(u_\lambda) \to -\infty$ as $\lambda \to \infty$, and the lemma is proved. □

Lemma 2.4 *For any $c > 0$, we have $e(c) \le 0$.*

Proof For any $u \in S_c$ and constant $\lambda > 0$, let $u_\lambda(x) = \lambda^{\frac{N}{2}}u(\lambda x)$. Then $u_\lambda \in S_c$, and from (7) we have

$$\begin{aligned} I_p(u_\lambda) &= \frac{a}{2}\lambda^{2s}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx + \frac{b}{4}\lambda^{4s}\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx\biggr)^2 \\ &\quad + \frac{1}{2+\gamma}\lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N} V\Bigl(\frac{x}{\lambda}\Bigr)\bigl|u(x)\bigr|^{2+\gamma}\,dx - \frac{1}{p+2}\lambda^{4s}\int_{\mathbb{R}^N}\bigl|u(x)\bigr|^{2+\frac{8s}{N}}\,dx. \end{aligned} \tag{19}$$

Hence $I_p(u_\lambda) \to 0$ as $\lambda \to 0$, which indicates that $e(c) \le 0$. □

Lemma 2.5 *Assume that the function $V(x)$ satisfies condition (10), $\frac{N\gamma}{2} + \alpha < 4s$, and a is small enough. Then $e(c) < 0$.*

Proof For fixed $|x_0| = 2$, assume that $\varphi(x) \in C_c^\infty(\mathbb{R}^N)$ is such that $\operatorname{supp}\varphi \in B_1(x_0)$ and $\int_{\mathbb{R}^N}\varphi^2(x)\,dx = c^2$. For constant $\lambda > 0$, take

$$\varphi_\lambda(x) = \lambda^{\frac{N}{2}}\varphi(\lambda x).$$

Then

$$\int_{\mathbb{R}^N}\varphi_\lambda^2(x)\,dx = \int_{\mathbb{R}^N}\varphi^2(x)\,dx = c^2, \tag{20}$$

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_\lambda^2(x)-\varphi_\lambda^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy &= \lambda^{2s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi^2(x)-\varphi^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\\ &= \lambda^{2s}\int_{\mathbb{R}^N}\big|(-\Delta)^{\frac{s}{2}}\varphi\big|^2\,dx,\end{aligned}\tag{21}$$

$$\int_{\mathbb{R}^N}\big|\varphi_\lambda(x)\big|^{2+\frac{8s}{N}}\,dx=\lambda^{4s}\int_{\mathbb{R}^N}\big|\varphi(x)\big|^{2+\frac{8s}{N}}\,dx,\tag{22}$$

and

$$\int_{\mathbb{R}^N}V(x)\big|\varphi_\lambda(x)\big|^{2+\gamma}\,dx=\lambda^{\frac{N\gamma}{2}}\int_{\mathbb{R}^N}V\left(\frac{x}{\lambda}\right)\big|\varphi(x)\big|^{2+\gamma}\,dx\le -C\lambda^{\frac{N\gamma}{2}+\alpha}\tag{23}$$

as $\lambda\to 0$.

From (20) we know that $\varphi_\lambda\in S_c$. Then (21)–(23) indicate that

$$\begin{aligned}I_p(\varphi_\lambda)\le{}&\frac{a\lambda^{2s}}{2}\int_{\mathbb{R}^N}\big|(-\Delta)^{\frac{s}{2}}\varphi\big|^2\,dx+\frac{b\lambda^{4s}}{4}\int_{\mathbb{R}^N}\big|(-\Delta)^{\frac{s}{2}}\varphi\big|^2\,dx\\ &-C\lambda^{\frac{N\gamma}{2}+\alpha}-\frac{N\lambda^{4s}}{2N+8s}\int_{\mathbb{R}^N}\big|\varphi(x)\big|^{2+\frac{8s}{N}}\,dx.\end{aligned}\tag{24}$$

For $2\le\gamma<\frac{8s}{N}$, we have $2s<N\le\frac{N\gamma}{2}<4s$. If $\frac{N\gamma}{2}+\alpha<4s$, then there is a small $\lambda_0>0$ such that

$$\frac{b\lambda_0^{4s}}{4}\int_{\mathbb{R}^N}\big|(-\Delta)^{\frac{s}{2}}\varphi\big|^2\,dx-C\lambda_0^{\frac{N\gamma}{2}+\alpha}-\frac{N\lambda_0^{4s}}{2N+8s}\int_{\mathbb{R}^N}\big|\varphi(x)\big|^{2+\frac{8s}{N}}\,dx<0.$$

Moreover, if

$$a<\frac{-\frac{b\lambda_0^{2s}}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\varphi|^2\,dx+2C\lambda_0^{\frac{N\gamma}{2}+\alpha-2s}+\frac{N\lambda_0^{2s}}{N+4s}\int_{\mathbb{R}^N}|\varphi(x)|^{2+\frac{8s}{N}}\,dx}{\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\varphi|^2\,dx},$$

then from (24) we can deduce that

$$e(c)\le\inf I_p(\varphi_\lambda)<0.\qquad\square$$

Lemma 2.6 *For any $c\in(0,c^*)$ and any $d\in(0,c)$, if $e(c)<0$, then*

$$e(c)<e(d)+e\big(\sqrt{c^2-d^2}\big).$$

Proof Let $\{u_n\}$ be any minimizing sequence. Then

$$\begin{aligned}\int_{\mathbb{R}^N}|u_n|^{\gamma+2}\,dx&=\int_{\mathbb{R}^N}|u_n|^{(\gamma+2)\theta}|u_n|^{(\gamma+2)(1-\theta)}\,dx\\ &\le\left(\int_{\mathbb{R}^N}|u_n|^2\,dx\right)^{\frac{(2+\gamma)\theta}{2}}\left(\int_{\mathbb{R}^N}|u_n|^{2^*(s)}\,dx\right)^{\frac{(2+\gamma)(1-\theta)}{2^*(s)}}\\ &\le C\left(\int_{\mathbb{R}^N}\big|(-\Delta)^{\frac{s}{2}}u\big|^2\,dx\right)^{\frac{\gamma N}{4s}},\end{aligned}\tag{25}$$

where $\theta=\frac{2s(2+\gamma)-\gamma N}{2(2+\gamma)s}$.

Using (12) and (25), we have

$$\begin{aligned} I_p(u_n) &= \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^2 \\ &\quad + \frac{1}{\gamma+2}\int_{\mathbb{R}^N}V(x)|u_n|^{\gamma+2}dx - \frac{N}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}dx \\ &\geq \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^2 \\ &\quad - C\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^{\frac{\gamma N}{4s}}, \end{aligned} \tag{26}$$

where $c^* = (b|U(x)|_2^{\frac{8s}{N}})^{\frac{N}{8s-2N}}$. Since $\gamma < \frac{8s}{N}$, we have that $\frac{\gamma N}{4s} < 2$. Since $\{u_n\}$ is a minimizing sequence and $c < c^*$, we have $e(c) = \lim_{n\to\infty} I_p(u_n)$, and the sequence $\{u_n\}$ is bounded in the space $H^s(\mathbb{R}^N)$. Moreover, from (26) we can deduce that $0 > e(c) > -\infty$ and

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\gamma+2}\int_{\mathbb{R}^N}V(x)|u_n|^{\gamma+2}dx \\ &\quad\leq e(c) - \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx - \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8s-2N}{N}}\right)\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^2 \\ &\quad< 0. \end{aligned} \tag{27}$$

For $\lambda > 1$, defining $\bar{u}_n = \lambda u_n$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\bar{u}_n^2\,dx = \lambda^2\int_{\mathbb{R}^N}u_n^2\,dx = \lambda^2c^2, \qquad \int_{\mathbb{R}^N}V(x)\bar{u}_n^{\gamma+2}dx = \lambda^{\gamma+2}\int_{\mathbb{R}^N}V(x)u_n^{\gamma+2}dx, \\ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{u}_n^2(x)-\bar{u}_n^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \lambda^2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n^2(x)-u_n^2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\qquad\qquad = \lambda^2\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx, \\ &\int_{\mathbb{R}^N}|\bar{u}_n|^{2+\frac{8s}{N}}dx = \lambda^{2+\frac{8s}{N}}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}dx. \end{aligned}$$

Then

$$\begin{aligned} I_p(\bar{u}_n) &= \frac{a\lambda^2}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx + \frac{b\lambda^4}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx\right)^2 \\ &\quad + \frac{\lambda^2}{\gamma+2}\int_{\mathbb{R}^N}V(x)|u_n|^{\gamma+2}dx - \frac{N\lambda^{2+\frac{8s}{N}}}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}dx \\ &\geq \lambda^4 I_p(u_n) + \left(\lambda^2-\lambda^4\right)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2dx \\ &\quad + \left(\lambda^4-\lambda^{2+\frac{8s}{N}}\right)\frac{N}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}dx \\ &\quad + \left(\lambda^{\gamma+2}-\lambda^4\right)\frac{1}{\gamma+2}\int_{\mathbb{R}^N}V(x)|u_n|^{\gamma+2}dx, \end{aligned} \tag{28}$$

which, together with $\lambda > 1$, $\gamma \geq 2$, and (27), indicates that

$$e(\lambda c) \leq \lim_{n\to\infty} I_p(\bar{u}_n) \leq \lambda^4 \lim_{n\to\infty} I_p(u_n) = \lambda^4 e(c). \tag{29}$$

Since $e(c) < 0$, this means that

$$e(\lambda c) < \lambda e(c).$$

Then for any $d \in [0, c)$, we have

$$e(c) < e(d) + e\left(\sqrt{c^2 - d^2}\right). \qquad \square$$

3 The proof of theorems

Proof of Theorem 1.1 (1) From Lemma 2.3 we know that $e(c) < -\infty$. Hence it is natural that for any $c > c^*$, there are no minimizers for problem (6).

(2) From Lemma 2.1 we know that since $V(x) \geq 0$, $e(c) \geq 0$. This, together with Lemma 2.4, indicates that $e(c) = 0$. Assume that there is $u_0 \in S_c$ such that

$$I_p(u_0) = e(c) = 0,$$

which contradicts with (15) since $I_p(u_0) > 0$ for any $V(x) \geq 0$. Thus there are no minimizers for problem (6).

(3) From Lemma 2.2 we have that $e(c) \geq 0$. This, together with Lemma 2.4, indicates that $e(c) = 0$. Similarly to the proof of (2), we can deduce that there are no minimizers for problem (6). $\square$

Proof of Theorem 1.2 Let $\{u_n\}$ be a minimizing sequence of $e(c)$. From (26) we get that $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2\,dx$ is bounded above, which, combined with $\int_{\mathbb{R}^N} |u_n|^2\,dx = c^2$, implies that $\{u_n\}$ is bounded in the space $H^s(\mathbb{R}^N)$. Hence there is $u \in H^s(\mathbb{R}^N)$ such that there is a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Then by the concentration-compactness principle [16] the sequence $\{u_n\}$ is compact. Hence the key point is excluding the case of vanishing (i.e., $u = 0$ in $H^s(\mathbb{R}^N)$) and dichotomy (i.e.m $u \neq 0$ in $H^s(\mathbb{R}^N)$ but $0 < |u|_2 < c$).

For any $0 < R < \infty$, set

$$\delta = \limsup_{n\to\infty, y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2\,dx.$$

If $\delta = 0$, then using the vanishing lemma (Lemma I.1 in [16]), we have

$$u_n \to 0, \quad \text{in } L^q\left(\mathbb{R}^N\right), q \in \left(2, 2^*(s)\right).$$

This indicates that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2+\frac{8s}{N}}\,dx = 0, \tag{30}$$

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N} V(x)|u_n|^{2+\gamma}\,dx\right| \le |V_\infty| \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2+\gamma}\,dx = 0. \tag{31}$$

Using (30) and (31), we can deduce that

$$\begin{aligned} e(c) &= \lim_{n\to\infty} I_p(u_n) \\ &= \lim_{n\to\infty}\left[\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx\right)^2 \right. \\ &\quad \left. + \frac{1}{\gamma+2}\int_{\mathbb{R}^N} V(x)|u_n|^{\gamma+2}\,dx - \frac{N}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}\,dx\right] \\ &= \lim_{n\to\infty}\left(\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 dx\right)^2\right) \ge 0, \end{aligned} \tag{32}$$

a contradiction to Lemma 2.5. Hence vanishing is impossible.

Now we assume that dichotomy occurs. Then there are $d \in (0, c)$ and bounded sequences $\{u_n^1\}$, $\{u_n^2\}$ in $H^s(\mathbb{R}^N)$ such that for any $q \in [2, 2^*(s))$, we have

$$\left|u_n - u_n^2 - u_n^2\right|_q \le \sigma_q(\varepsilon), \tag{33}$$

$$\left|\int_{\mathbb{R}^N}\left|u_n^1\right|^2 dx - d^2\right| \le \varepsilon, \qquad \left|\int_{\mathbb{R}^N}\left|u_n^2\right|^2 dx - \left(c^2 - d^2\right)\right| \le \varepsilon, \tag{34}$$

$$\operatorname{dist}\left(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2\right) \to \infty \quad \text{as } n \to \infty, \tag{35}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 - \left|(-\Delta)^{\frac{s}{2}}u_n^1\right|^2 - \left|(-\Delta)^{\frac{s}{2}}u_n^2\right|^2\right] dx. \tag{36}$$

Using (33)–(36), we can deduce that

$$\begin{aligned} e(c) &= \lim_{n\to\infty} I_p(u_n) \ge \lim_{n\to\infty}\left[I_p\left(u_n^1\right) + I_p\left(u_n^2\right)\right] + \sigma(\varepsilon) \\ &\ge e(d) + e\left(\sqrt{c^2 - d^2}\right) + \sigma(\varepsilon), \end{aligned} \tag{37}$$

where $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\varepsilon \to 0$. Then (37) contradicts to Lemma 2.6. Hence dichotomy cannot occur, and for any $\varepsilon > 0$, there exist $R_\varepsilon > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_{R_\varepsilon}(y_n)} |u_n|^2\,dx \ge c - \varepsilon. \tag{38}$$

Next, we discuss this problem for two cases: $\{y_n\}$ is bounded and $y_n \to \infty$ as $n \to \infty$.

(1) If $\{y_n\}$ is bounded from above, then (38) indicates that

$$u_n \to u \quad \text{in } L^2\left(\mathbb{R}^N\right).$$

Since $\{u_n\}$ is bounded in the space $H^s(\mathbb{R}^N)$, the Gagliardo–Nirenberg inequality gives that

$$\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}\,dx \le C\left(\int_{\mathbb{R}^N}|u_n|^2\,dx\right)^{\frac{4s-N}{2}}.$$

By Lebesgue's dominate convergence theorem we get that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}\,dx=\int_{\mathbb{R}^N}|u|^{2+\frac{8s}{N}}\,dx. \tag{39}$$

Similarly to the proof of (39), we obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}V(x)|u_n|^{2+\gamma}\,dx=\int_{\mathbb{R}^N}V(x)|u|^{2+\gamma}\,dx. \tag{40}$$

From [6] we know that the norm $\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2\,dx$ satisfies weak lower semi-continuity, that is,

$$\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx.$$

Then

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx\right)^2&\le\left(\liminf_{n\to\infty}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\right)^2\\&\le\liminf_{n\to\infty}\left(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\right)^2,\end{aligned}$$

which, together with (39) and (40), implies that

$$e(c)\le I_p(u)\le\liminf_{n\to\infty}I_p(u_n)=e(c).$$

This implies that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx=\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx,\\&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\right)^2=\left(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u\bigr|^2\,dx\right)^2.\end{aligned}$$

Then the sequence $\{u_n\}$ has a strongly convergent subsequence, which means that u is a minimizer of $e(c)$.

(2) If $y_n\to\infty$ as $n\to\infty$, then from the definition of $V(x)$ we know that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_{R_\varepsilon}(y_n)}V(x)|u_n|^{\gamma+2}\,dx=0. \tag{41}$$

From (25) we have

$$\begin{aligned}\lim_{n\to\infty}\left|\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(y_n)}V(x)|u_n|^{\gamma+2}\,dx\right|&\le C\lim_{n\to\infty}\left(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(y_n)}|u_n|^2\,dx\right)^{\frac{2s(\gamma+2)-\gamma N}{4s}}\\&\le C\varepsilon^{\frac{2s(\gamma+2)-\gamma N}{4s}},\end{aligned}$$

from which by letting $\varepsilon\to 0$ we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(y_n)}V(x)|u_n|^{\gamma+2}\,dx=0.$$

This, together with (41), indicates that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^{\gamma+2}\,dx = 0. \tag{42}$$

Using (42) and the Gagliardo–Nirenberg inequality (12), we deduce that

$$\begin{aligned}
e(c) = \lim_{n\to\infty}\biggl[&\frac{a}{2}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx + \frac{b}{4}\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\biggr)^2 \\
&+ \frac{1}{\gamma+2}\int_{\mathbb{R}^N} V(x)|u_n|^{\gamma+2}\,dx - \frac{N}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}\,dx\biggr] \\
= \lim_{n\to\infty}\biggl[&\frac{a}{2}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx + \frac{b}{4}\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\biggr)^2 \\
&- \frac{N}{2N+8s}\int_{\mathbb{R}^N}|u_n|^{2+\frac{8s}{N}}\,dx\biggr] \\
\geq \lim_{n\to\infty}\biggl[&\frac{a}{2}\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx + \frac{b}{4}\biggl(1-\biggl(\frac{c}{c^*}\biggr)^{\frac{8s-2N}{N}}\biggr)\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{\frac{s}{2}}u_n\bigr|^2\,dx\biggr)^2\biggr] \\
> 0,&
\end{aligned}$$

which contradicts to Lemma 2.5. Hence $y_n \to \infty$ as $n \to \infty$ cannot occur. □

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2. We omit it. □

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Authors' contributions
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References

1. Abdolrazaghi, F., Razani, A.: On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations. Miskolc Math. Notes **20**, 3–16 (2019)
2. Abdolrazaghi, F., Razani, A.: A unique weak solution for a kind of coupled system of fractional Schrödinger equations. Opusc. Math. **40**, 313–322 (2020)
3. Autuori, G., Fiscella, A., Pucci, P.: Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity. Nonlinear Anal. **125**, 699–714 (2015)

4. Bellazzini, J., Jeanjean, L., Luo, T.J.: Existence and instability of standing waves with prescribed norm for a class of Schrödinger–Poisson equations. Proc. Lond. Math. Soc. **107**(3), 303–339 (2013)
5. Deng, Y.B., Peng, S.J., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)
6. Frank, R.L., Lenzmann, E., Silvestre, L.: Uniqueness of radial solutions for the fractional Laplacian. Commun. Pure Appl. Math. **69**(9), 1671–1726 (2015)
7. Guo, H.L., Zhang, Y.M., Zhou, H.S.: Blow-up solutions for a Kirchhoff type elliptic equation with trapping potential. Commun. Pure Appl. Anal. **17**(5), 1875–1897 (2018)
8. He, X.M., Zou, W.M.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differ. Equ. **2**, 1813–1834 (2012)
9. He, Y., Li, G.B., Peng, S.J.: Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Adv. Nonlinear Stud. **14**, 441–468 (2014)
10. Huang, X.M., Zhang, Y.M.: Existence and uniqueness of minimizers for L^2-constrained problems related to fractional Kirchhoff equation. Math. Methods Appl. Sci. (2020, in press, 1–13). https://doi.org/10.1002/mma.6543
11. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28**, 1633–1659 (1997)
12. Jeanjean, L., Luo, T.J.: Sharp nonexistence results of prescribed L^2-norm solutions for some class of Schrödinger–Poisson and quasi-linear equations. Z. Angew. Math. Phys. **64**(4), 937–954 (2013)
13. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
14. Li, G.B., Luo, P., Peng, S.J., Wang, C.H., Xiang, C.L.: A singularly perturbed Kirchhoff problem revisited. J. Differ. Equ. **268**, 541–589 (2020)
15. Lions, J.L.: On some quations in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977). North-Holland Math. Stud., vol. 30, pp. 284–346. North-Holland, Amsterdam (1978)
16. Lions, J.L.: The concentration-compactness principle in the calculus of variations, the locally compact case, I. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 109–145 (1984)
17. Pohozaev, S.I.: A certain class of quasilinear hyperbolic equations. Mat. Sb. **96**, 152–168 (1975)
18. Polidoro, S., Ragusa, M.A.: Harnack inequality for hypoelliptic ultraparabolic equations with a singular lower order term. Rev. Mat. Iberoam. **24**(3), 1011–1046 (2008)
19. Pucci, P., Saldi, S.: Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators. Rev. Mat. Iberoam. **32**, 1–22 (2016)
20. Pucci, P., Xiang, M.Q., Zhang, B.L.: Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff equations involving the fractional p-Laplacian in $\mathbb{R}^N$. Calc. Var. Partial Differ. Equ. **54**, 2785–2806 (2015)
21. Safari, F., Razani, A.: Existence of radially positive solutions for Neumann problem on the Heisenberg group. Bound. Value Probl. **2020**, 88 (2020)
22. Wang, F.L., Xiang, M.Q.: Multiplicity of solutions for a class of fractional Choquard–Kirchhoff equations involving critical nonlinearity. Anal. Math. Phys. **9**(1), 1–16 (2019)
23. Wang, Z.Z., Zeng, X.Y., Zhang, Y.M.: Multi-peak solutions of Kirchhoff equations involving subcritical or critical Sobolev exponents. Math. Methods Appl. Sci. **43**, 5151–5161 (2020)
24. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolations estimates. Commun. Math. Phys. **87**, 567–576 (1983)
25. Ye, H.Y.: The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations. Math. Methods Appl. Sci. **38**, 2663–2679 (2015)
26. Ye, H.Y.: The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations. Z. Angew. Math. Phys. **66**, 1483–1497 (2015)
27. Ye, H.Y.: The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations. Z. Angew. Math. Phys. **67**(2), 29 (2016)
28. Zeng, X.Y., Zhang, Y.M.: Existence and uniqueness of normalized solutions for the Kirchhoff equation. Appl. Math. Lett. **74**, 52–59 (2017)
29. Zeng, X.Y., Zhang, Y.M.: Existence and asymptotic behavior for the ground state of quasi-linear elliptic equation. Adv. Nonlinear Stud. **18**(4), 725–744 (2018)
30. Zeng, X.Y., Zhang, Y.M.: Asymptotic behaviors of ground stats for a modified Gross–Pitaevskii equation. Discrete Contin. Dyn. Syst. **39**(9), 5263–5273 (2019)
31. Zeng, Y.L., Chen, K.S.: Remarks on normalized solutions for L^2-critical Kirchhoff problems. Taiwan. J. Math. **20**(3), 617–627 (2016)