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Well-posedness of the solution of the fractional semilinear pseudo-parabolic equation

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Abstract

This article concerns the Cauchy problem for the fractional semilinear pseudo-parabolic equation. Through the Green's function method, we prove the pointwise convergence rate of the solution. Furthermore, using this precise pointwise structure, we introduce a Sobolev space condition with negative index on the initial data and give the nonlinear critical index for blowing up.

Keywords: Fractional semilinear pseudo-parabolic equation; Green's function method; Pointwise; Negative index

1 Introduction

We consider the following Cauchy problem for the fractional semilinear pseudo-parabolic equation:

$$\begin{cases} u_t + k(-\Delta)^a u_t + (-\Delta)^a u = u^p, & x \in \mathbb{R}^n, t > 0, \\ u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

where $p > 0$, $k > 0$, $u_0(x)$ is sufficiently smooth and nonnegative. If $k = 0$, (1.1) is the classical heat equation, see [1]. If $k > 0$, (1.1) is a pseudo-parabolic equation, see [2].

The pseudo-parabolic equation is used in diverse fields such as seepage theory of homogeneous liquid through cracked rock [3] (the coefficient of the third-order term represents the degree of cracks in the rock, and its decrease corresponds to the increase in the degree of cracking), the unidirectional propagation of nonlinear dispersive long waves [4, 5] (where u is amplitude or curl), and the description of racial migration [6] (where u is the population density). Because of the wide range of applications of pseudo-parabolic equations, they attract great attention of mathematicians.

Ting, Showalter, and Gopala Rao proved the existence and uniqueness of the solution on the initial boundary value problem and the Cauchy problem of linear pseudo-parabolic equations, see [2, 7, 8]. Since then, many scholars have paid great attention to the study of nonlinear pseudo-parabolic equations, including about existence, asymptotic behavior, decay of regularity and solutions, etc., see [9–12]. Recently, Yang Cao et al. proved

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the existence and blowing up of the solution of equation (1.1) at $a = 1$, but the pointwise estimation of the solution was not discussed, see [13]. Later, Wang Wei et al. used the Green's function to improve [13] and p only needs to satisfy $p > 1 + \frac{2}{n+s}$ instead of $p > 1 + \frac{2}{n}$. More specifically, they proved the pointwise estimation of the solution of equation (1.1) at $a = 1$, also obtained the nonlinear critical index of the blowing up at $p > 1 + \frac{2}{n+s}$ by limiting the initial condition, see [14].

In the above research, they focused on integer order equations. The fractional dissipation operator $(-\Delta)^a$ can be regarded as the infinitesimal generators of Levy stable diffusion process. Compared with the integral differential equation, it can describe some physical phenomena more accurately, see [15–17]. Therefore, more and more scientists are devoted to the research of fractional differential equations, see [16–18].

Motivated by the above works, we study the pointwise estimate and exponential decay of the solution for problem (1.1) in the fractional order case. At present, there is little research on the pointwise estimate and exponential decay of the solution of this fractional equation, and the main difficulty stems from its fractional dissipation operator term. The structure of this article is organized as follows: In Sect. 2, we recall some preliminary results and show the main results of this paper. In Sect. 3, by Green's function method, we use the Green's function to express the solution of fractional equation (1.1) and get the pointwise estimate result of the Green's function. In Sect. 4, we obtain the pointwise estimate of fractional equation (1.1) with appropriate conditions p, u_0 . In Sect. 5, we prove that the exponential decay of equation (1.1) still exists without $a = 1$.

2 Preliminaries and main results

Let C represent a generic positive constant, which may change from line to line. The norm of $L^p(\Omega)$ is written as $\|\cdot\|_{L^p(\Omega)}$ ($1 \leq p \leq \infty$). The notation X is a Banach space with a norm $\|\cdot\|_X$.

Definition 2.1 Suppose $f(x, t) \in L^1(\mathbb{R}^n)$. Then the Fourier transform is as follows:

$$\widehat{f}(\xi, t) = \int_{\mathbb{R}^n} f(x, t) e^{-\sqrt{-1}x \cdot \xi} dx, \quad (2.1)$$

its inverse Fourier transform is

$$(\mathcal{F}^{-1}\widehat{f})(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi, t) e^{\sqrt{-1}x \cdot \xi} d\xi. \quad (2.2)$$

According to [19], we have the following two lemmas.

Lemma 2.1 If $\widehat{f}(\xi, t)$ has a compact support for ξ satisfying

$$|D_\xi^\beta (\xi^\alpha \widehat{f}(\xi, t))| \leq C(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{(|\alpha|+k)} t^{\frac{|\beta|}{2}})(1 + t|\xi|^2)^m e^{-b|\xi|^2 t}, \quad (2.3)$$

where $b > 0$, α, β are any multi-indexes and $|\beta| \leq 2N$, then

$$|D_x^\alpha f(x, t)| \leq C_N t^{-n+|\alpha|+\frac{k}{2}} B_N(|x|, t), \quad (2.4)$$

where k and m are any positive integers, $(a)_+ = \max(0, a)$, and

$$B_N(|x|, t) = \left(1 + \frac{|x|^2}{1+t}\right)^{-N}.$$

Lemma 2.2 Let $\text{supp} f(\xi) \subset O_R =: \{\xi, |\xi| > R\}$ and

$$|D_\xi^\beta \widehat{f}(\xi)| \leq C|\xi|^{-1-|\beta|}, \quad (2.5)$$

then there exist distributions $f_1(x)$ and $f_2(x)$ satisfying

$$f(x) = f_1(x) + f_2(x) + C_0 \delta(x), \quad (2.6)$$

where C_0 is a constant and $\delta(x)$ is the Dirac function. Furthermore, choosing ε_0 small enough, we have the estimate

$$\begin{aligned} |D_x^\alpha f_1(x)| &\leq C(1 + |x|^2)^{-N}, \\ \|f_2\|_{L^1} &\leq C, \quad \text{supp} f_2(x) \subset \{x; |x| < 2\varepsilon_0\}, \end{aligned}$$

for positive integer $2N > n + |\alpha|$.

We make the following assumptions:

(H1) $u_0 \in C^{\alpha+2}(R^n)$ for sufficiently small $u_0 > 0$;

(H2) $u_0 \in W^{-s,2}(R^n) \cap W^{-s,\infty}(R^n) \cap L^\infty(R^n) \cap L^2(R^n)$, $0 \leq s < n$, for sufficiently small $u_0 > 0$.

Based on the above assumptions, we draw the following conclusions.

Theorem 1 Let $p > p_c = 1 + \frac{2a}{n}$, (H1) be satisfied. Then Cauchy problem (1.1) has the point-wise estimate of the solution u , satisfying

$$\begin{aligned} |u| &\leq 2C(1+t)^{-\frac{n}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}, \\ |D_x^\alpha u| &\leq 2C(1+t)^{-\frac{n}{2a} - \frac{|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}, \end{aligned} \quad (2.7)$$

where $N > 0$ is an arbitrary constant, C depends on the initial value u_0 and the parameter p .

Theorem 2 Suppose $p > p_s = 1 + \frac{2a}{n+s}$, (H2) hold. Then problem (1.1) has solution u satisfying

$$\|u\|_{L^q(R^n)} \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q}) - \frac{s}{2a}}, \quad q \in [2, +\infty), \quad (2.8)$$

where C depends on the initial value data and p .

3 Pointwise estimate of the Green's function

In this section, we will consider the pointwise estimation of the solution to linear form of problem (1.1). We study the Green's function of Cauchy problem (1.1) and obtain the following:

$$\begin{cases} \partial_t G + k(-\Delta)^a G_t + (-\Delta)^a G = 0, & x \in \mathbb{R}^n, t > 0, \\ G|_{t=0} = \delta(x), \end{cases} \quad (3.1)$$

where $\delta(x) = \delta(x_1) \otimes \delta(x_2) \otimes \cdots \otimes \delta(x_n)$ is the Dirac function and \otimes represents the tensor product. Considering the Fourier transform of equation (3.1) with respect to x , we get

$$\begin{cases} \partial_t \widehat{G} + k|\xi|^{2a} \widehat{G}_t + |\xi|^{2a} \widehat{G} = 0, & \xi \in \mathbb{R}^n, \\ \widehat{G}|_{t=0} = 1. \end{cases} \quad (3.2)$$

By solving the above equation directly, we know that

$$\widehat{G}(\xi, t) = e^{\mu(\xi)t}, \quad (3.3)$$

where $\mu(\xi) = -\frac{|\xi|^{2a}}{1+k|\xi|^{2a}}$. Now we use frequency decomposition to obtain an estimate of the Green's function G . Let

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| \leq \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases} \quad \in C^\infty, \quad (3.4)$$

$$\chi_3(\xi) = \begin{cases} 1, & |\xi| \geq R, \\ 0, & |\xi| < R-1, \end{cases} \quad \in C^\infty, \quad (3.5)$$

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi), \quad \in C^\infty, \quad (3.6)$$

where $\chi_1(\xi)$ and $\chi_3(\xi)$ are the smooth cut-off functions, ε, R are any positive constants satisfying $2\varepsilon < R-1$. Define $\widehat{G}_i(t, \xi) = \chi_i \widehat{G}(t, \xi)$, $i = 1, 2, 3$. From the literature [20], we know that the attenuation of the solution of the linear problem is mainly related to the low frequency part of $\widehat{G}(t, \xi)$. We use cut-off functions to divide the solution into three parts: low frequency, intermediate frequency, and high frequency.

Proposition 3.1 *Let ε be a sufficiently small constant. Then there exists a constant $C > 0$ satisfying*

$$|D_x^\alpha G_1(x, t)| \leq C_N (1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t). \quad (3.7)$$

Proof In the case of low frequency, let $0 < |\xi| < 2R$, then \widehat{G} has compact support. Taking into account (3.3) and Lemma 2.1, there is

$$|D_\xi^\beta \xi^\alpha \chi_1(\xi) \widehat{G}(\xi, t)| \leq C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{\frac{|\beta|}{2a}}) (1+t|\xi|^{2a})^\beta e^{-b|\xi|^{2a}t} \quad (3.8)$$

for $\forall |\beta| \leq 2N$. Then is (3.7) established. \square

Actually, we can discuss obtaining (3.7) in two cases.

(1) If $|\beta| \leq |\alpha|$, we find

$$\begin{aligned} |x^\beta D_x^\alpha \chi_1(D)G| &= C \left| \int e^{ix\xi} D_\xi^\beta (\xi^\alpha \chi_1(\xi) \widehat{G}(\xi, t)) d\xi \right| \\ &\leq C \int (|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{\frac{|\beta|}{2a}}) (1+t|\xi|^{2a})^\beta |e^{ix\xi}| e^{-b|\xi|^{2a}t} d\xi \\ &\leq C \int |\xi|^{|\alpha|} (|\xi|^{-|\beta|_+} + t^{\frac{|\beta|}{2a}}) (1+t|\xi|^{2a})^\beta e^{-b|\xi|^{2a}t} d\xi \\ &\leq Ct^{-\frac{n+|\alpha|-|\beta|}{2a}}. \end{aligned} \quad (3.9)$$

(2) If $|\beta| > |\alpha|$, we obtain

$$\begin{aligned} |x^\beta D_x^\alpha \chi_1(D)G| &\leq C(1+t^{-\frac{|\alpha|-|\beta|}{2a}}) t^{-\frac{n}{2a}} \\ &\leq C(1+t)^{-\frac{|\alpha|-|\beta|}{2a}} t^{-\frac{n}{2a}} \\ &\leq C(1+t)^{\frac{|\beta|}{2a}} t^{-\frac{n+|\alpha|}{2a}}. \end{aligned} \quad (3.10)$$

On the other hand, if $|x|^{2a} \leq 1+t$, let $|\beta| = 0$, we have

$$|D_x^\alpha \chi_1(D)G| \leq Ct^{-\frac{n+|\alpha|}{2a}}. \quad (3.11)$$

If $|x|^{2a} > 1+t$, let $|\beta| = 2aN$, we see that

$$|D_x^\alpha \chi_1(D)G| \leq Ct^{-\frac{n+|\alpha|}{2a}} \min\left(1, \frac{(1+t)^N}{|x|^{2aN}}\right). \quad (3.12)$$

Because

$$1 + \frac{|x|^{2a}}{1+t} \leq \begin{cases} 2, & |x|^{2a} \leq 1+t, \\ 2\frac{|x|^{2a}}{1+t}, & |x|^{2a} > 1+t, \end{cases} \quad (3.13)$$

we know that

$$\min\left(1, \frac{(1+t)^N}{|x|^{2aN}}\right) \leq 2^N B_N(|x|^a, t). \quad (3.14)$$

Then, with the help of (3.11)–(3.12) and (3.14), we infer (3.7).

Next, we estimate $G_2(x, t)$.

Proposition 3.2 *Suppose that ε and R are fixed constants. Then*

$$|D_x^\alpha G_2(x, t)| \leq Ce^{-\frac{t}{2m_0}} B_N(|x|^a, t), \quad (3.15)$$

where m_0 is a positive constant.

Proof Choosing m sufficiently large and $m > \frac{1}{2}(\frac{1}{|\varepsilon|^{2a}} + k)$. If $\varepsilon \leq |\xi| \leq R$, it is easy to see that $\mu(\xi) \leq -\frac{1}{2m}$. This analysis reflects that

$$|\widehat{G_2}| = |\chi_2(\xi)\widehat{G}| \leq Ce^{-\frac{t}{2m}}. \quad (3.16)$$

From (3.16), there holds

$$\begin{aligned} |D_x^\alpha G_2(x, t)| &\leq C \left| \int_{\varepsilon \leq \xi \leq R} e^{ix\xi} (\xi^\alpha \widehat{G_2}(\xi, t)) d\xi \right| \\ &\leq Ce^{-\frac{t}{2m}} \int_{\varepsilon \leq \xi \leq R} |\xi|^\alpha d\xi \\ &\leq Ce^{-\frac{t}{2m}}. \end{aligned} \quad (3.17)$$

Now, we apply mathematical induction to prove the following inequality:

$$|D_\xi^\beta \widehat{G_2}(\xi, t)| \leq C(1+t)^{|\beta|} e^{-\frac{t}{2m}}. \quad (3.18)$$

Obviously, the above formula holds when $|\beta| = 0$. Suppose that $|\beta| \leq l-1$, the above formula still holds. Then we will prove that the formula of (3.18) also holds for $|\beta| \leq l$. Taking the Fourier transform of (3.1) with respect to x and multiplying it with $\chi_2(\xi)$, we get

$$\begin{cases} \partial_t \widehat{G_2}(\xi, t) - \mu(\xi) \widehat{G_2}(\xi, t) = 0, \\ \widehat{G_2}(\xi, 0) = \chi_2(\xi). \end{cases} \quad (3.19)$$

Using D_ξ^β to equation (3.19), we can obtain that

$$\begin{cases} \partial_t D_\xi^\beta \widehat{G_2}(\xi, t) - \mu(\xi) D_\xi^\beta \widehat{G_2}(\xi, t) = F(\xi, t), \\ \widehat{G}(\xi, 0) = a_0, \end{cases} \quad (3.20)$$

where $F(\xi, t) = \sum_{\beta_1 + \beta_2 = \beta, |\beta_1| \neq 0} \frac{\beta!}{\beta_1! \beta_2!} (D_\xi^{\beta_1} \mu(\xi) D_\xi^{\beta_2} \widehat{G_2}(\xi, t))$ and a_0 is a polynomial of $|\xi|$. Considering $|\beta| = l$, we see that

$$D_\xi^\beta \widehat{G_2}(\xi, t) = a_0 \widehat{G_2}(\xi, t) + \int_0^t \widehat{G_2}(\xi, t-s) F(\xi) ds. \quad (3.21)$$

By induction we have

$$\begin{aligned} |D_\xi^\beta \widehat{G_2}(\xi, t)| &\leq Ce^{-\frac{t}{2m}} + C \int_0^t e^{-\frac{t-s}{2m}} (1+t)^{|\beta|-1} e^{-\frac{s}{2m}} ds \\ &\leq C(1+t)^{|\beta|} e^{-\frac{t}{2m}}. \end{aligned} \quad (3.22)$$

Therefore, for each $1 \leq |\beta| \leq l$, we find

$$\begin{aligned} |x^\beta D_x^\alpha G_2(x, t)| &\leq C \left| \int_{R^n} e^{ix\xi} D_\xi^\beta (\xi^\alpha \widehat{G_2}(\xi, t)) \, d\xi \right| \\ &\leq C e^{-\frac{t}{2m}} (1+t)^{|\beta|} \int_{\varepsilon \leq |\beta| \leq R} |\xi|^{|\alpha|} + |\xi|^{||\alpha|-|\beta||} \, d\xi \\ &\leq C e^{-\frac{t}{2m}} (1+t)^{|\beta|} \\ &\leq C e^{-\frac{t}{2m_0}} (1+t)^{\frac{|\beta|}{2a}}, \end{aligned} \quad (3.23)$$

where $m_0 \in (0, m)$. Taking into account (3.17) and (3.23) (let $|\beta| = 2N$), we deduce

$$|D_x^\alpha G_2(x, t)| \leq C e^{-\frac{t}{2m_0}} \min \left(1, \frac{(1+t)^{2N}}{|x|^{2N}} \right). \quad (3.24)$$

Considering (3.14) and (3.24), we obtain the desired result. \square

Now considering the high frequency part $G_3(x, t)$.

Proposition 3.3 *Let R be a sufficiently small constant. Then*

$$|D_x^\alpha (G_3(x, t) - F_l)| \leq C(1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t), \quad (3.25)$$

where b is a positive constant and

$$F_l = \chi_3(D) \left[e^{-\frac{t}{k}} \left(\delta(x) + \sum_{j=1}^l p_j(t) (-\Delta)^{-aj} \right) \right], \quad l = \frac{|\alpha| + n}{2a}, \quad (3.26)$$

is the distribution.

Proof Denote

$$\rho = \frac{1}{|\xi|^{2a}}, \quad h(\rho) = e^{-\frac{t}{|\rho|+k}}. \quad (3.27)$$

Expanded in Taylor's series at $|\rho| \rightarrow 0$, we infer

$$h(\rho) = e^{-\frac{t}{k}} \left(1 + \frac{t\rho}{k^2} + \frac{t^2\rho^2}{2k^4} - \frac{t\rho^2}{2k^3} + \cdots \right). \quad (3.28)$$

Then

$$e^{-\frac{t|\xi|^{2a}}{1+k|\xi|^{2a}}} = e^{-\frac{t}{k}} \left(1 + \sum_{j=1}^l p_j(t) (|\xi|^{2a})^{-j} \right) \chi_3(\xi) + R(\xi, t), \quad (3.29)$$

where $p_j(t)$ is a polynomial of degree j .

Let

$$F_l = \chi_3(D) \left[e^{-\frac{t}{k}} \left(\delta(x) + \sum_{j=1}^l p_j(t) (-\Delta)^{-aj} \right) \right], \quad l = \frac{|\alpha| + n}{2a}. \quad (3.30)$$

With the help of Lemma 2.2 and choosing R big enough, it is easy to see that

$$|D_x^\alpha (G_3(x, t) - F_l)| \leq C(1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t). \quad (3.31)$$

□

In conclusion, we use the following lemma to explain the estimate of the regular part of G .

Lemma 3.1 *Let G be the solution of the linear form of Cauchy problem (1.1). Then*

$$|D_x^\alpha (G(x, t) - F_l)| \leq C(1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t), \quad (3.32)$$

where F_l is the distribution and

$$F_l = \chi_3(D) \left[e^{-\frac{t}{k}} \left(\delta(x) + \sum_{j=1}^l p_j(t) (-\Delta)^{-aj} \right) \right], \quad l = \frac{|\alpha| + n}{2a}. \quad (3.33)$$

Proof Note that

$$\begin{aligned} |D_x^\alpha (G(x, t) - F_l)(x, t)| &\leq |D_x^\alpha G_1(x, t)| + |D_x^\alpha G_2(x, t)| \\ &\quad + |D_x^\alpha (G_3(x, t) - F_l)(x, t)|. \end{aligned} \quad (3.34)$$

Considering Proposition 3.1, Proposition 3.2, Proposition 3.3, and (3.34), we have (3.32). Then Lemma 3.1 is proved. □

Lemma 3.2 *Assume that $t > 0$, $m > \frac{n}{2a}$, then*

$$\int_{R^n} \left(1 + \frac{|x|^{2a}}{1+t} \right)^{-m} dx \leq C(1+t)^{\frac{n}{2a}}. \quad (3.35)$$

Proof Let $w = |x|$, we infer

$$\begin{aligned} \int_{R^n} \left(1 + \frac{|x|^{2a}}{1+t} \right)^{-m} dx &\leq C \int_0^\infty \left(1 + \frac{w^{2a}}{1+t} \right)^{-m} w^{n-1} dw \\ &\leq C(1+t)^{\frac{n}{2a}} \int_0^\infty \left(1 + \frac{w}{(1+t)^{\frac{1}{2a}}} \right)^{-2am+n-1} d \frac{w}{(1+t)^{\frac{1}{2a}}} \\ &\leq C(1+t)^{\frac{n}{2a}}. \end{aligned} \quad (3.36)$$

□

Using Hausdorff–Young’s inequality, the following lemma of Green’s function is easily obtained.

Lemma 3.3 *If $p \in [1, \infty]$, then*

$$\|D_x^\alpha (G - F_l)(t, \cdot)\|_{L^p(R^n)} \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{p})-\frac{|\alpha|}{2a}} \quad (3.37)$$

for any multi-indexes α .

Proof If $p \in [1, \infty)$, it follows from (3.25) that

$$\begin{aligned} \|D_x^\alpha(G - F_l)(t, \cdot)\|_{L^p(\mathbb{R}^n)} &= \left| \int_{\mathbb{R}^n} |D_x^\alpha(G - F_l)(t, \cdot)|^p dx \right|^{\frac{1}{p}} \\ &\leq C(1+t)^{-\frac{n+|\alpha|}{2a}} \left(\int_{\mathbb{R}^n} |B_N(|x|^a, t)|^p dx \right)^{\frac{1}{p}} \\ &\leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{p})-\frac{|\alpha|}{2a}}. \end{aligned} \quad (3.38)$$

If $p = \infty$, by (3.25), we conclude that

$$\begin{aligned} \|D_x^\alpha(G - F_l)(t, \cdot)\|_{L^p(\mathbb{R}^n)} &= \text{ess sup} |D_x^\alpha(G - F_l)(t, \cdot)| \\ &\leq \text{ess sup} \{C(1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t)\} \\ &\leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{p})-\frac{|\alpha|}{2a}}. \end{aligned} \quad (3.39)$$

□

4 Pointwise estimation of the solution

In this section, we get the pointwise estimate of the solution under appropriate conditions of u_0, p .

Lemma 4.1 Assume that $|y| \leq M, t \geq 4M^2, N > 0$, then

$$\left(1 + \frac{|y-x|^{2a}}{1+t}\right)^{-N} \leq C_N \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}. \quad (4.1)$$

Proof (1) If $|x|^{2a} \leq 1+t$, then

$$\frac{|x|^{2a}}{1+t} \leq 1 \quad (4.2)$$

and

$$\left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N} \geq 2^{-N}. \quad (4.3)$$

On the other hand,

$$\left(1 + \frac{|x-y|^{2a}}{1+t}\right)^{-N} \leq 1. \quad (4.4)$$

It follows from (4.3) and (4.4) that

$$\left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N} \leq 2^N \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}. \quad (4.5)$$

(2) If $|x|^{2a} > 1+t$, then

$$|x-y| \geq |x| - |y| \geq \sqrt{1-t} - |y| \geq 0 \quad (4.6)$$

and

$$\begin{aligned} |x-y|^{2a} &\geq (|x|-|y|)^{2a} = (|x|^2 - 2|x||y| + |y|^2)^a \\ &= \left(\frac{|x|^2}{2} + \frac{(|x|-2|y|)^2}{2} - |y|^2 \right)^a \geq \left(\frac{|x|^2}{4} + \left(\frac{t}{4} - M^2 \right) \right)^a. \end{aligned} \quad (4.7)$$

From $\frac{t}{4} - M^2 \geq 0$, the proof is obtained. \square

Proof of Theorem 1 With the help of Lemma 3.3 and considering (1.1), from Green's function, we have

$$\begin{aligned} u &= \widehat{T}u = G(t) * u_0 + \int_0^t H(t-s, \cdot) * F(u(s, \cdot)) \, ds \\ &= G(t) * u_0 + \int_0^t \int_{R^n} H(x-y, t-s) * F(u(y, s)) \, dy \, ds, \end{aligned} \quad (4.8)$$

where the symbol $*$ represents convolution, $F(u) = u^p$, and H satisfies

$$\widehat{H} = \frac{1}{1 + k|\xi|^{2a}} \widehat{G}. \quad (4.9)$$

Applying the inverse Fourier transform, we deduce

$$H = K_k * G, \quad K_k(x) = (4\pi)^{-\frac{n}{2a}} \int_0^\infty e^{-\gamma_1 s - \frac{|x|^{2a}}{4s}} s^{-\frac{n}{2a}} \, ds. \quad (4.10)$$

Obviously, the estimated value on \widehat{G} is also correct on \widehat{H} .

By using D_x^α to equation (4.8), we get

$$\begin{aligned} D_x^\alpha u(x, t) &= D_x^\alpha G * u_0 + \int_0^t \int_{R^n} H(x-y, t-s) D_x^\alpha F(u(y, s)) \, dy \, ds \\ &= D_x^\alpha G * u_0 + \int_0^t \int_{R^n} D_x^\alpha H(x-y, t-s) F(u(y, s)) \, dy \, ds. \end{aligned} \quad (4.11)$$

Define

$$\begin{aligned} \phi &= (1+t)^{-\frac{n}{2a}} B_N(|x|^a, t), \\ \phi_\alpha &= (1+t)^{-\frac{n+|\alpha|}{2a}} B_N(|x|^a, t), \\ M(t) &= \sup_{0 \leq s \leq t, x \in R} |u(x, s)| \phi^{-1}(x, s). \end{aligned} \quad (4.12)$$

Then

$$\begin{aligned} u &\leq M \cdot \phi, \\ u(x, t) &\leq G * u_0 + M^p \int_0^t \int_{R^n} H(x-y, t-s) \phi^p(y, s) \, dy \, ds, \\ D_x^\alpha u(x, t) &\leq D_x^\alpha G * u_0 + M^p \int_0^t \int_{R^n} D_x^\alpha H(x-y, t-s) \phi^p(y, s) \, dy \, ds. \end{aligned} \quad (4.13)$$

First of all, we discuss the singular part. By (H2), following [20], we have

$$\begin{aligned} |D_x^\alpha F_l * u_0| &\leq C\varepsilon e^{-\frac{t}{2ak}} (1 + |x|^{2a})^{-N} \\ &\leq C\varepsilon (1+t)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N} \\ &\leq C\varepsilon \phi_\alpha. \end{aligned} \quad (4.14)$$

Secondly, we consider the nonsingular part. Since $|u_0| \leq (1 + |y|^{2a})^{-N}$, $\text{supp } u_0 \subset \{|y| \leq M\}$, according to the definition of tight support, we can know u_0 has a compact support. If t is large enough, we find

$$\begin{aligned} |D_x^\alpha (G - F_l) * u_0| &\leq C(1+t)^{-\frac{n+|\alpha|}{2a}} \int_{R^n} \left(1 + \frac{|y-x|^{2a}}{1+t}\right)^{-N} \left(1 + \frac{|y|^{2a}}{1+t}\right)^{-N} dy \\ &\leq C(1+t)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N} \int_{R^n} \left(1 + \frac{|y|^{2a}}{1+t}\right)^{-N} dy \\ &\leq C\varepsilon \phi_\alpha. \end{aligned} \quad (4.15)$$

Then

$$|D_x^\alpha G * u_0| \leq C\varepsilon \phi_\alpha. \quad (4.16)$$

Here we still divide the nonlinear term into a singular part and a nonsingular part. Define

$$\begin{aligned} \int_0^t \int_{R^n} D_x^\alpha H(x-y, t-s) \phi^p(y, s) dy ds &\leq \int_0^t \int_{R^n} D_x^\alpha (G - F_l) \phi^p(y, s) dy ds \\ &\quad + \int_0^t \int_{R^n} D_x^\alpha F_l \phi^p(y, s) dy ds \\ &= \psi_{1,1} + \psi_{1,2}, \end{aligned} \quad (4.17)$$

where $\psi_{1,1} = \int_0^t \int_{R^n} D_x^\alpha (G - F_l) \phi^p(y, s) dy ds$, $\psi_{1,2} = \int_0^t \int_{R^n} D_x^\alpha F_l \phi^p(y, s) dy ds$.

Estimating the nonsingular part. Recalling Lemma 3.1, it shows that

$$\begin{aligned} \psi_{1,1} &\leq C \int_0^t \int_{R^n} (1+t-s)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x-y|^{2a}}{1+t-s}\right)^{-N} \\ &\quad \times (1+s)^{-\frac{np}{2a}} \left(1 + \frac{|y|^{2a}}{1+s}\right)^{-Np} dy ds. \end{aligned} \quad (4.18)$$

Denote $\Omega = [0, t] \times R^n$, $\Omega^1 = \Omega \cap \{s \geq \frac{t}{2}\}$, $\Omega^2 = \Omega \cap \{s \leq \frac{t}{2}\}$. We will discuss it in two cases:

(1) If $|x|^{2a} \leq 1+t$, then

$$\begin{aligned} \psi_{1,1} &\leq C \int_{\Omega^1} (1+t-\tau)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x-y|^{2a}}{1+t-\tau}\right)^{-N} (1+\tau)^{-\frac{np}{2a}} \\ &\quad \times \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau + C \int_{\Omega^2} (1+t-\tau)^{-\frac{n}{2a}} \left(1 + \frac{|x-y|^{2a}}{1+t-\tau}\right)^{-N} \end{aligned}$$

$$\begin{aligned}
& \times (1+\tau)^{-\frac{np+|\alpha|}{2a}} \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau \\
& \leq C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{n+|\alpha|}{2a}} (1+\tau)^{-\frac{np}{2a}} (1+\tau)^{\frac{n}{2a}} d\tau \\
& \quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{2a}} (1+\tau)^{-\frac{np+|\alpha|}{2a}} (1+\tau)^{\frac{n}{2a}} d\tau \\
& \leq C(1+t)^{-\frac{n+np-n}{2a}+1-\frac{|\alpha|}{2a}} \\
& \leq C(1+t)^{-\frac{np}{2a}+1-\frac{|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-n}. \tag{4.19}
\end{aligned}$$

(2) If $|x|^{2a} > 1+t$, then

$$\begin{aligned}
\psi_{1,1} & \leq C \int_{\Omega^1} (1+t-\tau)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x-y|^{2a}}{1+t-\tau}\right)^{-N} (1+\tau)^{-\frac{np}{2a}} \\
& \quad \times \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau + C \int_{\Omega^2} (1+t-\tau)^{-\frac{n}{2a}} \left(1 + \frac{|x-y|^{2a}}{1+t-\tau}\right)^{-N} \\
& \quad \times (1+\tau)^{-\frac{np+|\alpha|}{2a}} \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau \\
& \leq C \int_{\Omega^1} (1+t-\tau)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t-\tau}\right)^{-N} (1+\tau)^{-\frac{np}{2a}} \\
& \quad \times \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau + C \int_{\Omega^2} (1+t-\tau)^{-\frac{n}{2a}} \left(1 + \frac{|x|^{2a}}{1+t-\tau}\right)^{-N} \\
& \quad \times (1+\tau)^{-\frac{np+|\alpha|}{2a}} \left(1 + \frac{|y|^{2a}}{1+\tau}\right)^{-Np} dy d\tau \\
& \leq C \left(1 + \frac{|x|^{2a}}{1+\tau}\right)^{-n} \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{n+|\alpha|}{2a}} (1+\tau)^{-\frac{np}{2a}} (1+\tau)^{\frac{n}{2a}} d\tau \\
& \quad + C \left(1 + \frac{|x|^{2a}}{1+\tau}\right)^{-n} \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{2a}} (1+\tau)^{-\frac{np+|\alpha|}{2a}} (1+\tau)^{\frac{n}{2a}} d\tau \\
& \leq C(1+t)^{-\frac{np}{2a}+1-\frac{|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-n}. \tag{4.20}
\end{aligned}$$

From (4.19)–(4.20), we can get

$$\psi_{1,1} \leq C(1+t)^{-\frac{np}{2a}+1-\frac{|\alpha|}{2a}} B_N(|x|^a, t). \tag{4.21}$$

Due to $p > 1 + \frac{2a}{n}$, we deduce

$$(1+t)^{-\frac{np}{2a}+1-\frac{|\alpha|}{2a}} \leq (1+t)^{-\frac{n}{2a}-\frac{|\alpha|}{2a}}. \tag{4.22}$$

Estimating the singular part. By [19], we know that

$$\psi_{1,2} = \int_0^t \int_{R^n} D_x^\alpha F_t \phi^p(y, s) dy ds$$

$$\begin{aligned}
&\leq \int_0^t e^{-\frac{t-s}{2ka}} \phi_\alpha^p \, ds \\
&\leq C\phi_\alpha.
\end{aligned} \tag{4.23}$$

Coming back to the whole solution, we find

$$\begin{aligned}
|u(x, t)| &\leq |G * u_0| + \left| \int_0^t \int_{R^n} H(x-y, t-s) F(u(y, s)) \, dy \, ds \right| \\
&\leq C(\varepsilon + M^p)\phi, \\
|D_x^\alpha u(x, t)| &\leq C(\varepsilon + M^p)\phi_\alpha.
\end{aligned} \tag{4.24}$$

Therefore,

$$M \leq C(\varepsilon + M^p). \tag{4.25}$$

Since $M(0) \leq C\varepsilon$, applying the continuity method, we get that

$$M(t) \leq 2C\varepsilon, \quad \forall t \in [0, +\infty). \tag{4.26}$$

By the above inequality, we have

$$\begin{aligned}
|D_x^\alpha u| &\leq M\phi_\alpha \leq 2C\varepsilon\phi_\alpha, \\
|u| &\leq 2C\varepsilon(1+t)^{-\frac{n}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}, \\
|D_x^\alpha u| &\leq 2C\varepsilon(1+t)^{-\frac{n+|\alpha|}{2a}} \left(1 + \frac{|x|^{2a}}{1+t}\right)^{-N}.
\end{aligned} \tag{4.27}$$

□

5 Improvement of the initial data

In this section we consider the Cauchy problem of (1.1). It shows that the limit of the parameter p can be weaker when the initial conditions become stronger. Since we have known the proof of the existence and uniqueness of the solution, we will not discuss it. Only for attenuation of the decay estimate.

Definition 5.1 Suppose that operator Λ^s , $s \in R^n$, satisfies

$$\Lambda^s f(x) = (2\pi)^{-n} \int_{R^n} |\xi|^s \widehat{f}(\xi) e^{\sqrt{-1}x \cdot \xi} \, d\xi, \tag{5.1}$$

and Sobolev space $W^{-s,p}$ indicates

$$W^{-s,p} = \{f \mid \|\Lambda^{-s} f\|_{L^p(R^n)} \leq C\}. \tag{5.2}$$

Lemma 5.1 Let $0 < s < n$, $1 < p < q < \infty$, $\frac{1}{q} + \frac{s}{n} = \frac{1}{p}$, then

$$\|\Lambda^{-s} f\|_{L^q(R^n)} \leq C_{p,q} \|f\|_{L^p(R^n)}, \tag{5.3}$$

where $C_{p,q}$ is a constant depending on p, q .

Through [21, p. 99, Theorem 1], Lemma 5.1 can be proved.

Proof of Theorem 2 We divide it into a singular part and a nonsingular part:

$$u = G * u_0 + \int_0^t H * u^p \, d\tau = u_N + u_S, \quad (5.4)$$

where

$$\begin{aligned} u_N &= (G - F_l) * u_0 + \int_0^t \frac{G - F_l}{1 + k(-\Delta)^a} * u^p \, d\tau, \\ u_S &= F_l * u_0 + \int_0^t \frac{F_l}{1 + k(-\Delta)^a} * u^p \, d\tau. \end{aligned} \quad (5.5)$$

Now, let us start from the nonsingular part, we obtain

$$\begin{aligned} u_N &= (G - F_l) * u_0 + \int_0^t \frac{G - F_l}{1 + k(-\Delta)^a} * u^p \, d\tau \\ &= \Lambda^s(G - F_l) * \Lambda^{-s}u_0 + \int_0^t \Lambda^s \frac{G - F_l}{1 + k(-\Delta)^a} * \Lambda^{-s}u^p \, d\tau. \end{aligned} \quad (5.6)$$

Using Young's inequality, we have

$$\begin{aligned} \|u_N\|_{L^q(R^n)} &\leq \|\Lambda^s(G - F_l) * \Lambda^{-s}u_0\|_{L^q(R^n)} \\ &\quad + \left\| \int_0^t \Lambda^s(G - F_l) * \Lambda^{-s}u^p \, d\tau \right\|_{L^q(R^n)} \\ &\leq \|\Lambda^s(G - F_l) * \Lambda^{-s}u_0\|_{L^q(R^n)} \\ &\quad + \int_0^t \|\Lambda^s(G - F_l)\|_{L^{\frac{nq}{n+q}}(R^n)} \|\Lambda^{-s}u^p\|_{L^{\frac{n}{n-1}}(R^n)} \, d\tau. \end{aligned} \quad (5.7)$$

Based on the initial condition and estimate of the linear part, we see that

$$\|\Lambda^s(G - F_l)\|_{L^q(R^n)} \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q})-\frac{s}{2a}} \quad (5.8)$$

and

$$\|\Lambda^s(G - F_l) * \Lambda^{-s}u_0\|_{L^q(R^n)} \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q})-\frac{s}{2a}}. \quad (5.9)$$

Recalling Lemma 5.1, there holds

$$\|\Lambda^{-s}u^p\|_{L^{\frac{n}{n-1}}(R^n)} \leq C\|u\|_{L^{\frac{np}{n+s-1}}(R^n)}^p \leq C\left[(1+t)^{-\frac{n}{2a}(1-\frac{n+s-1}{np})-\frac{s}{2a}}\right]^p. \quad (5.10)$$

Taking into account (5.8) and (5.10), we get

$$\begin{aligned} & \int_0^t \left\| \Lambda^s (G - F_l) \right\|_{L^{\frac{nq}{n+q}}(R^n)} \left\| \Lambda^{-s} u^p \right\|_{L^{\frac{n}{n-1}}(R^n)} ds \\ & \leq C \int_0^t (1+t-s)^{-\frac{n}{2a}(1-\frac{n+q}{nq})-\frac{s}{2a}} \left[(1+t)^{-\frac{n}{2a}(1-\frac{n+s-1}{np})-\frac{s}{2a}} \right]^p ds \\ & \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q})-\frac{s}{2a}}. \end{aligned} \quad (5.11)$$

Considering the singular part, we infer

$$\|F_l * u_0\|_{L^q(R^n)} \leq C e^{-\frac{t}{2ak}} \|u_0\|_{L^q(R^n)} \leq C \varepsilon e^{-\frac{t}{2ak}} \quad (5.12)$$

and

$$\begin{aligned} \int_0^t \left\| \frac{F_l}{1+k(-\Delta)^a} * u^p \right\|_{L^q(R^n)} d\tau & \leq C \int_0^t e^{-\frac{t-\tau}{2ak}} \|u\|_{L^{p,q}(R^n)}^p d\tau \\ & \leq C \int_0^t e^{-\frac{t-\tau}{2ak}} (1+\tau)^{-\frac{n}{2a}(1-\frac{1}{pq})p} d\tau \\ & \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q})-\frac{s}{2a}}. \end{aligned} \quad (5.13)$$

With the help of (5.9) and (5.11)–(5.13), we obtain

$$\|u\|_{L^q(R^n)} \leq C(1+t)^{-\frac{n}{2a}(1-\frac{1}{q})-\frac{s}{2a}}. \quad (5.14)$$

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Authors' contributions

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