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Properties of solutions to fractional p-subLaplace equations on the Heisenberg group

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Abstract

The aim of this paper is to study properties of solutions to the fractional p-subLaplace equations on the Heisenberg group. Based on the maximum principles and the generalization of the direct method of moving planes, we obtain the symmetry and monotonicity of the solutions on the whole group and the Liouville property of solutions on a half space.

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1 Introduction

Let

$$(-\Delta_{\mathbb{H}})_p^s u(\xi) = C_{Q,s,p} PV \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta \tag{1.1}$$

be the fractional p-subLaplacian on the Heisenberg group $\mathbb{H}^n$, where $0 < s < 1$, $Q = 2n + 2$, $C_{Q,s}$ is a positive constant, and PV is the Cauchy principal value. In this paper we study the properties of cylindrical solutions to the fractional p-subLaplace equation

$$(-\Delta_{\mathbb{H}})_p^s u(\xi) = f(u(\xi)), \tag{1.2}$$

where $2 \le p < \infty$.

Recall that the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudodifferential operator defined by

$$(-\Delta)^\alpha u(x) = C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} \, dy, \tag{1.3}$$

where $0 < \alpha < 1$, $C_{n,\alpha}$ is a constant, and u belongs to the Schwartz space. Since the non-local property of the operator $(-\Delta)^\alpha$ brings new difficulties to investigate, Caffarelli and

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Silvestre in [4] developed the extension method which can reduce the nonlocal problem relating to $(-\Delta)^{\alpha}$ to a local one in higher dimensions. This method has been applied to deal with equations involving the fractional Laplacian, and fruitful results have been obtained, see [3] and the references therein. Chen et al. [7] developed a direct method of moving planes to handle the problem involving $(-\Delta)^{\alpha}$ for $0 < \alpha < 1$, and this direct method has been used successfully to study symmetry, monotonicity, and nonexistence for many fractional Laplace equations, see [6, 7] and the references therein. Recently, Chen and Li [6] considered the fractional p-Laplacian

$$(-\Delta)_p^{\alpha} u(x) = C_{n,\alpha,p} PV \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+\alpha p}} \, dy \tag{1.4}$$

and obtained the radial symmetry and monotonicity of solutions to the equations involving operator (1.4).

To the elliptic equation

$$-\Delta u = g(u) \tag{1.5}$$

in $\mathbb{R}^n$, Li and Ni [19] proved that the positive solutions to (1.5) are radially symmetric with the assumptions that the limit of u is zero at the infinity and $g' \le 0$ if u is sufficiently small. Under the same conditions, the authors in [6] extended the result in [19] to the fractional p-Laplace equation

$$(-\Delta)_p^{\alpha} u = g(u), \tag{1.6}$$

and got the radial symmetry and monotonicity of the solutions. They also pointed out that the fractional p-Laplacian becomes p-Laplacian as $\alpha \to 1$ and, furthermore, it reduces to $-\Delta$ when $p = 2$.

There are many interesting results about subLaplace and p-subLaplace equations on the Heisenberg group (see [13, 15, 17, 18] and [10, 11, 20, 21, 25–27]). There have been several different definitions of the fractional power subLaplacian in $\mathbb{H}^n$ (see [12, 14, 22] etc.). The definition of fractional power subLaplacian given by Roncal and Thangavelu in [22] is indeed a generalization of the definition given by Cowling and Haagerup in [9] about the heat semigroup. The fractional power subLaplace equations can also be studied by generalizing the extension method in [4] to $\mathbb{H}^n$, although the fractional power subLaplacian $(-\Delta_{\mathbb{H}})^s$ $(0 < s < 1)$ does not have the concrete integral expression, for example, see [14] and [8] for $s = \frac{1}{2}$. There are also some results of the fractional power subLaplacian which are the extension of [8], see [23, 24]. Note that the expression of fractional power subLaplacian on $\mathbb{H}^n$ (see [22])

$$(-\Delta_{\mathbb{H}})^s u(\xi) = C_{Q,s} PV \int_{\mathbb{H}^n} \frac{u(\xi) - u(\eta)}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+2s}} \, d\eta \tag{1.7}$$

is the special form of fractional p-subLaplacian (1.1). By extending the method of moving planes in [5–7] to $\mathbb{H}^n$, in this paper, we study the properties of the solutions to (1.2) on $\mathbb{H}^n$ and $\mathbb{H}_+^n = \{\xi \in \mathbb{H}^n \mid t > 0\}$.

Our main results are the following.

Theorem 1.1 *Let $0 < s < 1$, $2 \le p < \infty$, and $u \in L_{sp}(\mathbb{H}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{H}^n)$ be a nonnegative cylindrical solution to* (1.2) *with*

$$\lim_{|\xi|_{\mathbb{H}} \to \infty} u(\xi) = 0, \tag{1.8}$$

and suppose that $f'(a)$ is nonpositive and locally bounded for a sufficiently small. Then u must be symmetric and monotone with respect to t about some point in $\mathbb{H}^n$.

Theorem 1.2 *Let $0 < s < 1$, $2 \le p < \infty$, and $u \in L_{sp}(\mathbb{H}^n_+) \cap C_{\text{loc}}^{1,1}(\mathbb{H}^n_+)$ be a nonnegative cylindrical solution to the problem*

$$\begin{cases} (-\Delta_{\mathbb{H}})_p^s u(\xi) = f(u(\xi)), & \xi \in \mathbb{H}^n_+, \\ u(\xi) = 0, & \xi \notin \mathbb{H}^n_+, \end{cases} \tag{1.9}$$

and suppose that u satisfies (1.8) *and is lower semicontinuous on $\bar{\mathbb{H}}^n_+$. If $f(0) = 0$, $f'(a)$ is nonpositive and locally bounded for a sufficiently small, then $u \equiv 0$.*

Observe that Theorem 1.1 is the extension of symmetry and monotonicity of solutions to the fractional p-Laplace equation on $\mathbb{R}^n$ in [6] to the Heisenberg group, and Theorem 1.2 is the Liouville property on a half space in $\mathbb{H}^n$. When $f(a) = -a + a^q$ $(q > 1)$, our results still hold.

The authors in [22] assumed that $u \in C_0^\infty(\mathbb{H}^n)$ in (1.7). We point out that (1.1) is also well defined for $u \in L_{sp}(\mathbb{H}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{H}^n)$, where

$$L_{sp}(\mathbb{H}^n) = \left\{ u : \mathbb{H}^n \to \mathbb{R} \,\Big|\, \int_{\mathbb{H}^n} \frac{|u(\xi)|^{p-1}}{1 + |\xi|_{\mathbb{H}}^{Q+sp}} \, d\xi < \infty \right\}.$$

The paper is organized as follows. Section 2 collects some well-known results on $\mathbb{H}^n$, and we show that (1.1) is well defined for $u \in L_{sp}(\mathbb{H}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{H}^n)$. In Sect. 3, we establish three maximum principles. Theorem 1.1 and Theorem 1.2 are proved in Sect. 4.

2 Preliminaries

The Heisenberg group $\mathbb{H}^n$ is the Euclidean space $\mathbb{R}^{2n+1}(n \ge 1)$ endowed with the group law $\circ$:

$$\bar{\xi} \circ \xi = \left(x + \bar{x}, y + \bar{y}, t + \bar{t} + 2 \sum_{i=1}^n (x_i \bar{y}_i - y_i \bar{x}_i) \right), \tag{2.1}$$

where $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t) := (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\bar{\xi} = (\bar{x}, \bar{y}, \bar{t})$. Denote by δ_κ the dilations on $\mathbb{H}^n$, i.e.,

$$\delta_\kappa(\xi) = (\kappa x, \kappa y, \kappa^2 t), \quad \kappa > 0, \tag{2.2}$$

which satisfy $\delta_\kappa(\bar{\xi} \circ \xi) = \delta_\kappa(\bar{\xi}) \circ \delta_\kappa(\xi)$.

The left invariant vector fields corresponding to $\mathbb{H}^n$ are

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n,$$

$$Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \ldots, n,$$

$$T = \frac{\partial}{\partial t}.$$

It is easy to check that X_i and Y_j satisfy

$$[X_i, Y_j] = -4T\delta_{ij}, \qquad [X_i, X_j] = [Y_i, Y_j] = 0, \quad i, j = 1, \ldots, n.$$

The Heisenberg gradient of a function u is defined by

$$\nabla_{\mathbb{H}} u = (X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u), \tag{2.3}$$

and the subLaplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}^n$ is

$$\Delta_{\mathbb{H}} := \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i{}^2} + \frac{\partial^2}{\partial y_i{}^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4\left(x_i{}^2 + y_i{}^2 \right) \frac{\partial^2}{\partial t^2} \right). \tag{2.4}$$

The family $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ satisfies Hörmander's rank condition (see [16]) which implies that $\Delta_{\mathbb{H}}$ is hypoelliptic and the maximum principle holds for solutions to the equation involving $\Delta_{\mathbb{H}}$ (see [2]).

The integer $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$. Denote by $|\xi|_{\mathbb{H}}$ the distance from ξ to the zero (see [13])

$$|\xi|_{\mathbb{H}} = \left(\sum_{i=1}^{n} \left(x_i{}^2 + y_i{}^2 \right)^2 + t^2 \right)^{\frac{1}{4}}. \tag{2.5}$$

Authors in [22] used the norm $|(z, w)| = \left(\sum_{i=1}^{n} (x_i{}^2 + y_i{}^2)^2 + 16t^2 \right)^{\frac{1}{4}}$ for $(x, y, t) := (z, w) \in \mathbb{H}^n$, which is equivalent to (2.5). The distance between two points of $\mathbb{H}^n$ is defined by

$$d_{\mathbb{H}}(\xi, \eta) = \left| \eta^{-1} \circ \xi \right|_{\mathbb{H}},$$

where η^{-1} denotes the inverse of η with respect to $\circ$, that is, $\eta^{-1} = -\eta$. The open ball of radius $R > 0$ centered at ξ is the set

$$B_{\mathbb{H}}(\xi, R) = \left\{ \eta \in \mathbb{H}^n \mid d_{\mathbb{H}}(\eta, \xi) < R \right\}.$$

It is well known that $\xi \to |\xi|_{\mathbb{H}}$ is homogeneous of degree one with respect to δ_κ and

$$\left| B_{\mathbb{H}}(\xi, R) \right| = \left| B_{\mathbb{H}}(0, R) \right| = \left| B_{\mathbb{H}}(0, 1) \right| R^Q,$$

where $| \cdot |$ denotes the Lebesgue measure.

A function u is called the cylindrical function if

$$u(x, y, t) = u(r, t),$$

where $(x, y, t) \in \mathbb{H}^n$, $r = (|x|^2 + |y|^2)^{\frac{1}{2}}$.

Proposition 2.1 *For $u \in L_{sp}(\mathbb{H}^n) \cap C_{\mathrm{loc}}^{1,1}(\mathbb{H}^n)$, the operator in* (1.1) *is well defined.*

Proof For any $\xi \in \mathbb{H}^n$,

$$
PV \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta
$$

$$
= \lim_{\varepsilon \to 0} \Bigg[\int_{B_{\mathbb{H}}(\xi,1) \setminus B_{\mathbb{H}}(\xi,\varepsilon)} \left(\left| \left\langle -(\nabla_{\mathbb{H}} u, Tu), \eta^{-1} \circ \xi \right\rangle + o\left(\left| \eta^{-1} \circ \xi \right|_{\mathbb{H}}^2 \right) \right|^{p-2} \right.
$$

$$
\left. \times \left(\left\langle -(\nabla_{\mathbb{H}} u, Tu), \eta^{-1} \circ \xi \right\rangle + o\left(\left| \eta^{-1} \circ \xi \right|_{\mathbb{H}}^2 \right) \right) \right) \left(\left| \eta^{-1} \circ \xi \right|_{\mathbb{H}}^{Q+sp} \right)^{-1} d\eta
$$

$$
+ \int_{\mathbb{H}^n \setminus B_{\mathbb{H}}(\xi,1)} \frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta \Bigg]
$$

$$
\leq C \lim_{\varepsilon \to 0} \Bigg[\int_{B_{\mathbb{H}}(\xi,1) \setminus B_{\mathbb{H}}(\xi,\varepsilon)} \frac{o(1)}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta + \int_{B_{\mathbb{H}}(\xi,1) \setminus B_{\mathbb{H}}(\xi,\varepsilon)} \frac{o(1)}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta
$$

$$
+ \int_{\mathbb{H}^n \setminus B_{\mathbb{H}}(\xi,1)} \frac{u^{p-1}(\xi)}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta + \int_{\mathbb{H}^n \setminus B_{\mathbb{H}}(\xi,1)} \frac{u^{p-1}(\eta)}{|\eta^{-1} \circ \xi|_{\mathbb{H}}^{Q+sp}} \, d\eta \Bigg]
$$

$$
:= C \lim_{\varepsilon \to 0} (I_1 + I_2 + I_3 + I_4),
$$

where ε is sufficiently small. Noting that $u \in C_{\mathrm{loc}}^{1,1}(\mathbb{H}^n)$, $Q + sp - p < Q$, and $Q + sp - 2p + 2 < Q$, we know that I_1 and I_2 are finite; I_3 is clearly convergent when $|\xi|_{\mathbb{H}} \to \infty$; and I_4 is finite from $u \in L_{sp}(\mathbb{H}^n)$. Hence, (1.1) is well defined.   □

3 Maximum principles

In this section, we prove three maximum principles which will be used in the process of moving planes. These maximum principles are on a bounded domain in $\mathbb{H}^n$, on a bounded domain in the left domain of some hyperplane, and on a narrow region.

Lemma 3.1 *Let Ω be a bounded domain in $\mathbb{H}^n$. Assume $u \in L_{sp}(\mathbb{H}^n) \cap C_{\mathrm{loc}}^{1,1}(\mathbb{H}^n)$ is lower semicontinuous on $\bar{\Omega}$ and satisfies*

$$
\begin{cases}
(-\Delta_{\mathbb{H}})_p^s u(\xi) \geq 0, & \xi \in \Omega, \\
u(\xi) \geq 0, & \xi \in \mathbb{H}^n \setminus \Omega,
\end{cases}
\tag{3.1}
$$

then

$$
u(\xi) \geq 0, \quad \xi \in \Omega.
\tag{3.2}
$$

Furthermore, if $u = 0$ at some point in Ω, then

$$
u(\xi) = 0 \quad \text{almost everywhere in } \mathbb{H}^n.
$$

These conclusions also hold on the unbounded region Ω if we further assume that

$$
\varliminf_{|\xi|_{\mathbb{H}} \to \infty} u(\xi) \geq 0.
$$

Proof Suppose that (3.2) is not true, then by the lower semicontinuity of u on $\bar{\Omega}$ there exists $\xi^0 \in \bar{\Omega}$ such that

$$u(\xi^0) = \min_{\bar{\Omega}} u < 0.$$

From (3.1), we know that ξ^0 is a point in Ω, and

$$(-\Delta_{\mathbb{H}})_p^s u(\xi^0) = C_{Q,s,p} PV \int_{\mathbb{H}^n} \frac{|u(\xi^0) - u(\eta)|^{p-2}(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta$$

$$\leq C_{Q,s,p} \int_{\mathbb{H}^n \setminus \Omega} \frac{|u(\xi^0) - u(\eta)|^{p-2}(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta$$

$$< 0,$$

which contradicts (3.1). This implies (3.2).

If there exists some point $\xi^0 \in \Omega$ such that $u(\xi^0) = 0$, then

$$0 \leq (-\Delta_{\mathbb{H}})_p^s u(\xi^0) = C_{Q,s,p} \int_{\mathbb{H}^n} \frac{|u(\eta)|^{p-2}(-u(\eta))}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta.$$

Using $u(\xi) \geq 0$, we have $u(\xi) = 0$ almost everywhere in $\mathbb{H}^n$.

For an unbounded region Ω, the condition $\underline{\lim}_{|\xi|_{\mathbb{H}} \to \infty} u(\xi) \geq 0$ implies that the negative minimum ξ^0 of u cannot be reached at infinity. Then the condition of lower semicontinuity ensures that the proof can go on as above. The proof is ended.   □

Let T_λ be a hyperplane in $\mathbb{H}^n$ defined by

$$T_\lambda = \left\{ \xi \in \mathbb{H}^n \mid t = \lambda, \lambda \in \mathbb{R} \right\}.$$

Denote by $\tilde{\xi} = (y, x, 2\lambda - t)$ the H-reflection of $\xi = (x, y, t)$ about the plane T_λ and by

$$\Sigma_\lambda = \left\{ \xi \in \mathbb{H}^n \mid t < \lambda \right\}$$

the region in the left of the plane T_λ. Letting

$$u_\lambda(\xi) = u_\lambda(|(x,y)|, t) := u(|(x,y)|, 2\lambda - t)$$

and using the H-refection (see [1]), we have

$$u_\lambda(\xi) = u(y, x, 2\lambda - t) = u(\xi^\lambda).$$

Set

$$w_\lambda(\xi) = u_\lambda(\xi) - u(\xi).$$

Lemma 3.2 *Let Ω be a bounded domain in Σ_λ. Assume that the cylindrical function $u \in L_{sp}(\mathbb{H}^n) \cap C^{1,1}_{\text{loc}}(\mathbb{H}^n)$ is lower semicontinuous on $\bar{\Omega}$ and satisfies*

$$
\begin{cases}
(-\Delta_{\mathbb{H}})^s_p u_\lambda(\xi) - (-\Delta_{\mathbb{H}})^s_p u(\xi) \geq 0, & \xi \in \Omega, \\
w_\lambda(\xi) \geq 0, & \xi \in \Sigma_\lambda \setminus \Omega, \\
w_\lambda(\xi^\lambda) = -w_\lambda(\xi), & \xi \in \Sigma_\lambda,
\end{cases}
\tag{3.3}
$$

then

$$
w_\lambda(\xi) \geq 0, \quad \xi \in \Omega.
\tag{3.4}
$$

Furthermore, if $w_\lambda = 0$ at some point in Ω, then

$$
w_\lambda(\xi) = 0 \quad \text{almost everywhere in } \mathbb{H}^n.
$$

These conclusions also hold for the unbounded region Ω if we further assume that

$$
\varliminf_{|\xi|_{\mathbb{H}} \to \infty} w_\lambda(\xi) \geq 0.
$$

Proof Suppose that (3.4) is incorrect. By the lower semicontinuity of w_λ on $\bar{\Omega}$, there exists $\xi^0 \in \bar{\Omega}$ such that

$$
w_\lambda\left(\xi^0\right) = \min_{\bar{\Omega}} w_\lambda < 0.
$$

For simplicity, we denote

$$
G(a) = |a|^{p-2}a, \quad a \geq 0.
$$

Note that $G(a)$ is increasing and $G'(a) = (p-1)|a|^{p-2} \geq 0$. A direct calculation gives

$$
\begin{aligned}
& (-\Delta_{\mathbb{H}})^s_p u_\lambda\left(\xi^0\right) - (-\Delta_{\mathbb{H}})^s_p u\left(\xi^0\right) \\
&= C_{Q,s,p} PV \int_{\mathbb{H}^n} \frac{G(u_\lambda(\xi^0) - u_\lambda(\eta)) - G(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta \\
&= C_{Q,s,p} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(\xi^0) - u_\lambda(\eta)) - G(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta \\
&\quad + C_{Q,s,p} PV \int_{\Sigma_\lambda} \frac{G(u_\lambda(\xi^0) - u(\eta)) - G(u(\xi^0) - u_\lambda(\eta))}{|(\eta^\lambda)^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \, d\eta \\
&= C_{Q,s,p} PV \int_{\Sigma_\lambda} \left(\frac{1}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} - \frac{1}{|(\eta^\lambda)^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} \right) \\
&\quad \times \left(G\left(u_\lambda\left(\xi^0\right) - u_\lambda(\eta)\right) - G\left(u\left(\xi^0\right) - u(\eta)\right) \right) d\eta \\
&\quad + C_{Q,s,p} PV \int_{\Sigma_\lambda} \left(G\left(u_\lambda\left(\xi^0\right) - u_\lambda(\eta)\right) - G\left(u\left(\xi^0\right) - u_\lambda(\eta)\right) + G\left(u_\lambda\left(\xi^0\right) - u(\eta)\right) \right)
\end{aligned}
$$

$$- G\big(u\big(\xi^0\big) - u(\eta)\big)\big)\big(\big|\big(\eta^\lambda\big)^{-1} \circ \xi^0\big|_{\mathbb{H}}^{Q+sp}\big)^{-1} d\eta$$
$$:= C_{Q,s,p}(J_1 + J_2). \tag{3.5}$$

For J_1, we have for any $\xi^0, \eta \in \Sigma_\lambda$,

$$\frac{1}{|\eta^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} - \frac{1}{|(\eta^\lambda)^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} > 0.$$

By the monotonicity of G and the fact that

$$\big(u_\lambda\big(\xi^0\big) - u_\lambda(\eta)\big) - \big(u\big(\xi^0\big) - u(\eta)\big) = w_\lambda\big(\xi^0\big) - w_\lambda(\eta)$$

is nonpositive but not identity to 0, we deduce that

$$G\big(u_\lambda\big(\xi^0\big) - u_\lambda(\eta)\big) - G\big(u\big(\xi^0\big) - u(\eta)\big)$$

is also nonpositive but not identity to 0. So we have

$$J_1 < 0. \tag{3.6}$$

For J_2, by the mean value theorem,

$$J_2 = \int_{\Sigma_\lambda} \frac{G(u_\lambda(\xi^0) - u_\lambda(\eta)) - G(u(\xi^0) - u_\lambda(\eta)) + G(u_\lambda(\xi^0) - u(\eta)) - G(u(\xi^0) - u(\eta))}{|(\eta^\lambda)^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} d\eta$$
$$= w_\lambda\big(\xi^0\big) \int_{\Sigma_\lambda} \frac{G'(g(\eta)) + G'(h(\eta))}{|(\eta^\lambda)^{-1} \circ \xi^0|_{\mathbb{H}}^{Q+sp}} d\eta$$
$$\leq 0. \tag{3.7}$$

In fact, if $u_\lambda(\eta) \geq u(\eta)$, then we have $w_\lambda(\eta) \geq 0$, i.e., (3.4) holds. If $u_\lambda(\eta) > u(\eta)$, we know G is strictly increasing, then $G'(g(\eta)) \geq 0$ and $G'(h(\eta)) \geq 0$. Hence, we have (3.7).

Putting (3.6) and (3.7) into (3.5) implies

$$(-\Delta_{\mathbb{H}})_p^s u_\lambda\big(\xi^0\big) - (-\Delta_{\mathbb{H}})_p^s u\big(\xi^0\big) < 0.$$

This contradicts (3.3) and we obtain (3.4).

If there exists some point $\xi^0 \in \Omega$ such that $w_\lambda(\xi^0) = 0$, then (3.5) holds and $J_2 \geq 0$. Hence from the first inequality in (3.3) we have $J_1 \geq 0$, and by the monotonicity of G,

$$G\big(u_\lambda\big(\xi^0\big) - u_\lambda(\eta)\big) - G\big(u\big(\xi^0\big) - u(\eta)\big) \geq 0.$$

We have, for almost all $\eta \in \Sigma_\lambda$,

$$\big(u_\lambda\big(\xi^0\big) - u_\lambda(\eta)\big) - \big(u\big(\xi^0\big) - u(\eta)\big) = w_\lambda\big(\xi^0\big) - w_\lambda(\eta) = -w_\lambda(\eta) \geq 0.$$

Using (3.4), we have

$$w_\lambda(\xi) = 0 \quad \text{almost everywhere in } \Sigma_\lambda.$$

From the antisymmetry of w_λ,

$$w_\lambda(\xi) = 0 \quad \text{almost everywhere in } \mathbb{H}^n. \qquad\qquad \square$$

Lemma 3.3 *Let Ω be a bounded narrow domain in Σ_λ and locate in $\{\xi \mid \lambda - l < t < \lambda\}$ for small l. Assume that the cylindrical function $u \in L_{sp}(\mathbb{H}^n) \cap C_{\text{loc}}^{1,1}(\mathbb{H}^n)$ is lower semicontinuous on $\bar{\Omega}$. If $c(x)$ is bounded from below in Ω and u satisfies*

$$\begin{cases} (-\Delta_\mathbb{H})_p^s u_\lambda(\xi) - (-\Delta_\mathbb{H})_p^s u(\xi) + c(\xi) w_\lambda(\xi) \geq 0, & \xi \in \Omega, \\ w_\lambda(\xi) \geq 0, & \xi \in \Sigma_\lambda \setminus \Omega, \\ w_\lambda(\xi^\lambda) = -w_\lambda(\xi), & \xi \in \Sigma_\lambda, \end{cases} \tag{3.8}$$

then

$$w_\lambda(\xi) \geq 0, \quad \xi \in \Omega. \tag{3.9}$$

Furthermore, if $w_\lambda = 0$ at some point in Ω, then

$$w_\lambda(\xi) = 0 \quad \text{almost everywhere in } \mathbb{H}^n.$$

These conclusions also hold for the unbounded region Ω if we further assume that

$$\varliminf_{|\xi|_\mathbb{H} \to \infty} w_\lambda(\xi) \geq 0.$$

Proof By the proof of Lemma 3.2, we have

$$\begin{aligned} &(-\Delta_\mathbb{H})_p^s u_\lambda(\xi^0) - (-\Delta_\mathbb{H})_p^s u(\xi^0) \\ &\quad = C_{Q,s,p} PV \int_{\Sigma_\lambda} \left(\frac{1}{|\eta^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} - \frac{1}{|(\eta^\lambda)^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} \right) \\ &\qquad \times \left(G(u_\lambda(\xi^0) - u_\lambda(\eta)) - G(u(\xi^0) - u(\eta)) \right) d\eta \\ &\qquad + C_{Q,s,p} PV \int_{\Sigma_\lambda} \left(G(u_\lambda(\xi^0) - u_\lambda(\eta)) - G(u(\xi^0) - u(\eta)) + G(u_\lambda(\xi^0) - u(\eta)) \right. \\ &\qquad \left. - G(u(\xi^0) - u_\lambda(\eta)) \right) \left(|(\eta^\lambda)^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp} \right)^{-1} d\eta \\ &\quad := C_{Q,s,p}(I_1 + I_2). \end{aligned} \tag{3.10}$$

Obviously,

$$I_2 \leq 0. \tag{3.11}$$

Similar to (3.6), we know

$$I_1 < 0.$$

Denote

$$\delta_{\xi^0} = \mathrm{dist}\left(\xi^0, \partial \Sigma_\lambda\right) = \left|\lambda - t^0\right|.$$

Combining (3.10), (3.11), and $I_1 < 0$, we have

$$\frac{(-\Delta_{\mathbb{H}})_p^s u_\lambda(\xi^0) - (-\Delta_{\mathbb{H}})_p^s u(\xi^0)}{\delta_{\xi^0}} < 0. \tag{3.12}$$

Noting that ξ^0 is a negative minimum of w_λ, we infer $\nabla w_{\lambda_0}(\xi^0) = 0$, and so

$$\frac{\partial w_\lambda}{\partial t}\left(\xi^0\right) = \lim_{\delta_k \to 0} \frac{w_\lambda(\xi^0)}{\delta_{\xi^0}} = 0,$$

i.e.,

$$\frac{c(\xi^0) w_\lambda(\xi^0)}{\delta_{\xi^0}} \le o(1). \tag{3.13}$$

Now (3.12) and (3.13) contradict (3.8), and then (3.9) is proved. □

4 Proof of the main results

Following the idea in [6], we first use Lemma 3.1, Lemma 3.2, and Lemma 3.3 to prove Theorem 1.1.

Proof of Theorem 1.1 First we check that for λ sufficiently negative it holds

$$w_\lambda(\xi) \ge 0, \quad \xi \in \Sigma_\lambda. \tag{4.1}$$

Indeed, suppose that (4.1) is violated, then by (1.8) there exists a point $\xi^0 \in \Sigma_\lambda$ such that

$$w_\lambda\left(\xi^0\right) = \min_{\Sigma_\lambda} w_\lambda < 0,$$

i.e., $u_\lambda(\xi^0) \le \varsigma_\lambda(\xi^0) \le u(\xi^0)$. Note by (1.2) that

$$(-\Delta_{\mathbb{H}})_p^s u_\lambda(\xi) - (-\Delta_{\mathbb{H}})_p^s u(\xi) = f'\left(\varsigma_\lambda(\xi)\right) w_\lambda(\xi), \tag{4.2}$$

where $\varsigma_\lambda(\xi)$ between $u_\lambda(\xi)$ and $u(\xi)$. For sufficiently negative λ, $u(\xi^0)$ is small by (1.8), hence so is $\varsigma_\lambda(\xi^0)$. Due to the condition of f', we have $f'(\varsigma_\lambda(\xi^0)) \le 0$. From (4.2),

$$(-\Delta_{\mathbb{H}})_p^s u_\lambda\left(\xi^0\right) - (-\Delta_{\mathbb{H}})_p^s u\left(\xi^0\right) \ge 0. \tag{4.3}$$

On the other hand, it follows by the proof of Lemma 3.2 that

$$(-\Delta_{\mathbb{H}})_p^s u_\lambda\left(\xi^0\right) - (-\Delta_{\mathbb{H}})_p^s u\left(\xi^0\right) < 0. \tag{4.4}$$

This contradicts (4.3), and hence (4.1) is proved.

The above result provides the starting point of moving planes. Let us move the plane T_λ to the right as long as (4.1) holds to its limiting position

$$\lambda_0 = \sup\{\lambda \mid w_\mu(\xi) \geq 0, \forall \xi \in \Sigma_\mu, \mu \leq \lambda\}.$$

We will show that

$$\lambda_0 = 0 \tag{4.5}$$

i.e.,

$$w_{\lambda_0}(\xi) = 0, \quad \xi \in \Sigma_{\lambda_0}. \tag{4.6}$$

In fact, suppose that (4.6) is false, we have by Lemma 3.2 that

$$w_{\lambda_0}(\xi) > 0, \quad \xi \in \Sigma_{\lambda_0}. \tag{4.7}$$

From the definition of λ_0, there exist a sequence $\lambda_k \to \lambda_0$ and a point $\xi^k \in \Sigma_{\lambda_k}$ such that

$$w_{\lambda_k}(\xi^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0, \qquad \nabla w_{\lambda_k}(\xi^k) = 0. \tag{4.8}$$

Note that

$$(-\Delta_{\mathbb{H}})^s_p u_{\lambda_k}(\xi^k) - (-\Delta_{\mathbb{H}})^s_p u(\xi^k) = f'(\varsigma_{\lambda_k}(\xi^k)) w_{\lambda_k}(\xi^k). \tag{4.9}$$

If $|\xi^k|_{\mathbb{H}}$ is sufficiently large, then $u(\xi^k)$ is small and so $\varsigma_{\lambda_k}(\xi^k)$ is also small, this implies $f'(\varsigma_{\lambda_k}(\xi^k)) \leq 0$ (because $f'(a) \leq 0$ for the sufficiently small a). It follows

$$(-\Delta_{\mathbb{H}})^s_p u_{\lambda_k}(\xi^k) - (-\Delta_{\mathbb{H}})^s_p u(\xi^k) \geq 0.$$

But this contradicts the fact that ξ^k is a negative minimum of w_{λ_k} (see Lemma 3.2). Hence, $\{\xi^k\}$ is bounded, i.e., the sequence $\{\xi^k\}$ is bounded.

It follows that the subsequence of $\{\xi^k\}$ converges to some point ξ^0. Then (4.8) means that, for $\xi^0 \in \partial \Sigma_{\lambda_0}$,

$$w_{\lambda_0}(\xi^0) \leq 0, \qquad \nabla w_{\lambda_0}(\xi^0) = 0.$$

Particularly,

$$\frac{\partial w_{\lambda_0}}{\partial t}(\xi^0) = \lim_{\delta_k \to 0} \frac{w_{\lambda_k}(\xi^k)}{\delta_k} = 0.$$

Applying (4.9), we have

$$\lim_{\delta_k \to 0} \frac{1}{\delta_k}\left((-\Delta_{\mathbb{H}})^s_p u_{\lambda_k}(\xi^k) - (-\Delta_{\mathbb{H}})^s_p u(\xi^k)\right) = \lim_{\delta_k \to 0} \frac{1}{\delta_k} f'(\varsigma_{\lambda_k}(\xi^k)) w_{\lambda_k}(\xi^k) = 0,$$

which is a contradiction with Lemma 3.3. Therefore,

$$w_{\lambda_0}(\xi) \geq 0, \quad \xi \in \Sigma_{\lambda_0}. \tag{4.10}$$

Similarly, we can move the plane from $+\infty$ to the left to get

$$w_{\lambda_0}(\xi) \leq 0, \quad \xi \in \Sigma_{\lambda_0}. \tag{4.11}$$

Then (4.6) follows by combining (4.10) and (4.11). Finally, we see that u must be symmetric and monotone with respect to t about some point. $\square$

Next, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 By condition (1.8) and $f(0) = 0$, we claim

$$u(\xi) > 0 \quad \text{or} \quad u(\xi) \equiv 0 \quad \text{for any } \xi \in \mathbb{H}^n_+.$$

In fact, suppose that the conclusion $u(\xi) > 0$ is not correct, we will verify $u(\xi) \equiv 0$. The lower semicontinuity of u on $\mathbb{H}^n_+$ implies that there exists $\xi^0 \in \bar{\mathbb{H}}^n_+$ such that

$$u(\xi^0) = \min_{\bar{\mathbb{H}}^n_+} u = 0,$$

and then

$$
\begin{aligned}
(-\Delta_\mathbb{H})^s_p u(\xi^0) &= C_{Q,s,p} PV \int_{\mathbb{H}^n} \frac{|u(\xi^0) - u(\eta)|^{p-2}(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} \, d\eta \\
&= C_{Q,s,p} PV \int_{\mathbb{H}^n_+} \frac{|u(\xi^0) - u(\eta)|^{p-2}(u(\xi^0) - u(\eta))}{|\eta^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} \, d\eta \\
&= C_{Q,s,p} PV \int_{\mathbb{H}^n_+} \frac{-u(\eta)|u(\eta)|^{p-2}}{|\eta^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} \, d\eta \\
&= f(u(\xi^0)) = 0.
\end{aligned}
$$

Hence $\int_{\mathbb{H}^n_+} \frac{-u(\eta)|u(\eta)|^{p-2}}{|\eta^{-1} \circ \xi^0|_\mathbb{H}^{Q+sp}} \, d\eta = 0$, and then $u(\xi) \equiv 0, \xi \in \mathbb{H}^n_+$.

In the sequel, we only need to treat the case $u > 0$ on $\mathbb{H}^n_+$. Let us employ the method of moving planes to u along the t direction and denote

$$T^+_\lambda = \left\{ \xi \in \mathbb{H}^n_+ \mid t = \lambda, \lambda \in \mathbb{R}^+ \right\}$$

and

$$\Sigma^+_\lambda = \left\{ \xi \in \mathbb{H}^n_+ \mid 0 < t < \lambda \right\}.$$

The H-reflection of $\xi = (x, y, t)$ about T^+_λ is $\xi^\lambda = (y, x, 2\lambda - t)$, and let

$$w_\lambda(\xi) = u_\lambda(\xi) - u(\xi).$$

If $\lambda > 0$ is sufficiently small, we deduce from Lemma 3.3 with $\Omega = \Sigma_\lambda^+$ and $\Sigma_\lambda = \Sigma_\lambda^+ \cup (\mathbb{H}^n \setminus \mathbb{H}_+^n)$ that on the narrow region Σ_λ^+,

$$w_\lambda(\xi) \geq 0. \tag{4.12}$$

This provides the starting point of moving planes. Now we will explain that the plane T_λ^+ can be moved to the infinity so that (4.12) holds. Let

$$\lambda_0 = \sup\{\lambda > 0 \mid w_\mu(\xi) \geq 0, \forall \xi \in \Sigma_\lambda^+, \mu \leq \lambda\},$$

and we will prove

$$\lambda_0 = \infty. \tag{4.13}$$

In fact, if $\lambda_0 < \infty$, then we claim that T_λ^+ can be moved further to the right, that is, there exists $\sigma > 0$ such that, for any $\lambda \in (\lambda_0, \lambda_0 + \sigma)$,

$$w_\lambda(\xi) \geq 0, \quad \xi \in \Sigma_\lambda^+. \tag{4.14}$$

This will contradict the definition of λ_0, and hence (4.13) holds.

At present, we prove (4.14). If $|\xi|_{\mathbb{H}}$ is sufficiently large, then (4.14) is true by using the similar proof to (4.1) in Theorem 1.1. This implies that there exists some $R_0 > 0$ such that (4.14) holds true on $\mathbb{H}_+^n \setminus B_{\mathbb{H}}(0, R_0)$. Next we point out that (4.14) is also true on $B_{\mathbb{H}}(0, R_0)$. Noting $\lambda_0 < \infty$ and using Lemma 3.2, we find that on $\xi \in \Sigma_{\lambda_0}^+ \cap B_{\mathbb{H}}(0, R_0)$,

$$w_{\lambda_0}(\xi) > 0 \tag{4.15}$$

or

$$w_{\lambda_0}(\xi) \equiv 0.$$

In the case $w_{\lambda_0}(\xi) \equiv 0$, we observe by the boundary conditions of u that $u(\xi) \equiv 0$. On the other hand, (4.15) implies that there exists small $\delta > 0$ such that

$$w_{\lambda_0}(\xi) \geq c > 0, \quad \xi \in \overline{\Sigma_{\lambda_0-\delta}^+ \cap B_{\mathbb{H}}(0, R_0)}. \tag{4.16}$$

Since w_λ relies continuously on λ, there exists $\sigma > \varepsilon > 0$ such that

$$w_{\lambda_0+\varepsilon}(\xi) \geq 0, \quad \xi \in \overline{\Sigma_{\lambda_0-\delta}^+ \cap B_{\mathbb{H}}(0, R_0)}. \tag{4.17}$$

Since $(\Sigma_{\lambda_0+\varepsilon}^+ \setminus \Sigma_{\lambda_0-\delta}^+) \cap B_{\mathbb{H}}(0, R_0)$ is a narrow region, we have by Lemma 3.3

$$w_{\lambda_0+\varepsilon}(\xi) \geq 0, \quad \xi \in \overline{\Sigma_{\lambda_0+\varepsilon}^+ \cap B_{\mathbb{H}}(0, R_0)},$$

and therefore (4.14) is proved.

Using (4.13) and Lemma 3.2 once again, it follows that, for any $\xi \in \Sigma_\lambda^+$ (here $0 \leq \lambda \leq \infty$),

$$w_\lambda(\xi) > 0, \tag{4.18}$$

or

$$w_\lambda(\xi) \equiv 0. \tag{4.19}$$

For (4.19), we can use the boundary condition of u to obtain $u(\xi) \equiv 0$. In addition, it follows from (4.18) that $u(\xi)$ is strictly increasing, which contradicts the boundary condition of u and (1.8). □

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