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Existence and multiplicity of solutions for nonlocal fourth-order elliptic equations with combined nonlinearities

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Abstract

This paper is concerned with the following nonlocal fourth-order elliptic problem:

$$\begin{cases} \Delta^2 u - m(\int_\Omega |\nabla u|^2\,dx)\Delta u = a(x)|u|^{s-2}u + f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

by using the mountain pass theorem, the least action principle, and the Ekeland variational principle, the existence and multiplicity results are obtained.

Keywords: Fourth-order elliptic equation; Nonlocal; Asymptotically linear; Mountain pass theorem; Critical point

1 Introduction

In this paper, we consider the following nonlocal fourth-order elliptic problem:

$$\begin{cases} \Delta^2 u - m(\int_\Omega |\nabla u|^2\,dx)\Delta u = a(x)|u|^{s-2}u + f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ $(N > 4)$ is a bounded smooth domain, $m(\cdot) \in C(R^+, R^+)$, $a(\cdot) \in C(\overline{\Omega}, R^+)$, $s \in (1,2)$, and $f \in C(\overline{\Omega} \times R, R)$.

Problem (1.1) is related to the stationary problems associated with

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \left(Q + \int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u,u_t).$$

This plate model was proposed by Berger [1] in 1955, as a simplification of the von Karman plate equation which describes large defection of a plate, where the parameter Q describes in-plane forces applied to the plate and the function f represents transverse loads which may depend on the displacement u and the velocity u_t.

Because of the important background, several researchers have considered problem (1.1) by using variational methods when $a(x) \equiv 0$,

$$\begin{cases} \Delta^2 u - m(\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

with the function m being bounded or unbounded and f having superlinear growth. We refer the readers to [2–11] and the references therein.

Recently, in [12], Ru et al. considered problem (1.1) with $m(t) = a + bt$ and a more general f such as

$$\begin{cases} \Delta^2 u - (a + b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u,\nabla u,\triangle u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases}$$

By using an iterative method based on the mountain pass lemma and truncation method developed by De Figueiredo et al. [13], they proved that the above problem has at least one nontrivial solution.

One of the important conditions in their work is that $f(x,t)$ satisfies the famous Ambrosetti–Rabinowitz type condition, for short, which is called the (AR) condition:

(AR condition) there exist $\Theta > 2$ and $t_1 > 0$, such that

$$0 < \Theta F(x,t,\xi_1,\xi_2) \le tf(x,t,\xi_1,\xi_2), \quad \forall |t| \ge t_1, x \in \Omega, (\xi_1,\xi_2) \in R^{N+1},$$

where $F(x,t,\xi_1,\xi_2) = \int_0^t f(x,s,\xi_1,\xi_2)\,ds$.

It is well known that (AR) is a important technical condition to apply the mountain pass theorem. This condition implies that

$$\lim_{u\to\infty} \frac{F(x,u)}{u^2} = \infty.$$

If $f(x,u)$ is asymptotically linear at $u = 0$ or $u = +\infty$. then $f(x,u)$ does not satisfy the (AR) condition. In [14], A. Bensedik and M. Bouchekif considered second-order elliptic equations of Kirchhoff type with an asymptotically linear potential

$$\begin{cases} -m(\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

On the other hand, the classical equation involving a biharmonic operator

$$\begin{cases} \Delta^2 u + c\Delta u = a(x)|u|^{s-2}u + f(x,u), & x \in \Omega, \\ u(x) = \Delta u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

has been extensively studied using the mountain pass theorem when $a(x) \equiv 0$ and $f(x,u)$ is asymptotically linear at $u = 0$ or $u = +\infty$. We refer the reader to [15, 16]. In particular, in [17], Pu et al. considered problem (1.2) when $a(x) \not\equiv 0$.

Until now, there are few works on problem (1.1) when $a(x) \not\equiv 0$ and $f(x,u)$ does not satisfy the (AR) condition. Inspired by these references, in this paper, we discuss the existence and multiplicity of solutions of problem (1.1) when $a(x) \not\equiv 0$ and the nonlinearity f is asymptotically linear at $u = 0$ or $u = +\infty$.

2 Preliminaries

Assume that the function $m(t)$ satisfies the following conditions:

(M) $m : R^+ \to R^+$ is continuous, nondecreasing, and there exists $m_1 \geq m_0 > 0$ such that

$$m_0 = \min_{t \in R^+} m(t) = m(0), \qquad m_1 = \sup_{t \in R^+} m(t).$$

Remark In [14] and [18], the function $m(t)$ is assumed that satisfy (M) and there exits $t_0 > 0$ such that $m(t) = m_1$, $\forall t > t_0$.

First, we study the nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 u - m(\int_\Omega |\nabla u|^2 \, dx)\Delta u = \Lambda u, & x \in \Omega, \\ u = 0, \qquad \Delta u = 0, & x \in \partial\Omega. \end{cases}$$

Let (λ_k, ϕ_k) be the eigenvalue and the corresponding eigenfunction of $(-\Delta, H_0^1(\Omega))$, namely

$$\begin{cases} -\Delta \phi_k = \lambda_k \phi_k, & x \in \Omega, \\ \phi_k(x) = 0, & x \in \partial\Omega. \end{cases}$$

Set

$$Lu = \Delta^2 u - m\left(\int_\Omega |\nabla u|^2 \, dx\right)\Delta u.$$

Via some simple computations, we get

$$\begin{aligned} L\phi_k &= \Delta^2 \phi_k - m\left(\int_\Omega |\nabla \phi_k|^2 \, dx\right)\Delta \phi_k \\ &= \left[\lambda_k^2 + \lambda_k m\left(\int_\Omega |\nabla \phi_k|^2 \, dx\right)\right]\phi_k \\ &= \left[\lambda_k^2 + \lambda_k m\left(\lambda_k \int_\Omega |\phi_k|^2 \, dx\right)\right]\phi_k. \end{aligned}$$

Set

$$\Lambda_k = \begin{cases} \lambda_k^2 + \lambda_k m(\int_\Omega |\nabla \phi_k|^2 \, dx), & \text{or} \\ \lambda_k^2 + \lambda_k m(\lambda_k \int_\Omega |\phi_k|^2 \, dx) \end{cases} \tag{2.1}$$

and so Λ_k $(k = 1, 2, \ldots)$ are the eigenvalues of the operator L associated to the eigenfunction ϕ_k.

Assume that the eigenfunctions ϕ_k are suitably normalized with respect to the $L^2(\Omega)$ inner product, namely

$$(\phi_i, \phi_j)_{L^2(\Omega)} = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

Expression (2.1) can be rewritten as

$$\Lambda_k = \lambda_k^2 + \lambda_k m\left(\lambda_k \int_\Omega |\phi_k|^2 \, dx\right) = \lambda_k^2 + \lambda_k m(\lambda_k).$$

For each eigenvalue λ_k being repeated as often as multiplicity, recall that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \to +\infty,$$

and if (M) holds, then

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \leq \Lambda_k \to +\infty.$$

Denote

$$\bar{\Lambda}_k = \lambda_k^2 + m_1 \lambda_k, \quad k = 1, 2, \ldots,$$

then we know that

$$\Lambda_k \leq \bar{\Lambda}_k, \quad k = 1, 2, \ldots.$$

It is well known that

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^2 \, dx : u \in \mathbf{H}_0^1(\Omega), \int_\Omega |u|^2 \, dx = 1\right\}.$$

Similarly, we have

Lemma 2.1 *Assume that* (M) *holds, then*

$$\begin{aligned} \Lambda_1 = \inf\bigg\{ & \int_\Omega |\Delta u|^2 \, dx + m\left(\int_\Omega |\nabla u|^2 \, dx\right) \int_\Omega |\nabla u|^2 \, dx : \\ & u \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \int_\Omega |u|^2 \, dx = 1 \bigg\}. \end{aligned}$$

Proof Denote

$$\begin{aligned} \inf\bigg\{ & \int_\Omega |\Delta u|^2 \, dx + m\left(\int_\Omega |\nabla u|^2 \, dx\right) \int_\Omega |\nabla u|^2 \, dx : \\ & u \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \int_\Omega |u|^2 \, dx = 1 \bigg\} = \Lambda_0, \end{aligned}$$

then it is clear that

$$\Lambda_1 = \lambda_1^2 + \lambda_1 m(\lambda_1) \geq \Lambda_0.$$

Let $u_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ achieve Λ_0, then $\int_\Omega |u_0|^2\,dx = 1$, $\int_\Omega |\nabla u_0|^2\,dx \geq \lambda_1$ and $u_0 = 0$ on $\partial\Omega$, therefore

$$\int_\Omega |\nabla u_0|^2\,dx = -\int_\Omega u_0 \Delta u_0\,dx,$$

which implies that

$$\left(\int_\Omega |\nabla u_0|^2\,dx\right)^2 = \left(-\int_\Omega u_0 \Delta u_0\right)^2 dx \leq \int_\Omega |u_0|^2\,dx \int_\Omega |\Delta u_0|^2\,dx = \int_\Omega |\Delta u_0|^2\,dx,$$

then

$$\begin{aligned}
\Lambda_0 &= \int_\Omega |\Delta u_0|^2\,dx + m\left(\int_\Omega |\nabla u_0|^2\,dx\right)\int_\Omega |\nabla u_0|^2\,dx \\
&\geq \left(\int_\Omega |\nabla u_0|^2\,dx\right)^2 + m\left(\int_\Omega |\nabla u_0|^2\,dx\right)\int_\Omega |\nabla u_0|^2\,dx \\
&\geq \lambda_1^2 + \lambda_1 m(\lambda_1) = \Lambda_1.
\end{aligned}$$

So $\Lambda_0 = \Lambda_1$.

Let $\mathbf{H} = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ be the Hilbert space equipped with the standard inner product

$$(u, v)_H = \int_\Omega (\Delta u \Delta v + \nabla u \nabla v)\,dx$$

and the deduced norm

$$\|u\|_H^2 = \int_\Omega |\Delta u|^2\,dx + \int_\Omega |\nabla u|^2\,dx.$$

It is well know that $\|u\|_H$ is equivalent to $(\int_\Omega |\Delta u|^2\,dx)^{\frac{1}{2}}$. And there exists $\tau > 0$ such that

$$\int_\Omega |\Delta u|^2\,dx \leq \|u\|_H^2 \leq \tau \int_\Omega |\Delta u|^2\,dx.$$

Denote

$$\|u\|^2 = \int_\Omega |\Delta u|^2\,dx + m_1 \int_\Omega |\nabla u|^2\,dx$$

and

$$\|u\|_{m_0}^2 = \int_\Omega |\Delta u|^2\,dx + m_0 \int_\Omega |\nabla u|^2\,dx.$$

It is obvious that the norms $\|u\|$ and $\|u\|_{m_0}$ are equivalent to the norm $\|u\|_H$ in $\mathbf{H}$. And since $m_0 < m_1$, we have

$$\|u\|^2 \geq \|u\|_{m_0}^2 \geq \theta \|u\|^2,$$

where $\theta = \frac{m_0}{m_1} \in (0.1)$.

Throughout this paper, we denote by C universal positive constants, unless otherwise specified, and

$$\|u\|_\infty = \|u\|_{L^\infty} \quad \text{for } u \in \mathbf{L}^\infty(\Omega) \text{ or } u \in \mathbf{C}(\overline{\Omega}),$$

$$\|u\|_q = \left(\int_\Omega |\nabla u|^q\,dx\right)^{\frac{1}{q}} \quad \text{for } u \in \mathbf{L}^q, 1 \le q < +\infty.$$

By the Sobolev embedding theorem, there is a positive K_q such that

$$\|u\|_q \le K_q\|u\| \quad \text{for } u \in \mathbf{H} \text{ and } 1 \le q < \frac{2N}{N-4}. \tag{2.2}$$

Specially, when condition (M) holds and $q = 2$, by Lemma 2.1, then

$$\|u\|_2^2 \le \frac{1}{\Lambda_1}\|u\|^2. \tag{2.3}$$

The mountain pass theorem and the Ekeland variational principle are our main tools, which can be found in [19]. □

Lemma 2.2 *Let E be a real Banach space, and $I \in C^1(E,R)$ satisfy (PS) condition. Suppose*

1 *There exist $\rho > 0$, $\alpha > 0$ such that*

$$I|_{\partial B_\rho} \ge I(0) + \alpha,$$

where $B_\rho = \{u \in E | \|u\| \le \rho\}$.

2 *There is an $e \in E$ with $\|e\| > \rho$ such that*

$$I(e) \le I(0).$$

Then $I(u)$ has a critical value c which can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I(u),$$

where $\Gamma = \{\gamma \in C([0,1],E) | \gamma(0) = 0, \gamma(1) = e\}$.

Lemma 2.3 *Let V be a complete metric space and $I : V \to R \cup \{+\infty\}$ be lower semicontinuous, bounded from below. Let $\varepsilon > 0$ be given and $v \in V$ be such that*

$$I(v) \le \inf_V I + \varepsilon.$$

Then there exists $u \in V$ such that

$$I(u) \le I(v), \qquad d(v,u) \le 1$$

and for all $w \ne u$ in V,

$$I(w) > F(u) - \varepsilon d(v,w).$$

3 Main results

A function $u \in \mathbf{H}$ is called a weak solution of (1.1) if

$$\int_\Omega \Delta u \Delta v\,dx + m\biggl(\int_\Omega |\nabla u|^2\,dx\biggr)\int_\Omega \nabla u \nabla v\,dx - \int_\Omega a(x)|u|^{s-2}uv\,dx = \int_\Omega f(x,u)v\,dx$$

holds for any $v \in \mathbf{H}$. Let $J : \mathbf{H} \to R$ be the functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}M\biggl(\int_\Omega |\nabla u|^2\,dx\biggr) - \frac{1}{s}\int_\Omega a(x)|u|^s\,dx - \int_\Omega F(x,u)\,dx,$$

where

$$M(t) = \int_0^t m(s)\,ds, \qquad F(t) = \int_0^t f(x,s)\,ds.$$

It is easy to see that $J \in C^1(\mathbf{H}, R)$ and the critical points of J in $\mathbf{H}$ correspond to the weak solutions of problem (1.1).

We make the following assumptions.

(A) $a(x) \in \mathbf{C}(\overline{\Omega})$, $a(x) \geq 0$, $\forall x \in \overline{\Omega}$ and $\|a(x)\|_\infty = \bar{a} > 0$;

(F_0) $tf(x,t) \geq 0$ for $x \in \overline{\Omega}$, $t \in \mathbf{R}$;

(F_1) $\lim_{|t|\to 0}\frac{f(x,t)}{t} = p(x)$ uniformly a.e. $x \in \Omega$, where $0 < p(x) \in L^\infty(\Omega)$, and $\|p(x)\|_\infty < \theta\Lambda_1$;

(F_2) $\lim_{|t|\to +\infty}\frac{f(x,t)}{t} = l$ $(-\infty < l < +\infty)$ uniformly a.e. $x \in \Omega$.

Our first main result is concluded as the following theorem:

Theorem 3.1 *Assume the function $m(t)$ satisfies (M), $a(x)$ satisfies (A), and the nonlinearity $f(x,t)$ satisfies (F_1) and (F_2), then problem (1.1) has at least one solution if $l < \Lambda_1$.*

Proof It is easy to see, from condition (F_1), that $f(x,0) = 0$ for $x \in \Omega$. So $u = 0$ is the trivial solution of (1.1). From condition (F_2), we can take $\varepsilon = \frac{1}{2}(\Lambda_1 - l) > 0$, and there exists $T > 0$ such that

$$f(x,t)t \leq (l+\varepsilon)t^2$$

for all $|t| \geq T$ and a.e. $x \in \Omega$. By the continuity of F, there exists $C > 0$ such that

$$\bigl|F(x,t)\bigr| \leq \tfrac{l+\varepsilon}{2}t^2 + C$$

for all $(x,t) \in \Omega \times R$. On the other hand, from (M) it follows that

$$m_0 t \leq M(t) = \int_0^t m(s)\,ds \leq m_1 t, \quad \text{for } t > 0. \tag{3.1}$$

Then we have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}M\biggl(\int_\Omega |\nabla u|^2\,dx\biggr) - \frac{1}{s}\int_0^t a(x)|u|^s\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}m_0\int_\Omega |\nabla u|^2\,dx - \frac{1}{s}\bar{a}\int_\Omega |u|^s\,dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{l+\varepsilon}{2}\int_\Omega |u|^2\,dx - C|\Omega| \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{s}K_s\bar{a}\|u\|^s - \frac{l+\varepsilon}{2\Lambda_1}\|u\|^2 - C|\Omega| \\
&= \frac{\Lambda_1 - l - \varepsilon}{2\Lambda_1}\|u\|^2 - \frac{1}{s}K_s\bar{a}\|u\|^s - C|\Omega|,
\end{aligned}
$$

which shows that J is coercive. Moreover, conditions (F_1) and (F_2) imply that J is weakly lower semicontinuous in $\mathbf{H}$. Therefore we get a global minimum u_1 of J.

Next, we prove $u_1 \neq 0$, so it is a nontrivial solution of (1.1). From condition (F_1), there exists $C > 0$ such that

$$
\left|f(x,t)\right| \leq C|t|,
$$

for all $|t|$ small enough and $x \in \Omega$. It follows that

$$
\left|F(x,t)\right| \leq \frac{C}{2}t^2,
$$

for all $|t|$ small enough and $x \in \Omega$. From condition (A), we can chose $v \in \mathbf{H}$ such that

$$
\int_\Omega a(x)|v|^s\,dx > 0.
$$

Then we have

$$
\begin{aligned}
&\limsup_{t\to 0}\frac{J(tv)}{t^s} \\
&\quad= \limsup_{t\to 0}\frac{\frac{1}{2}\int_\Omega |\Delta(tv)|^2\,dx + \frac{1}{2}M(\int_\Omega |\nabla(tv)|^2\,dx) - \frac{1}{s}\int_\Omega a(x)|tv|^s\,dx - \int_\Omega F(x,tv)\,dx}{t^s} \\
&\quad\leq \limsup_{t\to 0}\frac{\frac{1}{2}\int_\Omega |\Delta(tv)|^2\,dx + \frac{1}{2}m_1(\int_\Omega |\nabla(tv)|^2\,dx) - \frac{1}{s}\int_\Omega a(x)|tv|^s\,dx - \int_\Omega F(x,tv)\,dx}{t^s} \\
&\quad\leq \limsup_{t\to 0}\left(\frac{t^{2-s}}{2}\|v\|^2 - \frac{1}{s}\int_\Omega a(x)|v|^s\,dx + \frac{Ct^{2-s}}{2}\int_\Omega v^2\,dx\right) \\
&\quad< 0.
\end{aligned}
$$

Therefore, we get that $J(u_1) < 0$. It is clear that $J(0) = 0$. Thus, u_1 is a nontrivial solution of (1.1). □

Our second result is the following theorem:

Theorem 3.2 *Assume the function $m(t)$ satisfies (M), $a(x)$ satisfies (A), and the nonlinearity $f(x,t)$ satisfies (F_0), (F_1), and (F_2), then there exists a positive constant a_0 such that problem (1.1) has at least three nontrivial solutions if $\bar{a} < a_0$ and $\bar{\Lambda}_1 < l < +\infty$.*

Before proving Theorem 3.2, we give two lemmas.

Lemma 3.1 *Suppose the conditions of Theorem 3.2 hold, then there exists a positive constant a_0 such that J satisfies the following conditions for $\bar{a} < a_0$ and $\bar{\Lambda}_1 < l < +\infty$:*

1. *There exist constants* $\rho > 0$, $\alpha > 0$ *such that* $J|_{\partial B_\rho} \geq \alpha$ *with* $B_\rho = \{u \in \mathbf{H} : \|u\| \leq \rho\}$;
2. $J(t\varphi_1) \to -\infty$ *as* $t \to +\infty$.

Proof (Claim 1) By (F_1) and (F_2), there exists $C > 0$ such that for all $(x,t) \in \Omega \times R$ and $p \in (1, \frac{N+4}{N-4})$, we have

$$F(x,t) \leq \frac{1}{4}\left(\left\|p(x)\right\|_\infty + \theta\Lambda_1\right)t^2 + C|t|^{p+1}.$$

From inequalities (2.2), (2.3) and (3.1), we have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}m\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{1}{s}\int_0^t a(x)|u|^s\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}m_0\int_\Omega |\nabla u|^2\,dx - \frac{1}{s}\bar{a}\int_\Omega |u|^s\,dx \\ &\quad - \frac{1}{4}\left(\left\|p(x)\right\|_\infty + \theta\Lambda_1\right)\|u\|_2^2 - C\|u\|_{p+1}^{p+1} \\ &\geq \frac{\theta}{2}\|u\|^2 - \frac{1}{s}\bar{a}K_s\|u\|^s - \frac{1}{4}\frac{(\|p(x)\|_\infty + \theta\Lambda_1)}{\Lambda_1}\|u\|^2 - CK_{p+1}\|u\|^{p+1} \\ &= \left(\frac{\theta\Lambda_1 - \|p(x)\|_\infty}{4\Lambda_1} - \frac{1}{s}\bar{a}K_s\|u\|^{s-2} - CK_{p+1}\|u\|^{p-1}\right)\|u\|^2. \end{aligned}$$

Setting

$$a_0 = \frac{s}{2K_sK_{p+1}^{\frac{2-s}{p-1}}}\left(\frac{\theta\Lambda_1 - \|p(x)\|_\infty}{8\Lambda_1}\right)^{\frac{p-s+1}{p-1}}, \qquad \rho = \left(\frac{\theta\Lambda_1 - \|p(x)\|_\infty}{8\Lambda_1 CK_{p+1}}\right)^{\frac{1}{p-1}},$$

when $\bar{a} \leq a_0$ and $\|u\| = \rho$, it follows that

$$J(u) \geq \left(\frac{\theta\Lambda_1 - \|p(x)\|_\infty}{16\Lambda_1}\right)\|\rho\|^2 = \alpha > 0.$$

So, Claim 1 is proved.

(Claim 2) By (F_2) and for $l > \bar{\Lambda}_1$, there exists $C > 0$ such that

$$F(x,t) \geq \frac{1}{4}(l + \bar{\Lambda}_1)t^2 - C$$

for all $(x,t) \in \Omega \times R$. Let λ_1 and ϕ_1 be the first eigenvalue and eigenfunction of $(-\Delta, H_0^1(\Omega))$ with $\int_\Omega |\phi_1|^2\,dx = 1$. We know that

$$\bar{\Lambda}_1 = \int_\Omega |\Delta\phi_1|^2\,dx + m_1\int_\Omega |\nabla\phi_1|^2\,dx = \lambda_1^2 + m_1\lambda_1.$$

Then, we have

$$\begin{aligned} J(t\phi_1) &= \frac{1}{2}\int_\Omega \left|\Delta(t\phi_1)\right|^2\,dx + \frac{1}{2}m\left(\int_\Omega \left|\nabla(t\phi_1)\right|^2\,dx\right) \\ &\quad - \frac{1}{s}\int_\Omega a(x)|t\phi_1|^s\,dx - \int_\Omega F(x,t\phi_1)\,dx \end{aligned}$$

$$\begin{aligned}
&\le \frac{t^2}{2}\int_\Omega |\Delta\phi_1|^2\,dx + \frac{t^2}{2}m_1\int_\Omega |\nabla\phi_1|^2\,dx \\
&\quad - \frac{t^s}{s}\int_\Omega a(x)|\phi_1|^s\,dx - \frac{t^2}{4}(l+\bar{\Lambda}_1)\int_\Omega |\phi_1|^2\,dx + C|\Omega| \\
&= \frac{t^2}{4}(\bar{\Lambda}_1 - l) - \frac{t^s}{s}\int_\Omega a(x)|\phi_1|^s\,dx + C|\Omega|.
\end{aligned}$$

Hence, $J(t\psi_1)\to -\infty$, $t\to +\infty$.

The proof of Lemma 3.1 is completed. □

Let

$$f^+(x,t) = \begin{cases} f(x,t), & t\ge 0,\\ 0, & t<0,\end{cases}$$

and

$$f^-(x,t) = \begin{cases} f(x,t), & t\le 0,\\ 0, & t>0.\end{cases}$$

Define functionals $J^\pm : \mathbf{H}\to\mathbf{R}$ as follows:

$$J^\pm(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}m\left(\int_\Omega |\nabla u|^2\,dx\right) - \frac{1}{s}\int_\Omega a(x)|u|^s\,dx - \int_\Omega F^\pm(x,u)\,dx,$$

where $F^\pm(t) = \int_0^t f^\pm(x,s)\,ds$.

Lemma 3.2 *Assume that (M), (A) and (F_0)–(F_2) hold, and $\bar{\Lambda}_1 < l < +\infty$, then $J^\pm(u)$ satisfies the (PS) condition.*

Proof We just prove that $J^+(u)$ satisfies the (PS) condition. The proof for $J^-(u)$ is similar. Let $\{u_n\}\in\mathbf{H}$ be a (PS) sequence, namely

$$J^+(u_n)\to c, \tag{3.2}$$

$$\nabla J^+(u_n)\to 0. \tag{3.3}$$

Firstly, we claim that $\{u_n\}$ is bounded in $\mathbf{H}$. If not, we may assume that $\|u_n\|\to +\infty$ as $n\to+\infty$. Let $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$. Passing to a subsequence, we may assume that there exists $w\in\mathbf{H}$ such that

$$\begin{cases} w_n \rightharpoonup w & \text{in } \mathbf{H},\\ w_n\to w & \text{in } \mathbf{L}^r(\Omega), 1\le r\le \frac{2N}{N-4},\\ w_n\to w & \text{a.e. in } \Omega.\end{cases} \tag{3.4}$$

By (F_1) and (F_2), we see that there exist C_1 and C_2 such that

$$\left|\frac{f(x,t)}{t}\right| \le C_1, \qquad \left|\frac{F(x,t)}{t^2}\right| \le C_2 \tag{3.5}$$

for all $(x,t) \in \Omega \times \mathbf{R}$ and define

$$\frac{f(x,t)}{t}\bigg|_{t=0} = \lim_{t\to 0}\frac{f(x,t)}{t}, \qquad \frac{F(x,t)}{t^2}\bigg|_{t=0} = \lim_{t\to 0}\frac{F(x,t)}{t^2}.$$

Then we claim that $w \neq 0$. Otherwise, if $w \equiv 0$, we know that $w_n \to 0$ strongly in $\mathbf{L}^r(\Omega)$. Dividing (3.2) by $\|u_n\|^2$, we have

$$\begin{aligned}\frac{J^+(u_n)}{\|u_n\|^2} &= \frac{1}{2\|u_n\|^2}\left(\int_\Omega |\Delta u_n|^2\,dx + m\left(\int_\Omega |\nabla u_n|^2\,dx\right)\right)\\ &\quad - \frac{1}{s\|u_n\|^{2-s}}\int_\Omega a(x)\big|w_n(x)\big|^s\,dx - \int_\Omega \frac{F^+(x,u_n)}{\|u_n\|^2}\,dx\\ &= o(1).\end{aligned}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}\frac{\theta}{2} &\le \frac{1}{2\|u_n\|^2}\left(\int_\Omega |\Delta u_n|^2\,dx + m_0\int_\Omega |\nabla u_n|^2\,dx\right)\\ &\le \frac{1}{2\|u_n\|^2}\left(\int_\Omega |\Delta u_n|^2\,dx + m\left(\int_\Omega |\nabla u_n|^2\,dx\right)\right)\\ &= \frac{1}{s\|u_n\|^{2-s}}\int_\Omega a(x)\big|w_n(x)\big|^s\,dx + \int_\Omega \frac{F^+(x,u_n)}{\|u_n\|^2}\,dx + o(1)\\ &\le \frac{\bar{a}}{s\|u_n\|^{2-s}}\int_\Omega \big|w_n(x)\big|^s\,dx + C_2\int_\Omega \big|w_n(x)\big|^2\,dx + o(1) \to 0,\end{aligned}$$

which is impossible, so $w \neq 0$.

Let us define

$$\Omega_0 = \big\{x \in \Omega \,|\, w(x) = 0\big\}, \qquad \Omega_1 = \big\{x \in \Omega \,|\, w(x) \neq 0\big\}.$$

Then, for all $v \in \mathbf{H}$, we have

$$\begin{aligned}\left|\int_{\Omega_0} \frac{f^+(x,u_n)}{u_n} w_n v\,dx\right| &\le C_1\int_{\Omega_0} |w_n||v|\,dx\\ &\le C_1\left(\int_{\Omega_0} |w_n|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega_0} |v|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

So,

$$\lim_{n\to+\infty}\int_{\Omega_0} \frac{f^+(x,u_n)}{u_n} w_n v\,dx = 0 = \int_{\Omega_0} lw^+ v\,dx, \tag{3.6}$$

where $w^+(x) = \max\{w(x), 0\}$. On the other hand, since $\|u_n\| \to +\infty$, we have $|u_n(x)| = \|u_n\||w_n(x)| \to +\infty$ for $x \in \Omega_1$. Therefore, by (F_2) and the dominated convergence theorem, we get

$$\lim_{n\to+\infty}\int_{\Omega_1} \frac{f^+(x,u_n)}{u_n} w_n v\,dx = \int_{\Omega_1} \lim_{n\to+\infty}\frac{f^+(x,u_n)}{u_n} w_n v\,dx = \int_{\Omega_1} lw^+ v\,dx. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\lim_{n\to+\infty}\int_\Omega \frac{f^+(x,u_n)}{u_n}w_n v\,dx = \int_\Omega lw^+ v\,dx. \tag{3.8}$$

Now, (3.3) implies that, for all $v \in \mathbf{H}$, we have

$$\begin{aligned}\big(\nabla J^+(u_n),v\big) &= \int_\Omega \Delta u_n \Delta v\,dx + m\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega \nabla u_n \nabla v\,dx \\ &\quad - \int_\Omega a(x)\big|u_n(x)\big|^{s-1} v\,dx - \int_\Omega f^+(x,u_n)v\,dx \to 0.\end{aligned}$$

Dividing by $\|u_n\|$, we get

$$\begin{aligned}&\int_\Omega \Delta w_n \Delta v\,dx + m\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega \nabla w_n \nabla v\,dx \\ &\quad - \frac{1}{\|u_n\|^{2-s}}\int_\Omega a(x)\big|w_n(x)\big|^{s-1} v\,dx - \int_\Omega \frac{f^+(x,u_n)}{u_n} w_n v\,dx \to 0.\end{aligned} \tag{3.9}$$

Since

$$\|u_n\|^2 = \int_\Omega |\Delta u_n|^2\,dx + m_1\int_\Omega |\nabla u_n|^2\,dx \to +\infty$$

as $n \to +\infty$, we can suppose that there exists a subsequence, still denoted $\{\int_\Omega |\nabla u_n|^2\,dx\}$, such that

$$\int_\Omega |\nabla u_n|^2\,dx \to +\infty, \quad n \to +\infty, \tag{3.10}$$

otherwise, there exists $K > 0$ such that

$$\int_\Omega |\nabla u_n|^2\,dx \le K,$$

and furthermore, there exist a subsequence, still denoted $\{\int_\Omega |\nabla u_n|^2\,dx\}$, and a constant $t' \ge 0$ such that

$$\int_\Omega |\nabla u_n|^2\,dx \to t', \quad n \to +\infty. \tag{3.11}$$

In case (3.10) holds, by (M), we have

$$\lim_{n\to+\infty} m\left(\int_\Omega |\nabla u_n|^2\,dx\right) = m_1. \tag{3.12}$$

Combining (3.4), (3.8), (3.9) and (3.10), as $n \to +\infty$, we obtain

$$\int_\Omega \Delta w \Delta v\,dx + m_1\int_\Omega \nabla w \nabla v\,dx = \int_\Omega lw^+ v\,dx, \quad \forall v \in \mathbf{H}. \tag{3.13}$$

Taking $v = \phi_1$ in (3.13), we have

$$\int_\Omega \Delta w \Delta \phi_1 \, dx + m_1 \int_\Omega \nabla w \nabla \phi_1 \, dx = \int_\Omega l w^+ \phi_1 \, dx. \tag{3.14}$$

Noticing that ϕ_1 is the positive solution of

$$\begin{cases} \Delta^2 u + m_1 \Delta u = \bar{\Lambda}_1 u, & \text{in } \Omega, \\ u = 0, \qquad \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

we have

$$\int_\Omega \Delta w \Delta \phi_1 \, dx + m_1 \int_\Omega \nabla w \nabla \phi_1 \, dx = \int_\Omega \bar{\Lambda}_1 w \phi_1 \, dx. \tag{3.15}$$

Thus, from (3.14) and (3.15), we get

$$\int_\Omega l w^+ \phi_1 \, dx = \int_\Omega \bar{\Lambda}_1 w \phi_1 \, dx. \tag{3.16}$$

If $w(x) \geq 0$ a.e. in Ω, since $w(x) \not\equiv 0$, we have $\int_\Omega w\phi_1 \, dx > 0$. Then (3.15) implies that

$$\int_\Omega l w \phi_1 \, dx = \int_\Omega l w^+ \phi_1 \, dx = \int_\Omega \bar{\Lambda}_1 w \phi_1 \, dx,$$

which contradicts $l > \bar{\Lambda}_1$. Otherwise, let $\Omega_- = \{x \in \Omega | w(x) < 0\}$ and suppose $|\Omega_-| > 0$. Then $\int_{\Omega_-} -w\phi_1 \, dx > 0$ and $\int_\Omega w^+ \phi_1 \, dx > \int_\Omega w\phi_1 \, dx > 0$. It follows from (3.15) again that

$$\int_\Omega l w^+ \phi_1 \, dx = \int_\Omega \bar{\Lambda}_1 w \phi_1 \, dx < \int_\Omega \bar{\Lambda}_1 w^+ \phi_1 \, dx,$$

which contradicts $l > \bar{\Lambda}_1$.

So $\{u_n\}$ is bounded in $\mathbf{X}$.

In case (3.11) holds, by (M), we have

$$\lim_{n\to+\infty} m\left(\int_\Omega |\nabla u_n|^2 \, dx\right) = m(t') = m' \leq m_1. \tag{3.17}$$

Combining (3.4), (3.8), (3.9) and (3.17), as $n \to +\infty$, we obtain

$$\int_\Omega \Delta w \Delta v \, dx + m' \int_\Omega \nabla w \nabla v \, dx = \int_\Omega l w^+ v \, dx, \quad \forall v \in \mathbf{H}. \tag{3.18}$$

Taking $v = \phi_1$ in (3.18), we have

$$\int_\Omega \Delta w \Delta \phi_1 \, dx + m' \int_\Omega \nabla w \nabla \phi_1 \, dx = \int_\Omega l w^+ \phi_1 \, dx. \tag{3.19}$$

Notice that ϕ_1 is also the positive solution of

$$\begin{cases} \Delta^2 u + m' \Delta u = \Lambda'_1 u, & \text{in } \Omega, \\ u = 0, \qquad \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Lambda_1' = \lambda_1^2 + m'\lambda_1$. Then we have

$$\int_\Omega \Delta w \Delta \phi_1 \, dx + m' \int_\Omega \nabla w \nabla \phi_1 \, dx = \int_\Omega \Lambda_1' w \phi_1 \, dx. \tag{3.20}$$

From (3.19) and (3.20), we get

$$\int_\Omega l w^+ \phi_1 \, dx = \int_\Omega \Lambda_1' w \phi_1 \, dx. \tag{3.21}$$

Notice that for $\Lambda_1' \leq \bar{\Lambda}_1$, similar to the discussions in case (3.10) holds, (3.21) implies a contradiction to $l > \bar{\Lambda}_1$.

So $\{u_n\}$ is bounded in $\mathbf{X}$.

Now, since Ω is bounded and (F_1), (F_2) hold, by using the Sobolev embedding theorem and the standard procedures, we can easily prove that $\{u_n\}$ has a convergent subsequence. The proof of the lemma is completed. □

Proof of Theorem 3.2. From the proof of Lemma 3.1, it is easy to see that $J^+(u)$ and $J^-(u)$ satisfy the conditions of Lemma 3.1. So there exist $\rho > 0$, $\alpha > 0$, and $e \in \mathbf{H}$ with $\|e\| > \rho$ such that

$$J^\pm(u)\big|_{\partial B_\rho} \geq \alpha > 0, \qquad J^\pm(e) < 0.$$

It is clear that $J^\pm(0) = 0$. Moreover, by Lemma 3.2, the functionals $J^\pm$ satisfy the (PS) condition. By Lemma 2.2, we know that $J^\pm$ has the critical value $c^\pm$, respectively, which can be characterized as

$$c^\pm = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J^\pm(u),$$

where $\Gamma = \{\gamma \in C([0,1], \mathbf{H}) | \gamma(0) = 0, \gamma(1) = e\}$. So there exist critical points $u_1, u_2 \in \mathbf{H}$ such that

$$J^+(u_1) = c^+ > 0, \qquad J^-(u_2) = c^- > 0.$$

Since $f^+(x,t) \geq 0$ and $f^-(x,t) \leq 0$, by the comparison principles for some fourth order elliptic problems [20], u_1 is a positive solution of (1.1) and u_2 is a negative solution of (1.1).

Next, we prove that problem (1.1) has another solution $u_3 \in \mathbf{H}$ such that $J(u_3) < 0$. For $\rho > 0$ given by Lemma 3.1, define $B_\rho = \{u \in E : \|u\| \leq \rho\}$ and then B_ρ is a complete metric space with the distance $\operatorname{dist}(u,v) = \|u - v\|$ for $u, v \in B_\rho$. By Lemma 3.1, we know that

$$J(u)|_{\partial B_\rho} \geq \alpha > 0. \tag{3.22}$$

Clearly, $J \in C^1(B_\rho, R)$, so J is bounded from below on B_ρ. And we know that J is lower semicontinuous.

Similar to the proof of Theorem 3.1, there exists $v \in \mathbf{H}$ such that

$$\lim_{t \to 0} \frac{J(tv)}{t^p} < 0.$$

Then letting $c_1 = \inf\{J(u) : u \in B_\rho\}$, we get that $c_1 < 0$. By Lemma 2.3, for any $k > 0$, there is a $\{u_k\}$ such that

$$c_1 \leq J(u_k) \leq c_1 + \frac{1}{k}.$$

Now we claim that $\|u_k\| < \rho$ for k large enough. Otherwise, if $\|u_k\| = \rho$ for infinitely many k, and, without loss of generality, we may suppose that $\|u_k\| = \rho$ for all $k > 1$. It follows from (3.22) that $J(u_k) \geq \alpha > 0$. Letting $k \to \infty$, we see that $0 > c_1 \geq \alpha > 0$, which is a contradiction.

For any $u \in E$ with $\|u\| = 1$, let

$$w_k = u_k + tu$$

for any fixed $k \geq 1$. We get

$$\|w_k\| \leq \|u_k\| + t,$$

so $w_k \in B_\rho$ for $t > 0$ small enough. It follows from Lemma 2.3 that

$$J(w_k) = J(u_k + tu) \geq J(u_k) - \frac{t}{k}\|u\|.$$

Thus, we have

$$J'(u_k) = \lim_{t\to 0^+} \frac{J(u_k + tu) - J(u_k)}{t} \geq -\frac{1}{k}$$

and

$$J'(u_k) = \lim_{t\to 0^+} \frac{J(u_k - tu) - J(u_k)}{t} \leq \frac{1}{k}.$$

Then $|J'(u_k)| \leq \frac{1}{k} \to 0$ and $J(u_k) \to c_1$ as $k \to \infty$. Therefore $\{u_k\}$ is a (PS) sequence at level c_1. From Lemma 3.2, $\{u_k\}$ has a convergent subsequence. Hence, we see that there exists $u_3 \in \mathbf{H}$ such that $J'(u_3) = 0$ and $J(u_3) = c_1 < 0$. Thus, u_3 is a nontrivial weak solution of (1.1) and $u_3 \neq u_1$, $u_3 \neq u_2$. □

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