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Existence result for a Kirchhoff elliptic system with variable parameters and additive right-hand side via sub- and supersolution method

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On the occasion of the 80th birthday of the second author's mother, Mrs. Fatma Bint Al-Tayeb Zeghdoud

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Abstract

The paper deals with the study of the existence result for a Kirchhoff elliptic system with additive right-hand side and variable parameters by using the sub-/supersolution method. Our study is a natural extension result of our previous one in (Boulaaras and Guefaifia in Math. Methods Appl. Sci. 41:5203–5210, 2018), where we discussed only the simple case when the parameters are constant.

MSC: 35J60; 35B30; 35B40

Keywords: Kirchhoff elliptic systems; Existence; Positive solutions; Sub-/supersolution; Multiple parameters

1 Introduction

Consider the following system:

$$\begin{cases} -A(\int_\Omega |\nabla u|^2\,dx)\triangle u = \alpha(x)f(v) + \beta(x)g(u) & \text{in } \Omega, \\ -B(\int_\Omega |\nabla v|^2\,dx)\triangle v = \gamma(x)h(u) + \eta(x)l(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with further conditions to be given later, $\alpha, \beta, \gamma, \eta \in C(\overline{\Omega})$.

This nonlocal problem originates from the stationary version of Kirchhoff's work [16] in 1883, namely

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where Kirchhoff extended the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

Recently, Kirchhoff elliptic equations have been heavily studied, we refer to [1–7, 9, 11–15, 17–20].

In [2], Alves and Correa proved the validity of sub-/supersolution method for problems of Kirchhoff class involving a single equation and a boundary condition

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in C(\overline{\Omega} \times \mathbb{R})$.

By using a comparison principle that requires M to be nonnegative and nonincreasing in $[0, +\infty)$, with $H(t) := M(t^2)t$ increasing and $H(\mathbb{R}) = \mathbb{R}$, they managed to prove the existence of positive solutions assuming f was increasing in u for each $x \in \Omega$ fixed.

For systems involving similar equations, this result cannot be used directly, i.e., the existence of a subsolution and a supersolution does not guarantee the existence of the solution. Therefore, a further construction is needed. In [8], we studied the system

$$\begin{cases} -A(\int_\Omega |\nabla u|^2\,dx)\triangle u = \lambda_1 f(v) + \mu_1 g(u) & \text{in } \Omega, \\ -B(\int_\Omega |\nabla v|^2\,dx)\triangle v = \lambda_2 h(u) + \mu_2(x) l(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Using a weak positive supersolution as the first term of a constructed iterative sequence (u_n, v_n) in $H_0^1(\Omega) \times H_0^1(\Omega)$, and a comparison principle introduced in [2], the authors established the convergence of this sequence to a positive weak solution of the considered problem.

To complement our above work in [8], where we discussed only the simple case when the parameters are constant, in this paper we prove an existence result for problem (1.1) by considering the complicated case when the parameters α, β, γ, and η on the right-hand side are variable. We also give a better subsolution providing easier computations compared with the earlier work in [8].

2 Existence result

Definition 1 A pair $(u,v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$ is called a weak solution of (1.1) if it satisfies

$$A\left(\int_\Omega |\nabla u|^2\,dx\right)\int_\Omega \nabla u \nabla \phi\,dx = \int_\Omega \alpha(x) f(v)\phi\,dx + \int_\Omega \beta(x) g(u)\phi\,dx \text{ in } \Omega,$$
$$B\left(\int_\Omega |\nabla v|^2\,dx\right)\int_\Omega \nabla v \nabla \psi\,dx = \int_\Omega \gamma(x) h(u)\psi\,dx + \int_\Omega \eta(x) l(v)\psi\,dx \text{ in } \Omega$$

for all $(\phi, \psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Definition 2 Let $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ be pairs of nonnegative functions in $(H_0^1(\Omega) \times H_0^1(\Omega))$. They are called a positive weak subsolution and a positive weak supersolution, respectively, of

(1.1) if they satisfy the following:

$$A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega \nabla\underline{u}\nabla\phi\,dx \le \int_\Omega \alpha(x)f(\underline{v})\phi\,dx + \int_\Omega \beta(x)g(\underline{u})\phi\,dx,$$
$$B\left(\int_\Omega |\nabla\underline{v}|^2\,dx\right)\int_\Omega \nabla\underline{v}\nabla\psi\,dx \le \int_\Omega \gamma(x)h(\underline{u})\psi\,dx + \int_\Omega \eta(x)l(\underline{v})\psi\,dx,$$

and

$$A\left(\int_\Omega |\nabla\overline{u}|^2\,dx\right)\int_\Omega \nabla\overline{u}\nabla\phi\,dx \ge \int_\Omega \alpha(x)f(\overline{v})\phi\,dx + \int_\Omega \beta(x)g(\overline{u})\phi\,dx,$$
$$B\left(\int_\Omega |\nabla\overline{v}|^2\,dx\right)\int_\Omega \nabla\overline{v}\nabla\psi\,dx \ge \int_\Omega \gamma(x)h(\overline{u})\psi\,dx + \int_\Omega \eta(x)l(\overline{v})\psi\,dx$$

for all $(\phi,\psi)\in(H_0^1(\Omega)\times H_0^1(\Omega))$, with $\phi\ge 0$ and $\psi\ge 0$, and $(\underline{u},\underline{v}),(\overline{u},\overline{v})=(0,0)$ on $\partial\Omega$.

Lemma 1 (Comparison principle [2]) *Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous nonincreasing function such that*

$$M(s)>m_0>0,\quad \textit{for all } s\ge s_0, \tag{2.1}$$

and $H(t)=tM(t^2)$ increasing on $\mathbb{R}^+$.

If u_1, u_2 are two nonnegative functions verifying

$$\begin{cases} -M(\int_\Omega |\nabla u_1|^2\,dx)\triangle u_1 \ge -M(\int_\Omega |\nabla u_2|^2\,dx)\triangle u_2 & \textit{in } \Omega,\\ u=v=0 & \textit{on } \partial\Omega, \end{cases} \tag{2.2}$$

then $u_1\ge u_2$ a.e. in Ω.

Before stating and proving our main result, here are the conditions we need:

(H1) $A,B:\mathbb{R}^+\to\mathbb{R}^+$ are two continuous and increasing functions that satisfy the monotonicity conditions of Lemma 1 so that we can use the comparison principle, and assume further that there exist $a_i,b_i>0$, $i=1,2$,

$$a_1\le A(t)\le a_2,\qquad b_1\le B(t)\le b_2\quad \text{for all } t\in\mathbb{R}^+.$$

(H2) $\alpha,\beta,\gamma,\eta\in C(\overline{\Omega})$ and

$$\alpha(x)\ge\alpha_0>0,\qquad \beta(x)\ge\beta_0>0,\qquad \gamma(x)\ge\gamma_0>0,\qquad \eta(x)\ge\eta_0>0$$

for all $x\in\Omega$.

(H3) f, g, h, and l are continuous on $[0,+\infty[$, C^1 on $(0,+\infty)$, and increasing functions of infinite growth

$$\lim_{t\to+\infty} f(t)=+\infty,\qquad \lim_{t\to+\infty} l(t)=+\infty,\qquad \lim_{t\to+\infty} g(t)=+\infty,\qquad \lim_{t\to+\infty} h(t)=+\infty.$$

(H4) For all $K > 0$,

$$\lim_{t\to+\infty} \frac{f(K(h(t)))}{t} = 0.$$

(H5)

$$\lim_{t\to+\infty} \frac{g(t)}{t} = \lim_{t\to+\infty} \frac{l(t)}{t} = 0.$$

Theorem 1 *For large values of $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$, system* (1.1) *admits a large positive weak solution if conditions* (H1)–(H5) *are satisfied.*

Proof of Theorem 1 Consider σ to be the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\| = 1$ and $\phi_1 \in C^\infty(\overline{\Omega})$ (see [10]).

Let $S = \sup_{x\in\Omega}\{\sigma\phi_1^2 - |\nabla\phi_1|^2\}$, then from growth condition (H3)

$$f(t) \geq S, \qquad g(t) \geq S, \qquad h(t) \geq S, \qquad l(t) \geq S, \quad \text{for } t \text{ large enough.}$$

For each $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$ large, let us define

$$\underline{u} = \frac{\alpha_0 + \beta_0}{2a_2}\phi_1^2$$

and

$$\underline{v} = \frac{\gamma_0 + \eta_0}{2b_2}\phi_1^2,$$

where a_2, b_2 are given by condition (H1). Let us show that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1) for $\alpha_0 + \beta_0$ and $\gamma_0 + \eta_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By (H1)–(H3), we get

$$\begin{aligned}
A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega \nabla\underline{u}\nabla\phi\,dx &= A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\frac{\alpha_0+\beta_0}{a_2}\int_\Omega \phi_1\nabla\phi_1\nabla\phi\,dx \\
&= \frac{\alpha_0+\beta_0}{a_2}A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right) \\
&\quad \times \left\{\int_\Omega \nabla\phi_1\nabla(\phi_1\phi)\,dx - \int_\Omega |\nabla\phi_1|^2\phi\,dx\right\} \\
&= \frac{\alpha_0+\beta_0}{a_2}A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega \left(\sigma\phi_1^2 - |\nabla\phi_1|^2\right)\phi\,dx \\
&\leq (\alpha_0+\beta_0)\int_\Omega S\phi\,dx \\
&\leq \int_\Omega \alpha(x)f(\underline{v})\phi\,dx + \int_\Omega \beta(x)g(\underline{u})\phi\,dx
\end{aligned}$$

for $\alpha_0 + \beta_0 > 0$ large enough, and all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω.

Similarly,

$$B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\int_\Omega \nabla \underline{v}\nabla \psi\,dx \le \int_\Omega \gamma(x)h(\underline{u})\psi\,dx + \int_\Omega \eta(x)\text{ł}(\underline{v})\psi\,dx \quad \text{in } \Omega$$

for $\gamma_0 + \eta_0 > 0$ large enough and all $\psi \in H_0^1(\Omega)$ with $\psi \ge 0$ in Ω.

Also notice that $\underline{u} > 0$ and $\underline{v} > 0$ in Ω, $\underline{u} \to +\infty$ and $\underline{v} \to +\infty$ as $\alpha_0 + \beta_0 \to +\infty$ and $\gamma_0 + \eta_0 \to +\infty$.

For the supersolution part, consider e the solution of the following problem:

$$\begin{cases} -\triangle e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We give the supersolution of problem (1.1) by

$$\overline{u} = Ce, \qquad \overline{v} = \left(\|\gamma\|_\infty + \|\eta\|_\infty\right)h\left(C\|e\|_\infty\right)e,$$

where $C > 0$ is a large positive real number to be given later.

Indeed, for all $\phi \in H_0^1(\Omega)$ with $\phi \ge 0$ in Ω, we get from (2.3) and the condition (H1)

$$\begin{aligned} A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)\int_\Omega \nabla \overline{u}\nabla \phi\,dx &= A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)C\int_\Omega \nabla e\nabla \phi\,dx \\ &= A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)C\int_\Omega \phi\,dx \\ &\ge a_1 C\int_\Omega \phi\,dx. \end{aligned}$$

By (H4) and (H5), we can choose C large enough so that

$$a_1 C \ge \|\alpha\|_\infty f\left[\left(\|\gamma\|_\infty + \|\eta\|_\infty\right)h\left(C\|e\|_\infty\right)\|e\|_\infty\right] + \|\beta\|_\infty g\left(C\|e\|_\infty\right).$$

Therefore,

$$\begin{aligned} &A\left(\int_\Omega |\nabla \overline{u}|^2\,dx\right)\int_\Omega \nabla \overline{u}.\nabla \phi\,dx \\ &\quad \ge \left[\|\alpha\|_\infty f\left[\left(\|\gamma\|_\infty + \|\eta\|_\infty\right)h\left(C\|e\|_\infty\right)\|e\|_\infty\right] + \|\beta\|_\infty g\left(C\|e\|_\infty\right)\right]\int_\Omega \phi\,dx \\ &\quad \ge \|\alpha\|_\infty \int_\Omega f\left[\left(\|\gamma\|_\infty + \|\eta\|_\infty\right)h\left(C\|e\|_\infty\right)\|e\|_\infty\right]\phi\,dx + \|\beta\|_\infty \int_\Omega g\left(C\|e\|_\infty\right)\phi\,dx \\ &\quad \ge \int_\Omega \alpha(x)f(\overline{v})\phi\,dx + \int_\Omega \beta(x)g(\overline{u})\phi\,dx. \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla \overline{v}\nabla \psi\,dx &= \left(\|\gamma\|_\infty + \|\eta\|_\infty\right)\int_\Omega h\left(C\|e\|_\infty\right)\psi\,dx \\ &\ge \int_\Omega \gamma(x)h(\overline{u})\psi\,dx + \int_\Omega \eta(x)h\left(C\|e\|_\infty\right)\psi\,dx. \end{aligned} \tag{2.5}$$

Using (H4) and (H5) again for C large enough, we get

$$h\big(C\|e\|_\infty\big) \geq l\big[\big(\|\gamma\|_\infty + \|\eta\|_\infty\big)h\big(C\|e\|_\infty\big)\|e\|_\infty\big] \geq l(\overline{v}). \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$B\left(\int_\Omega |\nabla\overline{v}|^2\,dx\right)\int_\Omega \nabla\overline{v}\nabla\psi\,dx \geq \int_\Omega \gamma(x)h(\overline{u})\psi\,dx + \int_\Omega \eta(x)l(\overline{v})\psi\,dx. \tag{2.7}$$

By (2.4) and (2.7), we conclude that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1).

Furthermore, $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ for C chosen large enough.

Now, we use a similar argument to that in [8] in order to obtain a weak solution of our problem. Consider the following sequence $\{(u_n, v_n)\} \subset (H_0^1(\Omega) \times H_0^1(\Omega))$ where $u_0 := \overline{u}$, $v_0 = \overline{v}$, and (u_n, v_n) is the unique solution of

$$\begin{cases} -A(\int_\Omega |\nabla u_n|^2\,dx)\triangle u_n = \alpha(x)f(v_{n-1}) + \beta(x)g(u_{n-1}) & \text{in } \Omega, \\ -B(\int_\Omega |\nabla v_n|^2\,dx)\triangle v_n = \gamma(x)h(u_{n-1}) + \eta(x)l(v_{n-1}) & \text{in } \Omega, \\ u_n = v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Since A and B satisfy (H1) and $\alpha(x)f(v_{n-1})$, $\beta(x)g(u_{n-1})$, $\gamma(x)h(u_{n-1})$, and $\eta(x)l(v_{n-1}) \in L^2(\Omega)$ (in x), we deduce from a result in [2] that system (2.8) has a unique solution $(u_n, v_n) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Using (2.8) and the fact that (u_0, v_0) is a supersolution of (1.1), we get

$$\begin{cases} -A(\int_\Omega |\nabla u_0|^2\,dx)\triangle u_0 \geq \alpha(x)f(v_0) + \beta(x)g(u_0) = -A(\int_\Omega |\nabla u_1|^2\,dx)\triangle u_1, \\ -B(\int_\Omega |\nabla v_0|^2\,dx)\triangle v_0 \geq \gamma(x)h(u_0) + \eta(x)l(v_0) = -B(\int_\Omega |\nabla v_1|\,dx)\triangle v_1. \end{cases}$$

Then by Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and due to the monotonicity of f, g, h, and l, one has

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2\,dx\right)\triangle u_1 &= \alpha(x)f(v_0) + \beta(x)g(u_0) \\ &\geq \alpha(x)f(\underline{v}) + \beta(x)g(\underline{u}) \geq -A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\triangle\underline{u}, \\ -B\left(\int_\Omega |\nabla v_1|^2\,dx\right)\triangle v_1 &= \gamma(x)h(u_0) + \eta(x)l(v_0) \\ &\geq \gamma(x)h(\underline{u}) + \eta(x)l(\underline{v}) \geq -B\left(\int_\Omega |\nabla\underline{v}|^2\,dx\right)\triangle\underline{v}. \end{aligned}$$

According to Lemma 1 again, we obtain $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$.

Repeating the same argument for u_2, v_2, observe that

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2\,dx\right)\triangle u_1 &= \alpha(x)f(v_0) + \beta(x)g(u_0) \\ &\geq \alpha(x)f(v_1) + \beta(x)g(u_1) = -A\left(\int_\Omega |\nabla u_2|^2\,dx\right)\triangle u_2, \end{aligned}$$

$$
\begin{aligned}
-B\left(\int_\Omega |\nabla v_1|\,dx\right)\triangle v_1 &= \gamma(x)h(u_0) + \eta(x)l(v_0) \\
&\geq \gamma(x)h(u_1) + \eta(x)l(v_1) = -B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2,
\end{aligned}
$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, we get $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ from

$$
\begin{aligned}
-A\left(\int_\Omega |\nabla u_2|^2\,dx\right)\triangle u_2 &= \alpha(x)f(v_1) + \beta(x)g(u_1) \\
&\geq \alpha(x)f(\underline{v}) + \beta(x)g(\underline{u}) \geq -A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\triangle \underline{u}, \\
-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2 &= \gamma(x)h(u_1) + \eta(x)l(v_1) \\
&\geq \gamma(x)h(\underline{u}) + \eta(x)l(\underline{v}) \geq -B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\triangle \underline{v}.
\end{aligned}
$$

By repeating the same arguments, we construct a bounded decreasing sequence $\{(u_n, v_n)\} \subset (H_0^1(\Omega) \times H_0^1(\Omega))$ verifying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq \underline{u} > 0, \tag{2.9}$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_n \geq \cdots \geq \underline{v} > 0. \tag{2.10}$$

By continuity of functions f, g, h, and l and the definition of the sequences (u_n) and (v_n), there exist positive constants $C_i > 0$, $i = 1, \ldots, 4$ such that

$$\bigl|f(v_{n-1})\bigr| \leq C_1, \qquad \bigl|g(u_{n-1})\bigr| \leq C_2, \qquad \bigl|h(u_{n-1})\bigr| \leq C_3 \tag{2.11}$$

and

$$\bigl|l(u_{n-1})\bigr| \leq C_4 \quad \text{for all } n.$$

From (2.11), multiplying the first equation of (2.8) by u_n, integrating, using Hölder inequality and Sobolev embedding, we check that

$$
\begin{aligned}
a_1 \int_\Omega |\nabla u_n|^2\,dx \leq A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega |\nabla u_n|^2\,dx \\
&= \int_\Omega \alpha(x)f(v_{n-1})u_n\,dx + \int_\Omega \beta(x)g(u_{n-1})u_n\,dx \\
&\leq \|\alpha\|_\infty \int_\Omega \bigl|f(v_{n-1})\bigr||u_n|\,dx + \|\beta\|_\infty \int_\Omega \bigl|g(u_{n-1})\bigr||u_n|\,dx \\
&\leq C_1 \int_\Omega |u_n|\,dx + C_2 \int_\Omega |u_n|\,dx \\
&\leq C_5 \|u_n\|_{H_0^1(\Omega)}
\end{aligned}
$$

or

$$\|u_n\|_{H_0^1(\Omega)} \leq C_5, \quad \forall n, \tag{2.12}$$

where $C_5 > 0$ is a constant independent of n. Similarly, there exists $C_6 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \le C_6, \quad \forall n. \tag{2.13}$$

From (2.12) and (2.13), we deduce that $\{(u_n, v_n)\}$ admits a weakly converging subsequence in $H_0^1(\Omega, \mathbb{R}^2)$ to a limit (u, v) satisfying $u \ge \underline{u} > 0$ and $v \ge \underline{v} > 0$. Being monotone, by using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v). Now, letting $n \to +\infty$ in (2.8), we conclude that (u, v) is a positive weak solution of system (1.1). □

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