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Existence of solutions for tripled system of fractional differential equations involving cyclic permutation boundary conditions

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Abstract

In this paper, we introduce and study a tripled system of three associated fractional differential equations. Prior to proceeding to the main results, the proposed system is converted into an equivalent integral form by the help of fractional calculus. Our approach is based on using the addressed tripled system with cyclic permutation boundary conditions. The existence and uniqueness of solutions are investigated. We employ the Banach and Krasnoselskii fixed point theorems to prove our main results. Illustrative examples are presented to explain the theoretical results.

MSC: 34A08; 34A128

Keywords: Tripled fractional differential system; Cyclic permutation; Tripled boundary conditions; Banach and Krasnoselskii fixed point theorems

1 Introduction

Fractional derivatives and integrals find numerous applications in many branches of physics and engineering ranging from quantum optics to astro-physics and cosmology, dynamics of materials to biophysics and medicine, dynamical chaos to control, signal processing to communications, and more. For recent comprehensive reviews on fractional derivatives and their applications, we refer the reader to the monographs [25, 34, 36] and the recent undermentioned papers [1–3, 5, 6, 9, 11, 13, 17, 23, 26–29, 31, 32, 35]. Due to their widespread applications, a system of fractional differential equations subject to boundary conditions has received much attention amongst researchers who accommodate various numerical methods to establish their results; see for instance the papers [18, 22, 33].

Particularly, coupled fractional boundary systems, which study interaction between two quantities, have been under consideration as they provide adequate interpretations for models describing chaotic behavior, anomalous diffusion, ecological effects, and biological models. Many relevant results have been reported in this direction with different boundary conditions; see [4, 7, 8, 14–16, 21, 30, 37–39] and the references therein.

Tripled fractional boundary systems, which are considered as a generalization of coupled fractional systems, are governed by three associated differential equations with three initial or boundary conditions [12, 24]. In [12], Berinde and Borcut introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained existence and uniqueness theorems for contractive type mappings. Karakaya et al.[24] gave some results concerning the existence of tripled fixed points for a class of condensing operators in Banach spaces.

The cyclic boundary conditions have many applications on channel flow with fully developed flow at inlet as well as outlet using simple foam. In addition, some researchers introduced a railway track coupled dynamics model based on cyclic boundary conditions (see [10] and the references therein).

Unlike coupled fractional systems, the investigations of tripled fractional systems have gained less attention amongst researchers. To the best of authors' observation, indeed, there is no analytical literature on studying the existence of tripled systems of fractional differential equations.

Motivated by these research works, we investigate in this paper a tripled fractional abstract system with cyclic tripled boundary conditions that has the following form:

$$\begin{cases} {}^cD_0^{\alpha_k} x_k(t) = f_k(t, x(t)), & 1 < \alpha_k \leq 2, \\ x_k^{(j)}(0) = a_{k,j} x_{\sigma(k)}^{(j)}(T), & k = 1,2,3; j = 0,1, \end{cases} \tag{1.1}$$

where ${}^cD_0^{\alpha_k}$ denotes the Caputo fractional derivative of order α_k, $t \in J = [0,T]$, $f_k : J \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\sigma = (1\ 2\ 3)$ is a cycle permutation, and $a_{k,j} \in \mathbb{R}$, $k = 1,2,3$, $j = 0,1$, such that $\prod_{k=1}^3 a_{k,j} \neq 1$, $j = 0,1$. System (1.1) is converted into an equivalent integral form by the help of fractional calculus. The existence and uniqueness of solutions with cyclic permutation of tripled boundary conditions are investigated. We employ the Banach and Krasnoselskii fixed point theorems to prove our main results.

The railway track coupled system investigated in [10] can be modeled as a classical tripled system if it undergoes an external influence, by which many researchers can be prompted to generalize this idea using fractional differential models. We emphasize that the problem considered in the present settings is new and has novel approach that will provide further insight into the analytical study of tripled fractional systems with cyclic boundary conditions.

2 Preliminary assertions

In this section, we recall some basic definitions of fractional calculus [25]. Meanwhile, the integral form of the solution of system (1.1) as well as the definition of permutation group are presented. The notations and terminologies herein will be used in the subsequent section.

Definition 2.1 ([25]) The Riemann–Liouville fractional integral of a real-valued function $f \in C(J)$ is defined by

$$I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds, \quad t \in J, \alpha > 0,$$

provided the integral exists, and $I_0^0 f(t) := f(t)$. The Caputo fractional derivative of $f \in C^{(n)}(J)$ is given by

$$^cD_0^\alpha f(t) = I_0^{n-\alpha} f^{(n)}(t),$$

where $n = [\alpha]$ is the greatest integer function.

Lemma 2.2 ([25]) *Let* $[\alpha] = n \in \mathbb{N}$, *and* $f, {}^cD_0^\alpha f \in C(J)$. *Then*

$$I_0^{\alpha\, c}D_0^\alpha f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

for $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$.

For convenience, we introduce the following notations:

$$b_{1,1} = b_{2,2} = b_{3,3} = \frac{a_{1,1}a_{2,1}a_{3,1}}{1 - a_{1,1}a_{2,1}a_{3,1}}, \qquad b_{1,2} = \frac{a_{1,1}}{1 - a_{1,1}a_{2,1}a_{3,1}},$$
$$b_{1,3} = \frac{a_{1,1}a_{2,1}}{1 - a_{1,1}a_{2,1}a_{3,1}}, \qquad b_{2,1} = \frac{a_{2,1}a_{3,1}}{1 - a_{1,1}a_{2,1}a_{3,1}}, \qquad b_{2,3} = \frac{a_{2,1}}{1 - a_{1,1}a_{2,1}a_{3,1}},$$
$$b_{3,1} = \frac{a_{3,1}}{1 - a_{1,1}a_{2,1}a_{3,1}}, \qquad b_{3,2} = \frac{a_{3,1}a_{1,1}}{1 - a_{1,1}a_{2,1}a_{3,1}},$$
$$d_{1,1} = d_{2,2} = d_{3,3} = \frac{a_{1,0}a_{2,0}a_{3,0}}{1 - a_{1,0}a_{2,0}a_{3,0}}, \qquad d_{1,2} = \frac{a_{1,0}}{1 - a_{1,0}a_{2,0}a_{3,0}},$$
$$d_{1,3} = \frac{a_{1,0}a_{2,0}}{1 - a_{1,0}a_{2,0}a_{3,0}}, \qquad d_{2,1} = \frac{a_{2,0}a_{3,0}}{1 - a_{1,0}a_{2,0}a_{3,0}}, \qquad d_{2,3} = \frac{a_{2,0}}{1 - a_{1,0}a_{2,0}a_{3,0}},$$
$$d_{3,1} = \frac{a_{3,0}}{1 - a_{1,0}a_{2,0}a_{3,0}}, \qquad d_{3,2} = \frac{a_{3,0}a_{1,0}}{1 - a_{1,0}a_{2,0}a_{3,0}},$$
$$e_{1,1} = \frac{a_{1,0}a_{3,1}T(a_{2,0}a_{3,0}a_{1,1}a_{2,1} + a_{2,0} + a_{2,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{1,2} = \frac{a_{1,0}a_{1,1}T(a_{2,0}a_{3,0} + a_{2,0}a_{3,1} + a_{2,1}a_{3,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{1,3} = \frac{a_{1,0}a_{2,1}T(a_{2,0}a_{3,0}a_{1,1} + a_{2,0}a_{1,1}a_{3,1} + 1)}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{2,1} = \frac{a_{2,0}a_{3,1}T(a_{1,0}a_{2,1}a_{3,0} + a_{1,1}a_{2,1}a_{3,0} + 1)}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{2,2} = \frac{a_{2,0}a_{1,1}T(a_{1,0}a_{3,0}a_{2,1}a_{3,1} + a_{3,0} + a_{3,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{2,3} = \frac{a_{2,0}a_{2,1}T(a_{1,0}a_{3,0} + a_{3,0}a_{1,1} + a_{3,1}a_{1,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{3,1} = \frac{a_{3,0}a_{3,1}T(a_{1,0}a_{2,0} + a_{1,0}a_{2,1} + a_{1,1}a_{2,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{3,2} = \frac{a_{3,0}a_{1,1}T(a_{1,0}a_{2,0}a_{3,1} + a_{1,0}a_{2,1}a_{3,1} + 1)}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})},$$
$$e_{3,3} = \frac{a_{3,0}a_{2,1}T(a_{1,0}a_{2,0}a_{1,1}a_{3,1} + a_{1,0} + a_{1,1})}{(1 - a_{1,0}a_{2,0}a_{3,0})(1 - a_{1,1}a_{2,1}a_{3,1})}.$$

Lemma 2.3 *Let $f_k \in C(J,\mathbb{R})$ and $\prod_{k=1}^{3} a_{k,j} \neq 1, j = 0, 1$. Then the solution of the linear fractional differential system*

$$^{c}D_0^{\alpha_k} x(t) = f_k(t), \quad 1 < \alpha_k \leq 2, t \in (0,T), \tag{2.1}$$

subject to the conditions

$$x_k^{(j)}(0) = a_{k,j} x_{\sigma(k)}^{(j)}(T), \quad k = 1,2,3, j = 0,1, \tag{2.2}$$

is given by

$$x_k(t) = \sum_{m=1}^{3} \big(d_{k,m} I_0^{\alpha_m} f_m(T) + (e_{k,m} + t b_{k,m}) I_0^{\alpha_m - 1} f_m(T) \big) + I_0^{\alpha_k} f_k(t). \tag{2.3}$$

Proof Applying the fractional integral to both sides of (2.1) and using Lemma 2.2, we obtain

$$x_k(t) = c_{k,0} + c_{k,1} t + I_0^{\alpha_k} f_k(t). \tag{2.4}$$

Hence, we deduce that

$$x_k'(t) = c_{k,1} + I_0^{\alpha_k - 1} f_k(t).$$

The boundary conditions in (2.2) imply that

$$\begin{cases} c_{1,0} = a_{1,0}(c_{2,0} + c_{2,1} T + I_0^{\alpha_2} f_2(T)), \\ c_{2,0} = a_{2,0}(c_{3,0} + c_{3,1} T + I_0^{\alpha_3} f_3(T)), \\ c_{3,0} = a_{3,0}(c_{1,0} + c_{1,1} T + I_0^{\alpha_1} f_1(T)), \end{cases} \tag{2.5}$$

and

$$\begin{cases} c_{1,1} = a_{1,1}(c_{2,1} + I_0^{\alpha_2 - 1} f_2(T)), \\ c_{2,1} = a_{2,1}(c_{3,1} + I_0^{\alpha_3 - 1} f_3(T)), \\ c_{3,1} = a_{3,1}(c_{1,1} + I_0^{\alpha_1 - 1} f_1(T)). \end{cases} \tag{2.6}$$

By direct substitutions of the equations in (2.5), we get

$$c_{1,1} = b_{1,1} I_0^{\alpha_1 - 1} f_1(T) + b_{1,2} I_0^{\alpha_2 - 1} f_2(T) + b_{1,3} I_0^{\alpha_3 - 1} f_3(T), \tag{2.7}$$

$$c_{2,1} = b_{2,1} I_0^{\alpha_1 - 1} f_1(T) + b_{2,2} I_0^{\alpha_2 - 1} f_2(T) + b_{2,3} I_0^{\alpha_3 - 1} f_3(T), \tag{2.8}$$

and

$$c_{3,1} = b_{3,1} I_0^{\alpha_1 - 1} f_1(T) + b_{3,2} I_0^{\alpha_2 - 1} f_2(T) + b_{3,3} I_0^{\alpha_3 - 1} f_3(T). \tag{2.9}$$

Similarly, the equations in (2.6) together with the last constants lead to

$$c_{1,0} = d_{1,1} I_0^{\alpha_1} f_1(T) + d_{1,2} I_0^{\alpha_2} f_2(T) + d_{1,3} I_0^{\alpha_3} f_3(T)$$

$$+ e_{1,1} I_0^{\alpha_1 - 1} f_1(T) + e_{1,2} I_0^{\alpha_2 - 1} f_2(T) + e_{1,3} I_0^{\alpha_3 - 1} f_3(T), \tag{2.10}$$

$$\begin{aligned} c_{2,0} &= d_{2,1} I_0^{\alpha_1} f_1(T) + d_{2,2} I_0^{\alpha_2} f_2(T) + d_{2,3} I_0^{\alpha_3} f_3(T) \\ &\quad + e_{2,1} I_0^{\alpha_1 - 1} f_1(T) + e_{2,2} I_0^{\alpha_2 - 1} f_2(T) + e_{2,3} I_0^{\alpha_3 - 1} f_3(T), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} c_{3,0} &= d_{3,1} I_0^{\alpha_1} f_1(T) + d_{3,2} I_0^{\alpha_2} f_2(T) + d_{3,3} I_0^{\alpha_3} f_3(T) \\ &\quad + e_{3,1} I_0^{\alpha_1 - 1} f_1(T) + e_{3,2} I_0^{\alpha_2 - 1} f_2(T) + e_{3,3} I_0^{\alpha_3 - 1} f_3(T). \end{aligned} \tag{2.12}$$

Substituting the values of $c_{k,j}, k = 1,2,3, j = 0,1$, in (2.4), we get (2.3). This completes the proof. □

We adopt the following definition of permutation groups.

Definition 2.4 ([19]) A permutation of a set A is a function $\sigma : A \to A$ that is one to one and onto.

This defines the so-called permutation group (A,σ). Let $A = \{1,2,3\}$, then the cardinality of this group is $3! = 6$ permutations. For instance, one of such permutations is given by $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$, and this constitutes a cycle $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) = (1\ 2\ 3)$. However, we use this cycle in the boundary conditions of system (1.1) such that, for a triple (x_1, x_2, x_3), we have $x_1^{(j)}(0) = a_{1,j} x_2^{(j)}(T)$, $x_2^{(j)}(0) = a_{2,j} x_3^{(j)}(T)$, and $x_3^{j}(0) = a_{3,j} x_1^{j}(T)$, $j = 1,2$. The other five permutations can be used and another integral solution can be obtained which is isomorphic to the one in (2.3) with constant differences. To explain this more, we consider permutation (13)(2). As a consequence of Lemma 2.3, we find the same solution as (2.3) but with different coefficients. Indeed, we find the following:

$$\begin{aligned}
& b_{1,1} = b_{3,3} = \frac{a_{1,1} a_{3,1}}{1 - a_{1,1} a_{3,1}}, \qquad b_{2,2} = \frac{a_{2,1}}{1 - a_{2,1}}, \\
& b_{1,3} = \frac{a_{1,1}}{1 - a_{1,1} a_{3,1}}, \qquad b_{3,1} = \frac{a_{3,1}}{1 - a_{3,1} a_{1,1}}, \\
& b_{1,2} = b_{2,1} = b_{2,3} = b_{3,2} = 0, \\
& d_{1,1} = \frac{1}{1 - a_{1,0} a_{3,0}}, \qquad d_{2,2} = \frac{a_{2,0}}{1 - a_{2,0}}, \qquad d_{3,3} = \frac{a_{3,0} a_{1,0}}{1 - a_{1,0} a_{3,0}}, \\
& d_{1,3} = \frac{a_{1,0}}{1 - a_{1,0} a_{3,0}}, \qquad d_{3,1} = a_{3,0} \left(\frac{2 - a_{1,0} a_{3,0}}{1 - a_{1,0} a_{3,0}} \right), \\
& d_{1,2} = d_{2,1} = d_{2,3} = d_{3,2} = 0, \\
& e_{1,1} = a_{3,1} a_{1,0} T \left(\frac{1 + a_{1,1} a_{3,0}}{(1 - a_{1,0} a_{3,0})(1 - a_{1,1} a_{3,1})} \right), \\
& e_{1,3} = a_{1,1} a_{1,0} T \left(\frac{a_{3,1} + a_{3,0}}{(1 - a_{1,0} a_{3,0})(1 - a_{1,1} a_{3,1})} \right), \\
& e_{2,2} = \frac{a_{2,1} a_{2,0} T}{(1 - a_{2,0})(1 - a_{2,1})}, \\
& e_{3,1} = a_{3,0} a_{3,1} T \left(\frac{a_{1,0} + a_{1,1}}{(1 - a_{1,0} a_{3,0})(1 - a_{1,1} a_{3,1})} \right),
\end{aligned}$$

$$e_{3,3} = a_{3,0}T\left(\frac{a_{3,1}a_{1,1}a_{1,0}}{(1-a_{1,0}a_{3,0})(1-a_{3,1}a_{1,1})}\right),$$

$$e_{2,1} = 0 = e_{1,2} = e_{2,3} = e_{3,2} = 0.$$

A tripled fixed point of a mapping is given next.

Definition 2.5 ([12]) An element $(x_1,x_2,x_3) \in X \times X \times X$ is called a tripled fixed point of a mapping $\digamma : X \times X \times X \to X$ if $\digamma(x_1,x_2,x_3) = x_1$, $\digamma(x_2,x_1,x_3) = x_2$, and $\digamma(x_3,x_2,x_1) = x_3$.

Define an operator $\Psi : X \times X \times X \to X \times X \times X$ such that

$$\Psi(x_1,x_2,x_3) = \bigl(\digamma(x_1,x_2,x_3), \digamma(x_2,x_1,x_3), \digamma(x_3,x_2,x_1)\bigr).$$

Then (x_1,x_2,x_3) is a tripled fixed point of $\digamma$ iff (x_1,x_2,x_3) is a fixed point of Ψ, that is, $\Psi(x_1,x_2,x_3) = (x_1,x_2,x_3)$.

For completeness, we recall the following tools of fixed point theory.

Theorem 2.6 (Banach fixed point theorem [20]) *Let D be a nonempty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Theorem 2.7 (Krasnoselskii fixed point theorem [20]) *Let B be a closed convex and nonempty subset of a Banach space X. Let Ψ_1, Ψ_2 be operators defined on B such that*

(i) *$\Psi_1 x + \Psi_2 y \in B$ whenever $x, y \in B$;*

(ii) *Ψ_1 is a contraction mapping;*

(iii) *Ψ_2 is compact and continuous.*

Then there exists $z \in B$ such that $z = \Psi_1 z + \Psi_2 z$.

3 Main results

In this section we use the Banach and Krasnoselskii fixed point theorems to ensure the existence of solution for tripled system (1.1).

The Banach space $X = C(J,\mathbb{R})$ of continuous real-valued functions is defined on J with the usual maximum norm. Hence, we obtain a Banach space $X^3 = X \times X \times X$ equipped with the norm $\|x\|_{X^3} = \|(x_1,x_2,x_3)\|_{X^3} = \|x_1\| + \|x_2\| + \|x_3\|$. Using the result of Lemma 2.3, we define the operator $\Psi : X^3 \to X^3$ by

$$\Psi x(t) = \bigl(\Psi_1 x_1(t), \Psi_2 x_2(t), \Psi_3 x_3(t)\bigr),$$

where

$$\begin{aligned}\Psi_k x_k(t) = {}& I_0^{\alpha_k} f_k\bigl(t,x(t)\bigr) + \sum_{m=1}^{3}\bigl(d_{k,m} I_0^{\alpha_m} f_m\bigl(T,x(T)\bigr) \\ &+ (e_{k,m} + t b_{k,m}) I_0^{\alpha_m - 1} f_m\bigl(T,x(T)\bigr)\bigr).\end{aligned} \tag{3.1}$$

If the operator $\Psi_k : X \to X$ given by (3.1) has a fixed point in X, then $\Psi_k x_k = x_k$, $k = 1,2,3$. Hence in connection with Definition 2.5, we let $\Psi_1 x_1 = \digamma(x_1,x_2,x_3)$, $\Psi_2 x_2 = \digamma(x_2,x_1,x_3)$, and $\Psi_3 x_3 = \digamma(x_3, x_2, x_1)$. This assumption connects the definition of the tripled fixed point introduced in Definition 2.5 with the fixed point of the tripled operator $\Psi = (\Psi_1, \Psi_2, \Psi_3)$. By this idea, we obtain the main results later.

We make use of the following assumption:

(Λ) Let $f_k : J \times X^3 \to X, k = 1, 2, 3,$ *be a jointly continuous function, and there exists a positive constant* L_k *such that*

$$\left|f_k(t,x) - f_k(t,y)\right| \le L_k \|x - y\|_{X^3}$$

for all $t \in J$ *and* $x, y \in X^3$.

Theorem 3.1 *Let condition* (Λ) *be satisfied. Then tripled system* (1.1) *has a unique solution whenever*

$$\eta = \sum_{k=1}^{3}\left(\frac{L_k T^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} L_m\left(\left(\frac{|d_{k,m}|}{\alpha_m} + |b_{k,m}|\right)T + |e_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right) < 1.$$

Proof Let $B_r = \{x \in X^3 : \|x\|_{X^3} \le r\}$ be a closed subset in X^3 such that

$$r > (1-\eta)^{-1}\sum_{k=1}^{3}\left(\frac{N_k T^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} N_m\left(\left(\frac{|d_{k,m}|}{\alpha_m} + |b_{k,m}|\right)T + |e_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right).$$

Firstly, we show that $\Psi(B_r) \subset B_r$. For this, define $\sup_{t\in J}|f_k(t,0)| = N_k < \infty, k = 1, 2, 3,$ then $|f_k(t,x)| \le L_k\|x\|_{X^3} + N_k$ for any $t \in J$. Therefore

$$\begin{aligned}
\left|\Psi_k x_k(t)\right| &\le \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)}\left(L_k\|x\|_{X^3} + N_k\right) \\
&\quad + \sum_{m=1}^{3}\left(\frac{|d_{k,m}|T^{\alpha_m}}{\Gamma(\alpha_m+1)} + \left(|e_{k,m}| + t|b_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right)\left(L_m\|x\|_{X^3} + N_m\right) \\
&\le \left(\frac{L_k t^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} L_m\left(\frac{|d_{k,m}|T}{\alpha_m} + \left(|e_{k,m}| + t|b_{k,m}|\right)\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right)r \\
&\quad \times \left(\frac{N_k t^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} N_m\left(\frac{|d_{k,m}|T}{\alpha_m} + \left(|e_{k,m}| + t|b_{k,m}|\right)\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\Psi x\|_{X^3} &\le \sum_{k=1}^{3}\left(\frac{N_k T^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} N_m\left(\left(\frac{|d_{k,m}|}{\alpha_m} + |b_{k,m}|\right)T + |e_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right) \\
&\quad + \sum_{k=1}^{3}\left(\frac{L_k T^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} L_m\left(\left(\frac{|d_{k,m}|}{\alpha_m} + |b_{k,m}|\right)T + |e_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right)r \\
&\le r.
\end{aligned}$$

Next, we show that the operator Ψ is a contraction. For this, let $x, y \in X^3$, then for any $t \in J$ we get

$$\begin{aligned}&\left|\Psi_k x_k(t) - \Psi_k y_k(t)\right| \\ &\quad \le \left(\frac{L_k t^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} L_m\left(\frac{|d_{k,m}|T}{\alpha_m} + \left(|e_{k,m}| + t|b_{k,m}|\right)\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right)\|x-y\|_{X^3}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\Psi x - \Psi y\|_{X^3} &\le \sum_{k=1}^{3}\left(\frac{L_k T^{\alpha_k}}{\Gamma(\alpha_k+1)} + \sum_{m=1}^{3} L_m\left(\left(\frac{|d_{k,m}|}{\alpha_m} + |b_{k,m}|\right)T + |e_{k,m}|\right)\frac{T^{\alpha_m-1}}{\Gamma(\alpha_m)}\right) \\ &\le \eta\|x-y\|_{X^3}.\end{aligned}$$

Since $\eta < 1$, therefore Ψ is a contraction operator. Then, by the Banach fixed point theorem, the operator Ψ has a unique fixed point which is the unique solution of problem (1.1). This completes the proof. □

In the next result, we apply the Krasnoselskii fixed point theorem (Theorem 2.7) to prove the existence of at least one solution of the tripled fractional system (1.1). For this purpose, we decompose the triple operator $\Psi : X^3 \to X^3$ into two triple operators Ψ_1 and Ψ_2 such that

$$\Psi x(t) = \Psi_1 x(t) + \Psi_2 x(t),$$

where $\Psi_i x(t) = (\Psi_{1,i}x_1(t), \Psi_{2,i}x_2(t), \Psi_{3,i}x_3(t))$, $i = 1, 2$, and

$$\begin{cases}\Psi_{k,1}x_k(t) = I_0^{\alpha_k} f_k(t, x(t)), \quad k = 1,2,3, \\ \Psi_{k,2}x_k(t) = \sum_{m=1}^{3}(d_{k,m} I_0^{\alpha_m} f_m(T, x(T)) + (e_{k,m} + t b_{k,m}) I_0^{\alpha_m-1} f_m(T, x(T))).\end{cases}$$

Theorem 3.2 *Let $f_k : J \times X^3 \to X$, $k = 1,2,3$, be a jointly continuous function, and there exist nonnegative fractional integrable real-valued functions φ_k and μ_k such that*

$$\begin{cases}|f_k(t,x) - f_k(t,y)| \le \varphi_k(t)\|x-y\|_{X^3}, \\ |f_k(t,0)| \le \mu_k(t), \quad t \in J, k = 1,2,3,\end{cases}$$

where $x, y \in X^3$. Then tripled system (1.1) has a solution provided that

$$\sum_{k=1}^{3} m_k \max_{t\in J} I_0^{\alpha_k}\varphi_k(t) + n_k \max_{t\in J} I_0^{\alpha_k-1}\varphi_k(t) < 1,$$

where $m_k = 1 + \sum_{m=1}^{3}|d_{m,k}|$ and $n_k = \sum_{m=1}^{3}|e_{m,k}| + T|b_{m,k}|$.

Proof Let $B_r = \{x \in X^3 : \|x\|_{X^3} \le r\}$ be a closed convex nonempty subset in X^3 such that

$$r \ge \frac{\sum_{k=1}^{3} m_k \max_{t\in J} I_0^{\alpha_k}\mu_k(t) + n_k \max_{t\in J} I_0^{\alpha_k-1}\mu_k(t)}{1 - \sum_{k=1}^{3} m_k \max_{t\in J} I_0^{\alpha_k}\varphi_k(t) + n_k \max_{t\in J} I_0^{\alpha_k-1}\varphi_k(t)}.$$

We show that Ψ_1 is a contraction and Ψ_2 is compact on B_r. Before doing these two steps, we show that $\Psi_1 x + \Psi_2 y \in B_r$ whenever $x, y \in B_r$. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be any elements of B_r, then for $t \in J$ we have

$$\left|\Psi_{k,1} x_k(t)\right| \leq I_0^{\alpha_k} \left|f_k\big(t, x(t)\big)\right| \leq I_0^{\alpha_k} \varphi_k(t) \|x\|_{X^3} + I_0^{\alpha_k} \mu_k(t),$$

and

$$\begin{aligned}
\left|\Psi_{k,2} y_k(t)\right| &\leq \sum_{m=1}^{3} \Big(|d_{k,m}| I_0^{\alpha_m} \left|f_m\big(T, y(T)\big)\right| + \big(|e_{k,m}| + t|b_{k,m}|\big) I_0^{\alpha_m - 1} \left|f_m\big(T, y(T)\big)\right|\Big) \\
&\leq \|y\|_{X^3} \sum_{m=1}^{3} |d_{k,m}| I_0^{\alpha_m} \varphi_m(t) + \big(|e_{k,m}| + t|b_{k,m}|\big) I_0^{\alpha_m - 1} \varphi_m(t) \\
&\quad + \sum_{m=1}^{3} |d_{k,m}| I_0^{\alpha_m} \mu_m(t) + \big(|e_{k,m}| + t|b_{k,m}|\big) I_0^{\alpha_m - 1} \mu_m(t).
\end{aligned}$$

In consequence, we obtain

$$\|\Psi_1 x\|_{X^3} \leq \|x\|_{X^3} \sum_{k=1}^{3} \max_{t \in J} I_0^{\alpha_k} \varphi_k(t) + \sum_{k=1}^{3} \max_{t \in J} I_0^{\alpha_k} \mu_k(t)$$

and

$$\begin{aligned}
\|\Psi_2 y\|_{X^3} \leq{}& \|y\|_{X^3} \sum_{m=1}^{3} \sum_{k=1}^{3} |d_{k,m}| \max_{t \in J} I_0^{\alpha_m} \varphi_m(t) + \big(|e_{k,m}| + T|b_{k,m}|\big) \max_{t \in J} I_0^{\alpha_m - 1} \varphi_m(t) \\
&+ \sum_{m=1}^{3} |d_{k,m}| \max_{t \in J} I_0^{\alpha_m} \mu_m(t) + \big(|e_{k,m}| + t|b_{k,m}|\big) \max_{t \in J} I_0^{\alpha_m - 1} \mu_m(t).
\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned}
\|\Psi_1 x + \Psi_2 y\|_{X^3} \leq{}& r \sum_{k=1}^{3} m_k \max_{t \in J} I_0^{\alpha_k} \varphi_k(t) + n_k \max_{t \in J} I_0^{\alpha_k - 1} \varphi_k(t) \\
&+ \sum_{k=1}^{3} m_k \max_{t \in J} I_0^{\alpha_k} \mu_k(t) + n_k \max_{t \in J} I_0^{\alpha_k - 1} \mu_k(t).
\end{aligned}$$

In accordance with the previous estimates and the value of r, we deduce that $\Psi_1 x + \Psi_2 y \in B_r$.

Next we show the contraction of Ψ_1. Let $x, y \in X^3$, then

$$\begin{aligned}
\left|\Psi_{k,1} x_k(t) - \Psi_{k,1} y_k(t)\right| &\leq I_0^{\alpha_k} \left|f_k\big(t, x(t)\big) - f_k\big(t, y(t)\big)\right| \\
&\leq I_0^{\alpha_k} \varphi_k(t) \|x - y\|_{X^3} \\
&\leq \max_{t \in J} I_0^{\alpha_k} \varphi_k(t) \|x - y\|_{X^3}.
\end{aligned}$$

Hence

$$\|\Psi_1 x - \Psi_1 y\|_{X^3} \leq \left(\sum_{k=1}^{3} \max_{t\in J} I_0^{\alpha_k}\varphi_k(t)\right)\|x-y\|_{X^3}.$$

Since $\max_{t\in J} I_0^{\alpha_k}\varphi_k(t) \leq m_k \max_{t\in J} I_0^{\alpha_k}\varphi_k(t) < 1$, we deduce the contraction.

The last step shows the compactness of Ψ_2. It is obvious by (3.2) that Ψ_2 maps bounded sets into bounded sets. On the other hand, the continuity of f_k and its fractional integral would imply the continuity of the operator Ψ_2. The only thing we add is the equicontinuity of the family $\Psi_2 B_r$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, then we have

$$\left|\Psi_{k,2}x_k(t_2) - \Psi_{k,2}x_k(t_1)\right| \leq (t_2 - t_1)\sum_{m=1}^{3}|b_{k,m}|\bigl(I_0^{\alpha_m-1}\bigl(r\varphi_m(T)+\mu_m(T)\bigr)\bigr).$$

Accordingly, we find that

$$\|\Psi_2 x\|_{X^3} \leq (t_2 - t_1)\sum_{m=1}^{3}\left(\bigl(I_0^{\alpha_m-1}\bigl(r\varphi_m(T)+\mu_m(T)\bigr)\bigr)\sum_{m=1}^{3}|b_{k,m}|\right),$$

which tends to zero as $t_1 \to t_2$ independently of x. Hence, by the Arzelà–Ascoli theorem, the operator Ψ_2 is compact. Using Krasnoselskii Theorem 2.7, there exists a fixed point $x \in B_r \subset X^3$ satisfying the operator equation $x = \Psi_1 x + \Psi_2 x$, which is the solution of tripled system (1.1). This completes the proof. □

4 Application

Corresponding to system (1.1), we consider the following tripled fractional system:

$$\begin{cases} {}^cD_0^{\frac{6}{5}}x_1(t) = \frac{t}{20} + \frac{1}{\sqrt{169+t^2}}\bigl(\frac{|x_1(t)|}{1+|x(t)|} + \frac{|x_2(t)|}{1+|y(t)|} + \frac{|x_3(t)|}{1+|z(t)|}\bigr), \\ {}^cD_0^{\frac{3}{2}}x_2(t) = \frac{1}{\sqrt{64+t^2}} + \frac{t|x_1(t)|}{20} + \frac{t|x_2(t)|}{15(t^3+1)} + \frac{t|x_3(t)|}{15(2+t)}, \\ {}^cD_0^{\frac{9}{8}}x_3(t) = \frac{e^{-t}}{10+t} + \frac{|x_1(t)|}{\sqrt{169+t^2}} + \frac{|x_2(t)|}{12+t} + \frac{|x_3(t)|}{15\sqrt{1+t^2}}, \end{cases} \tag{4.1}$$

where $t \in [0,1]$ with

$$\begin{cases} x_1(0) = x_2(1), \quad 5x_1'(0) = 2x_2'(1), \\ 2x_2(0) = x_3(1), \quad 7x_2'(0) = 2x_3'(1), \\ 3x_3(0) = x_1(1), \quad 4x_3'(0) = 3x_1'(1). \end{cases} \tag{4.2}$$

Using the given data, we find the following constants:

$$a_{1,0} = 1, \qquad a_{1,1} = \frac{2}{5}, \qquad a_{2,0} = \frac{1}{2}, \qquad a_{2,1} = \frac{2}{7},$$

$$a_{3,0} = \frac{1}{3}, \qquad a_{3,1} = \frac{3}{4},$$

$$b_{1,1} = b_{2,2} = b_{3,3} = 0.09, \qquad b_{1,2} = 0.4375, \qquad b_{1,3} = 0.125,$$

$$b_{2,1} = 0.2344, \qquad b_{2,3} = 0.3125, \qquad b_{3,1} = 0.8203, \qquad b_{3,2} = 0.3281,$$

$$\begin{aligned} &d_{1,1} = d_{2,1} = d_{2,2} = d_{3,3} = 0.2, \qquad d_{1,2} = 1.2, \qquad d_{1,3} = d_{2,3} = 0.6, \\ &d_{3,1} = 0.4 = d_{3,2}, \qquad e_{1,1} = 0.7921, \qquad e_{1,2} = 0.397, \qquad e_{1,3} = 0.4562, \\ &e_{2,1} = 0.5578, \qquad e_{2,2} = 0.3031, \qquad e_{2,3} = 0.1437, \\ &e_{3,1} = 0.2953, \qquad e_{3,2} = 0.2781, \qquad e_{3,3} = 0.1937, \\ &L_1 = \frac{1}{13}, \qquad L_2 = \frac{1}{15}, \qquad L_3 = \frac{1}{12}. \end{aligned} \tag{4.3}$$

Then we deduce that $\eta \approx 0.95 < 1$. Hence by Theorem 3.1 there is a unique solution for system (4.1). Furthermore, we have

$$\begin{aligned} &\varphi_1(t) = \frac{1}{\sqrt{169 + t^2}}, \qquad \varphi_2(t) = \frac{t}{15}, \qquad \varphi_3(t) = \frac{1}{12}, \\ &\mu_1(t) = \frac{t}{20}, \qquad \mu_2(t) = \frac{1}{\sqrt{64 + t^2}}, \qquad \mu_3(t) = \frac{e^{-t}}{10 + t}, \end{aligned}$$

and

$$\begin{aligned} &m_1 = 1.8, \qquad m_2 = 2.8, \qquad m_3 = 2.4, \\ &n_1 = 2.79, \qquad n_2 = 1.834, \qquad n_3 = 1.3211. \end{aligned}$$

Hence

$$\sum_{k=1}^{3} m_k \max_{t\in J} I_0^{\alpha_k} \varphi_k(t) + n_k \max_{t\in J} I_0^{\alpha_k - 1} \varphi_k(t) = 0.812 < 1.$$

Therefore, using Theorem 3.2, there exists a solution of system (4.1). The reduction of the condition value from 0.95 to 0.812 is substantial. However, we lose the uniqueness property of the solution.

The used permutation in the boundary condition (4.2) has the form (1 2 3). Let us use another permutation of the boundary conditions for system (4.1) that has the form (1 3)(2) such that

$$\begin{cases} x_1(0) = x_3(1), & 5x_1'(0) = 2x_3'(1), \\ 2x_2(0) = x_2(1), & 7x_2'(0) = 2x_2'(1), \\ 3x_3(0) = x_1(1), & 4x_3'(0) = 3x_1'(1) \end{cases}$$

with the same constants as (4.3). Hence we deduce the same results as in the previous example. Furthermore, one can use four other permutations, namely (1 2)(3), (1)(2 3), (1 3 2), and identity (1)(2)(3).

5 Conclusion

In this paper, we investigate a tripled system of three fractional differential equations of order $\alpha \in (1, 2]$. The existence and uniqueness of solutions of the proposed system associated with cyclic permutation boundary conditions are established. The Banach and Krasnoselskii fixed point theorems are used as tools to prove our main results. We present examples to illustrate the applicability of the main results.

We study a fractional system consisting of three associated equations together with a new type of boundary conditions that is related to permutation groups. This might be a novel approach that will provide substantial potential for developing more new ideas in this field.

The results of this paper can be extended to a tripled system of fractional equations with impulsive effects and nonlocal conditions. Indeed, a tripled fractional system along with different boundary conditions can be considered and discussed. Finally, the results of this paper can be extended to m-tuple fractional systems. We leave investigation of these topics as future work for interested readers.

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Authors' contributions
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