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On decay and blow-up of solutions for a nonlinear Petrovsky system with conical degeneration

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Abstract

This paper deals with a class of Petrovsky system with nonlinear damping

$$w_{tt} + \Delta_{\mathbb{B}}^2 w - k_2 \Delta_{\mathbb{B}} w_t + a w_t |w_t|^{m-2} = b w |w|^{p-2}$$

on a manifold with conical singularity, where $\Delta_{\mathbb{B}}$ is a Fuchsian-type Laplace operator with totally characteristic degeneracy on the boundary $x_1 = 0$. We first prove the global existence of solutions under conditions without relation between m and p, and establish an exponential decay rate. Furthermore, we obtain a finite time blow-up result for local solutions with low initial energy $E(0) < d$.

MSC: 35B44; 35K30; 35K59

Keywords: Petrovsky system; Cone Sobolev spaces; Global existence; Decay rate; Blow-up

1 Introduction

Due to the frequent occurrence of high order nonlinear wave equations in many branches of engineering, physics, chemistry, material science, and other sciences, the study of wave equations plays a key role in mathematical analysis. For more details, see [1, 2]. In [3] and [4], the original Petrovsky model has the following form:

$$w_{tt} + \Delta^2 w - \Delta w_t + w_t |w_t|^{m-2} = w|w|^{p-2}, \quad x \in \Omega, t > 0, \tag{1.1}$$

$$w = 0, \qquad \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega, t \geq 0, \tag{1.2}$$

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \quad x \in \bar{\Omega}, \tag{1.3}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

Equation (1.1) is an important physical model that appears in many applications to mathematical physics as well as in the theory of vibrating plates, geophysics, and ocean acoustics [5, 6]. Some further physical interpretations are given in [7, 8].

For Equation (1.1), many results for global existence, nonexistence, and asymptotic behavior of solutions have been obtained [3–11]. Li et al. [3] studied problem (1.1)–(1.3) and derived that the solution is global without the relation between m and p. Moreover, the decay estimates of the energy function and the estimates of the lifespan of solution were given. Later, under suitable conditions decay estimates of the solutions for Equation (1.1) have been established by using Nakao's inequality in [4]. Messaoudi [9] proved the solution for problem (1.1)–(1.3) without Δw_t blows up in finite time if $p > m$ and the energy is negative. Wu [10] proved the blow-up result for problem (1.1)–(1.3) without Δw_t if $p > m$ and the energy is nonnegative. Recently, Chen et al. [11] proved that the solution of problem (1.1)–(1.3) without Δw_t blows up with positive initial energy and claimed that the solution blows up in finite time for even vanishing initial energy for $m = 2$. More recently, Philippin et al. [12] used a differential inequality technique to obtain a lower bound on blow-up time for Equation (1.1) without Δw_t. In recent years, lower bounds for the blow-up time in a superlinear hyperbolic equation with damping term have been derived [13]. For other related works, we refer the readers to [14–18] and the references therein.

In 2011 to 2012, Chen et al. established the corresponding Sobolev inequality on the cone Sobolev spaces in [19, 20]. And on this basis, they studied the initial boundary value problem of a semilinear parabolic equation on a manifold with conical singularity [21] and obtained the existence and nonexistence results by introducing a family of potential wells. Li et al. [22] proved the global existence, exponential decay, and finite time blow-up of solution for a class of semilinear pseudo-parabolic equations with conical degeneration. Recently, Alimohammady et al. [23] studied a class of semilinear degenerate hyperbolic equations on the cone Sobolev spaces

$$w_{tt} - \Delta_{\mathbb{B}} w + V(x)w + \gamma w_t = g_t(x)w|w|^{p-1}, \quad x \in \operatorname{int}\mathbb{B}, t > 0, \tag{1.4}$$

$$w(t,x) = 0, \quad x \in \partial\mathbb{B}, t \geq 0, \tag{1.5}$$

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \quad x \in \operatorname{int}\mathbb{B}, \tag{1.6}$$

where $\mathbb{B} = [0,1) \times X$, X is an $(n-1)$-dimensional closed compact manifold, which is regarded as the local model near the conical points and $\partial\mathbb{B} = \{0\} \times X$. $\Delta_{\mathbb{B}} = (x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$.

They discussed the invariance of some sets, global existence, nonexistence, and asymptotic behavior of solutions with initial energy $J(w_0) < d$ by introducing a family of potential wells which was first proposed by Sattinger [24]. More works on equations with conical degeneration can be seen in the literature [25–28] and the references therein.

If we consider Equation (1.1) on a manifold with conical singularity, that is, when the standard Laplace operator Δ of Equation (1.1) is replaced by Fuchsian-type Laplace operator $\Delta_{\mathbb{B}}$, what will happened for the initial boundary value problem? For this kind of Petrovsky equation with conical degeneration, the existence and nonexistence of global solutions to both the initial boundary value problem and the initial value problem remain open.

Inspired by the ideas of [3, 4, 23] and [29–31], we study the initial boundary value problem for the following Petrovsky equation:

$$w_{tt} + \Delta_{\mathbb{B}}^2 w - k_2 \Delta_{\mathbb{B}} w_t + a w_t |w_t|^{m-2} = b w|w|^{p-2}, \quad x \in \operatorname{int}\mathbb{B}, t > 0, \tag{1.7}$$

$$w = 0, \qquad \nabla_{\mathbb{B}} w \cdot \nu = 0, \quad x \in \partial\mathbb{B}, t \geq 0, \tag{1.8}$$

$$w(x,0) = w_0(x), \qquad w_t(x,0) = w_1(x), \quad x \in \operatorname{int}\mathbb{B}, \tag{1.9}$$

where $w_0(x)$, $w_1(x)$ are suitable initial data and k_2, a, b, m, p are constants such that k_2 and b are positive, a is nonnegative, and $m \geq 2$, $2 < p < \frac{2n}{n-2} = p^*$, where p^* is the critical Sobolev exponents. Here $\mathbb{B}$ is defined as above, and ν is the unit normal vector pointing toward the exterior of $\mathbb{B}$. Moreover, the operator $\Delta_{\mathbb{B}}$ in (1.7) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with conical degeneration on the boundary $x_1 = 0$ (we also called it Fuchsian-type Laplace operator), and the divergence operator $\operatorname{div}_{\mathbb{B}}$ is defined by $x_1\partial_{x_1} + \partial_{x_2} + \cdots + \partial_{x_n}$, the corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$. In the neighborhood of $\partial\mathbb{B}$ we will use coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1$, $x' \in X$.

Our main aim in this paper is to find the existence and nonexistence of solutions for problem (1.7)–(1.9) with cone degeneration by introducing a family of potential wells. Firstly, under the condition of low initial energy, we establish the existence of global solution in the cone Sobolev space by a combination of Galerkin method and potential well theory. Then, using the energy perturbation technique, we obtain the exponential decay result of the global solution. Finally, we show that the solution of the problem blows up in a finite time and give the estimates for lower and upper bounds of blow-up time. It is worth mentioning that two types of lower bounds of the blow-up time $T_{\max}$ for the weak solution of (1.7)–(1.9) are given, respectively.

The rest of this article is organized as follows. In Sect. 2, we recall the cone Sobolev spaces and the corresponding properties. In Sect. 3, we establish a global existence result and show the decay rates. In Sect. 4, we prove the blow-up properties of local solution.

2 Preliminaries

In this section, we recall the manifold with conical singularities and the cone Sobolev spaces which were introduced in [19, 20] and introduce some lemmas and notations.

We assume that the manifold B has only one conical point on the boundary. Thus, near the conical point, we have a stretched manifold $\mathbb{B}$ associated with B. Here $\mathbb{B} = [0,1) \times X$, $\partial\mathbb{B} = \{0\} \times X$ and X is a closed compact manifold of dimension $n-1$. Also, in the neighborhood of the conical point, we use coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1$, $x' \in X$.

Definition 2.1 Let $\mathbb{B} = [0,1) \times X$ be a stretched manifold of the manifold B with conical singularity. Then the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, and $1 < p < \infty$ is defined as

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \left\{u \in W_{\mathrm{loc}}^{m,p}(\operatorname{int}\mathbb{B}) | \omega u \in \mathcal{H}_p^{m,\gamma}\left(X^{\Lambda}\right)\right\}$$

for any cut-off function ω supported by a collar neighborhood of $(0,1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) = [\omega]\mathcal{H}_{p,0}^{m,\gamma}\left(X^{\Lambda}\right) + [1-\omega]W_0^{m,p}(\operatorname{int}\mathbb{B}),$$

where $X^{\Lambda} = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base X, $W_0^{m,p}(\operatorname{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\operatorname{int}\mathbb{B})$ in Sobolev spaces $W^{m,p}(\bar{X})$ when $\bar{X}$ is a closed compact C^∞ manifold of dimension n that contains B as a submanifold with boundary.

Remark 2.1 ([32]) We have the following properties:

(1) $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is a Banach space for $1 \le p < \infty$ and is a Hilbert space for $p = 2$.

(2) $L_p^\gamma(\mathbb{B}) := \mathcal{H}_p^{0,\gamma}(\mathbb{B})$.

(3) $L_p(\mathbb{B}) := \mathcal{H}_p^{0,0}(\mathbb{B})$.

(4) The embedding $\mathcal{H}_p^{m,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{m',\gamma'}(\mathbb{B})$ is continuous if $m \ge m'$, $\gamma \ge \gamma'$; and is compact embedding if $m > m'$, $\gamma > \gamma'$.

Definition 2.2 Let $\mathbb{B} = [0,1) \times X$. Then $u(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < \infty$ and $\gamma \in \mathbb{R}$ if

$$\|u(x)\|^p_{L_p^\gamma(\mathbb{B})} = \int_{\mathbb{B}} x_1^n \left|x_1^{-\gamma} u(x)\right|^p \frac{dx_1}{x_1}\, dx' < +\infty.$$

Observe that if $u(x) \in L_p^{\frac{n}{p}}(\mathbb{B})$, $v(x) \in L_q^{\frac{n}{q}}(\mathbb{B})$ with $p, q \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have the following Hölder inequality:

$$\int_{\mathbb{B}} \left|u(x)v(x)\right| \frac{dx_1}{x_1}\, dx' \le \left(\int_{\mathbb{B}} \left|u(x)\right|^p \frac{dx_1}{x_1}\, dx'\right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} \left|v(x)\right|^q \frac{dx_1}{x_1}\, dx'\right)^{\frac{1}{q}}. \tag{2.1}$$

In the sequel, for convenience we denote

$$(u,v)_2 = \int_{\mathbb{B}} u(x)v(x)\frac{dx_1}{x_1}\, dx', \qquad \|u\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} = \int_{\mathbb{B}} \left|u(x)\right|^p \frac{dx_1}{x_1}\, dx'.$$

$$\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) := \left\{u(x) \in \mathcal{H}_2^{1,\frac{n}{2}}(\mathbb{B}) | u = 0 \text{ on } \partial\mathbb{B}\right\},$$

$$\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}) := \left\{u(x) \in \mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B}) | u = \nabla_{\mathbb{B}} u \cdot \nu = 0 \text{ on } \partial\mathbb{B}\right\},$$

$$\|u\|^2_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} = \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})},$$

$$\|u\|^2_{\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})} = \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\Delta_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

The spaces $\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})$ with norms $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$, $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})}$ are Banach spaces, where the norms $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$, $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})}$ are equivalent to the norms $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$, $\|\Delta_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$, respectively.

Lemma 2.1 *Let* $u(x), v(x) \in \tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. *Then*

$$\int_{\mathbb{B}} v \Delta_{\mathbb{B}} u \frac{dx_1}{x_1}\, dx' = -\int_{\mathbb{B}} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \frac{dx_1}{x_1}\, dx'. \tag{2.2}$$

Proof Here we first suppose $u(x), v(x) \in C_0^\infty(\mathbb{B})$. From the definition of $\Delta_{\mathbb{B}}$, it follows that

$$\begin{aligned}
&\int_{\mathbb{B}} v\Delta_{\mathbb{B}} u \frac{dx_1}{x_1}\,dx' \\
&= \int_{\mathbb{B}} x_1\partial_{x_1}(x_1\partial_{x_1}u)\cdot v\frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} \left(\partial_{x_2}^2 u + \cdots + \partial_{x_n}^2 u\right)\cdot v\frac{dx_1}{x_1}\,dx' \\
&= \int_{\mathbb{B}} \partial_{x_1}(x_1\partial_{x_1}u)\cdot v\,dx + \int_{\mathbb{B}} \left(\partial_{x_2}^2 u + \cdots + \partial_{x_n}^2 u\right)\cdot v\frac{dx_1}{x_1}\,dx' \\
&= \int_{\mathbb{B}} \operatorname{div}\left(x_1\partial_{x_1}u, \frac{\partial_{x_2}u}{x_1}, \dots, \frac{\partial_{x_n}u}{x_1}\right)\cdot v\,dx \\
&= -\int_{\mathbb{B}} \left(x_1\partial_{x_1}u, \frac{\partial_{x_2}u}{x_1}, \dots, \frac{\partial_{x_n}u}{x_1}\right)\cdot \nabla v\,dx \\
&= -\int_{\mathbb{B}} \left(x_1^2\partial_{x_1}u, \partial_{x_2}u, \dots, \partial_{x_n}u\right)\cdot \nabla v\frac{dx_1}{x_1}\,dx' \\
&= -\int_{\mathbb{B}} (x_1\partial_{x_1}u, \partial_{x_2}u, \dots, \partial_{x_n}u)\cdot(x_1\partial_{x_1}v, \partial_{x_2}v, \dots, \partial_{x_n}v)\frac{dx_1}{x_1}\,dx' \\
&= -\int_{\mathbb{B}} \nabla_{\mathbb{B}}u\cdot\nabla_{\mathbb{B}}v\frac{dx_1}{x_1}\,dx'.
\end{aligned} \tag{2.3}$$

Finally, since $C_0^\infty(\mathbb{B})$ is dense in $\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, the equation above holds in the case of $u(x), v(x) \in \tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. □

Lemma 2.2 ([21], Poincaré inequality) *Let $\mathbb{B} = [0,1) \times X$ be a bounded subspace in $\mathbb{R}_+^n$ with $X \subset \mathbb{R}^{n-1}$, and $1 < p < +\infty$, $\gamma \in \mathbb{R}$. If $u(x) \in \tilde{\mathcal{H}}_{p,0}^{1,\gamma}(\mathbb{B})$, then*

$$\|u(x)\|_{L_p^\gamma(\mathbb{B})} \le c_\star \|\nabla_{\mathbb{B}}u(x)\|_{L_p^\gamma(\mathbb{B})}, \tag{2.4}$$

where $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$ and the constant $c_\star$ depends only on $\mathbb{B}$.

Lemma 2.3 ([21]) *For $1 < p < \frac{2n}{n-2}$, the embedding $\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \tilde{\mathcal{H}}_{p,0}^{0,\frac{n}{p}}(\mathbb{B})$ is continuous.*

From Lemma 2.2 and Lemma 2.3, we obtain the following lemma.

Lemma 2.4 *For $1 < p < p^*$, we have*

$$\|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})} \le C_0 \|\Delta_{\mathbb{B}}u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \tag{2.5}$$

for $u \in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})$ holds, where constant C_0 depends only on $\mathbb{B}$ and p.

3 Global existence and energy decay

In this section, we discuss the global existence and decay of the solution for problem (1.7)–(1.9).

Similar to the classical case, we introduce the following functionals on cone Sobolev space $\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})$:

$$J(w) = \frac{1}{2}\int_{\mathbb{B}} |\Delta_{\mathbb{B}}w|^2 \frac{dx_1}{x_1}\,dx' - \frac{b}{p}\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx', \tag{3.1}$$

$$I(w) = \int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' - b\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx'. \tag{3.2}$$

We also define the energy function as follows:

$$E(t) = \frac{1}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' + \frac{1}{2}\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' - \frac{b}{p}\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx'. \tag{3.3}$$

Finally, we introduce the potential well

$$W = \left\{w \in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}))|I(w) > 0\right\} \cup \{0\} \tag{3.4}$$

and the outside sets of the corresponding potential well

$$V = \left\{w \in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}))|I(w) < 0\right\}. \tag{3.5}$$

Remark 3.1 By (3.3) and Lemma 2.4, we know that

$$E(t) \geq \frac{1}{2}\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' - \frac{b}{p}\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx' \geq g\big(\|\Delta_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\big), \tag{3.6}$$

where $g(\lambda) = \frac{1}{2}\lambda^2 - \frac{bC_0^p}{p}\lambda^p$ and C_0 is given in Lemma 2.4. A direct calculation shows that $g(\lambda)$ has the maximum value at

$$\lambda_1 = \left(\frac{1}{bC_0^p}\right)^{\frac{1}{p-2}}$$

and the maximum value is

$$d = g(\lambda_1) = \frac{p-2}{2p}\left(\frac{1}{bC_0^p}\right)^{\frac{2}{p-2}} = \frac{p-2}{2p}\lambda_1^2 > 0. \tag{3.7}$$

By the definition of $g(\lambda)$ and $J(w)$, we can give another definition of d as follows:

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda w), w \in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}), \int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' \neq 0\right\} > 0, \tag{3.8}$$

and the Nehari manifold

$$\mathcal{N} = \left\{w \in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}) \,\middle|\, I(w) = 0, \int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' \neq 0\right\}. \tag{3.9}$$

Similar to the results in [29], one has $0 < d = \inf_{w\in\mathcal{N}} J(w)$.

The next lemma shows that our energy functional $E(t)$ is a nonincreasing function along the solution of (1.7)–(1.9).

Lemma 3.1 *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt}E(t) = -k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' - a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' \leq 0. \tag{3.10}$$

Proof Multiplying (1.7) by w_t and integrating it over $\mathbb{B}\times[0,t)$, we obtain

$$E(t)-E(0)=-\int_0^t\left(k_2\int_{\mathbb{B}}|\nabla_{\mathbb{B}}w_\tau|^2\frac{dx_1}{x_1}\,dx'+a\int_{\mathbb{B}}|w_\tau|^m\frac{dx_1}{x_1}\,dx'\right)d\tau \tag{3.11}$$

for $t\geq 0$. Thus, the proof is completed. □

Lemma 3.2 *Assume that* $E(0)<d$. *Then*:

(i) *If* $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_1$, *then* $\|\Delta_{\mathbb{B}}w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_1$ *for* $t\geq 0$.

(ii) *If* $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}>\lambda_1$, *then there exists* $\lambda_2>\lambda_1$ *such that* $\|\Delta_{\mathbb{B}}w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\geq\lambda_2$ *for* $t\geq 0$.

Proof From the definition of $g(\lambda)$, we see that $g(\lambda)$ is increasing in $(0,\lambda_1)$, decreasing in (λ_1,∞), and $g(\lambda)\to-\infty$ as $\lambda\to\infty$. Since $E(0)<d$, so there exist λ_2 and λ_2' such that $\lambda_2'<\lambda_1<\lambda_2$ and $g(\lambda_2')=g(\lambda_2)=E(0)$.

(i) When $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_1$, by (3.6), we have

$$g\bigl(\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr)\leq E(0)=g\bigl(\lambda_2'\bigr).$$

It implies $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_2'$. We claim that $\|\Delta_{\mathbb{B}}w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_2'$ for $t>0$. If not, then there exists $t_0>0$ such that $\|\Delta_{\mathbb{B}}w(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}>\lambda_2'$. If $\lambda_2'<\|\Delta_{\mathbb{B}}w(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_2$, then $g(\|\Delta_{\mathbb{B}}w(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})})>E(0)\geq E(t_0)$. It contradicts (3.6). If $\|\Delta_{\mathbb{B}}w(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\geq\lambda_2$, then by the continuity of $\|\Delta_{\mathbb{B}}w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$, there exists $0<t_1<t_0$ such that $\lambda_2'<\|\Delta_{\mathbb{B}}w(t_1)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}<\lambda_2$, then $g(\|\Delta_{\mathbb{B}}w(t_1)\|_{L_2^{\frac{n}{2}}(\mathbb{B})})>E(0)\geq E(t_1)$. This is a contradiction.

(ii) When $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}>\lambda_1$, as in case (i) we also deduce that $\|\Delta_{\mathbb{B}}w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})}>\lambda_1$ implies $\|\Delta_{\mathbb{B}}w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\geq\lambda_2$ for $t\geq 0$. □

Lemma 3.3 *Suppose that* $2<p<p^*$, $w_1\in L_2^{\frac{n}{2}}(\mathbb{B})$, *and* $E(0)<d$. *Let* $w_0\in W$ *such that*

$$\beta=bC_0^p\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}<1. \tag{3.12}$$

Then $w\in W$ *for each* $t\geq 0$.

Proof When $w=0$, we get $w\in W$ easily, so we just need to prove the case $w\neq 0$. Since $I(w_0)>0$, it follows from the continuity of w that

$$I(w)\geq 0 \tag{3.13}$$

for some interval near $t=0$. Let $T_m>0$ be a maximal time (possibly $T_m=T$) when (3.13) holds on $[0,T_m)$.

From (3.1)–(3.2), it follows that

$$\begin{aligned}J(w)&=\frac{p-2}{2p}\int_{\mathbb{B}}|\Delta_{\mathbb{B}}w|^2\frac{dx_1}{x_1}\,dx'+\frac{1}{p}I(w)\\&\geq\frac{p-2}{2p}\int_{\mathbb{B}}|\Delta_{\mathbb{B}}w|^2\frac{dx_1}{x_1}\,dx',\quad\text{on }t\in[0,T_m).\end{aligned} \tag{3.14}$$

By using (3.14), (3.3), and Lemma 3.1, we get

$$\begin{aligned}\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' &\le \frac{2p}{p-2} J(w)\\ &\le \frac{2p}{p-2} E(t)\\ &\le \frac{2p}{p-2} E(0).\end{aligned} \tag{3.15}$$

Then, by Lemma 2.4 and (3.15), we obtain

$$\begin{aligned} b\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} &\le bC_0^p \|\Delta_{\mathbb{B}} w\|^p_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ &\le bC_0^p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ &= \beta \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\\ &< \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \end{aligned} \tag{3.16}$$

on $t \in [0, T_m)$. Therefore, by using (3.2), we conclude that $I(w) > 0$ for all $t \in [0, T_m)$. By repeating the procedure, T_m is extended to T. The proof is completed. □

Remark 3.2 From Lemma 3.3, we can deduce that

$$\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \le \frac{1}{1-\beta} I(w). \tag{3.17}$$

Theorem 3.1 *Suppose that $2 < p < p^\star$, $w_1 \in L_2^{\frac{n}{2}}(\mathbb{B})$, and $E(0) < d$, let $w_0 \in W$ and w satisfy the assumption of Lemma 3.3. Then problem (1.7)–(1.9) admits a global weak solution $w(x,t) \in L^\infty([0,T]; \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}))$ with $w_t(x,t) \in L^2([0,T]; \tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})) \cap L^m([0,T]; L_m^{\frac{n}{m}}(\mathbb{B})) \cap L^\infty([0,T]; L_2^{\frac{n}{2}}(\mathbb{B}))$. Moreover, $w(t) \in W$ for $0 \le t < \infty$.*

Proof Let $\{\omega_j(x)\}$ be a system of base functions in $\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})$. Now we construct the following approximate solution $w_s(x,t)$ of problem (1.7)–(1.9):

$$w_s(x,t) = \sum_{j=1}^{s} g_{js}(t)\omega_j(x), \quad s = 1, 2, \ldots,$$

which satisfies

$$\begin{aligned}&(w_{stt}, \omega_j)_2 + (\Delta_{\mathbb{B}} w_s, \Delta_{\mathbb{B}} \omega_j)_2 + k_2(\nabla_{\mathbb{B}} w_{st}, \nabla_{\mathbb{B}} \omega_j)_2 + a\big(w_{st}|u_{st}|^{m-2}, \omega_j\big)_2\\ &\quad = b\big(w_s|u_s|^{p-2}, \omega_j\big)_2, \quad s = 1, 2, \ldots,\end{aligned} \tag{3.18}$$

$$w_s(x,0) = \sum_{j=1}^{s} g_{js}(0)\omega_j(x) \to w_0(x) \quad \text{in } \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}), \tag{3.19}$$

$$w_{st}(x,0) = \sum_{j=1}^{s} g'_{js}(0)\omega_j(x) \to w_1(x) \quad \text{in } L_2^{\frac{n}{2}}(\mathbb{B}). \tag{3.20}$$

Multiplying (3.18) by $g'_{js}(t)$, summing for j ($j = 1, 2, \dots, s$), and integrating from 0 to t, we obtain

$$k_2 \int_0^t \|\nabla_{\mathbb{B}} w_{s\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \, d\tau + a \int_0^t \|w_{s\tau}\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \, d\tau + E\big(w_s(t)\big)$$
$$= E\big(w_s(0)\big), \quad 0 \le t < \infty. \tag{3.21}$$

By (3.19) we can get $E(w_s(0)) \to E(w_0)$, then for sufficiently large m, we have

$$k_2 \int_0^t \|\nabla_{\mathbb{B}} w_{s\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \, d\tau + a \int_0^t \|w_{s\tau}\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \, d\tau + E\big(w_s(t)\big) < d,$$
$$0 \le t < \infty. \tag{3.22}$$

From (3.22) and the proof of Lemma 3.3, we can get $w_s(t) \in W$ for $0 \le t < \infty$ and sufficiently large s. Hence, by (3.22) and

$$E(w_s) = \frac{1}{2}\|w_{st}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{p-2}{2p}\|\Delta_{\mathbb{B}} w_s\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p} I(w_s), \tag{3.23}$$

we obtain

$$k_2 \int_0^t \|\nabla_{\mathbb{B}} w_{s\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \, d\tau + a \int_0^t \|w_{s\tau}\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \, d\tau + \frac{1}{2}\|w_{st}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$$
$$+ \frac{p-2}{2p}\|\Delta_{\mathbb{B}} w_s\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < d, \quad 0 \le t < \infty, \tag{3.24}$$

for sufficiently large s, which yields

$$\|\Delta_{\mathbb{B}} w_s\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < \frac{2p}{p-2} d, \quad 0 \le t < \infty, \tag{3.25}$$

$$\int_0^t \|\nabla_{\mathbb{B}} w_{s\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \, d\tau < \frac{d}{k_2}, \quad 0 \le t < \infty, \tag{3.26}$$

$$\int_0^t \|w_{s\tau}\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \, d\tau < \frac{d}{a}, \quad 0 \le t < \infty, \tag{3.27}$$

$$\|w_{st}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < 2d, \quad 0 \le t < \infty, \tag{3.28}$$

$$\int_{\mathbb{B}} \big||w_s|^{p-2} w_s\big|^{\frac{p}{p-1}} \frac{dx_1}{x_1} \, dx' = \int_{\mathbb{B}} |w_s|^p \frac{dx_1}{x_1} \, dx' = \|w_s\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}$$
$$\le C_0^p \|\Delta_{\mathbb{B}} w_s\|^p_{L_2^{\frac{n}{2}}(\mathbb{B})} \le C_0^p \left(\frac{2p}{p-2} d\right)^{\frac{p}{2}}, \tag{3.29}$$

$$\int_0^t \int_{\mathbb{B}} \big||w_{s\tau}|^{m-2} w_{s\tau}\big|^{\frac{m}{m-1}} \frac{dx_1}{x_1} \, dx' \, d\tau$$
$$= \int_0^t \int_{\mathbb{B}} |w_{s\tau}|^m \frac{dx_1}{x_1} \, dx' \, d\tau = \int_0^t \|w_{s\tau}\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \, d\tau < \frac{d}{a}. \tag{3.30}$$

Therefore, there exist w and a subsequence still denoted by $\{w_s\}$ for which, as $s \to \infty$,

$$
\begin{aligned}
&w_s \to w \quad \text{in } L^\infty\bigl(0,\infty;\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})\bigr) \text{ weakly star and a.e. in } \operatorname{int}\mathbb{B}\times[0,\infty),\\
&w_{st} \to w_t \quad \text{in } L^2\bigl(0,\infty;\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})\bigr) \text{ weakly},\\
&w_{st} \to w_t \quad \text{in } L^m\bigl(0,\infty;L_m^{\frac{n}{m}}(\mathbb{B})\bigr) \text{ weakly},\\
&w_{st} \to w_t \quad \text{in } L^\infty\bigl(0,\infty;L_2^{\frac{n}{2}}(\mathbb{B})\bigr) \text{ weakly star},\\
&w_s^{p-1} \to w^{p-1} \quad \text{in } L^\infty\bigl(0,\infty;L_{\frac{p}{p-1}}^{\frac{n(p-1)}{p}}(\mathbb{B})\bigr) \text{ weakly star},\\
&w_{st}^{m-1} \to w_t^{m-1} \quad \text{in } L^{\frac{m}{m-1}}\bigl(0,\infty;L_{\frac{m}{m-1}}^{\frac{n(m-1)}{m}}(\mathbb{B})\bigr) \text{ weakly}.
\end{aligned}
$$

In (3.18), we fix j, letting $s \to \infty$ and integrating from 0 to t. Then we have

$$
\begin{aligned}
&(w_t,\omega_j)_2 + \int_0^t (\Delta_{\mathbb{B}} w,\Delta_{\mathbb{B}}\omega_j)_2\,d\tau + k_2\int_0^t (\nabla_{\mathbb{B}} w_\tau,\nabla_{\mathbb{B}}\omega_j)_2\,d\tau\\
&\quad + a\int_0^t \bigl(w_\tau|u_\tau|^{m-2},\omega_j\bigr)_2\,d\tau = b\int_0^t \bigl(w|w|^{p-2},\omega_j\bigr)_2\,d\tau + (w_1,\omega_j)_2
\end{aligned}
\tag{3.31}
$$

and

$$
\begin{aligned}
&(w_t,v)_2 + \int_0^t (\Delta_{\mathbb{B}} w,\Delta_{\mathbb{B}} v)_2\,d\tau + k_2\int_0^t (\nabla_{\mathbb{B}} w_\tau,\nabla_{\mathbb{B}} v)_2\,d\tau\\
&\quad + a\int_0^t \bigl(w_\tau|u_\tau|^{m-2},v\bigr)_2\,d\tau = b\int_0^t \bigl(w|w|^{p-2},v\bigr)_2\,d\tau + (w_1,v)_2,\\
&\quad \forall v\in \tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}).
\end{aligned}
\tag{3.32}
$$

From (3.19) we obtain $w(x,0) = w_0(x)$ in $\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B})$ and $w_t(x,0) = w_1(x)$ in $L_2^{\frac{n}{2}}(\mathbb{B}), t \in (0,T)$. By density we obtain $w \in L^\infty([0,T];\tilde{\mathcal{H}}_{2,0}^{2,\frac{n}{2}}(\mathbb{B}))$ with $w_t \in L^2([0,T];\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})) \cap L^m([0,T];L_m^{\frac{n}{m}}(\mathbb{B})) \cap L^\infty([0,T];L_2^{\frac{n}{2}}(\mathbb{B}))$ is a global weak solution of problem (1.7)–(1.9). It is obvious that $w(t) \in W$ for $0 \le t < \infty$. □

Now, we use the following "modified" functional:

$$
G(t) = E(t) + \varepsilon\left(\int_{\mathbb{B}} w w_t \frac{dx_1}{x_1}\,dx' + \frac{k_2}{2}\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right).
$$

Lemma 3.4 *Let w satisfy the assumption of Theorem 3.1. For ε small enough, we have*

$$
\alpha_1 G(t) \le E(t) \le \alpha_2 G(t) \tag{3.33}
$$

holds for two positive constants α_1 and α_2.

Proof Making use of (3.23), straightforward computations lead to

$$
\begin{aligned}
G(t) &= E(t) + \varepsilon\left(\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\,dx' + \frac{k_2}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx'\right) \\
&\le E(t) + \frac{\varepsilon}{2}\int_{\mathbb{B}} |w|^2 \frac{dx_1}{x_1}\,dx' + \frac{\varepsilon}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\quad + \frac{\varepsilon k_2}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' \\
&\le E(t) + \frac{\varepsilon}{2}\left(k_2 + c_\star^2\right)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' + \frac{\varepsilon}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\le E(t) + \frac{\varepsilon}{2}C_1\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' + \frac{\varepsilon}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\le E(t) + \frac{\varepsilon}{2}\frac{2pC_1}{p-2}E(t) + \varepsilon E(t) \\
&\le \frac{1}{\alpha_1}E(t),
\end{aligned} \tag{3.34}
$$

and in the same way, we get

$$
\begin{aligned}
G(t) &\ge E(t) - \frac{\varepsilon}{2}C_1\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' - \frac{\varepsilon}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\ge E(t) - \frac{\varepsilon}{2}\frac{2pC_1}{p-2}E(t) - \varepsilon E(t) \\
&\ge \frac{1}{\alpha_2}E(t)
\end{aligned} \tag{3.35}
$$

for ε small enough. □

Theorem 3.2 *Suppose that* $2 \le m < m^* = \frac{n-2}{2n}$. *Let* $w(x,t)$ *satisfy the assumption of Theorem* 3.1. *Then we have the following decay estimates*:

$$
E(t) \le Ke^{-kt}, \quad t \ge 0, \tag{3.36}
$$

where K *and* k *are positive constants which will be defined later.*

Proof From the definition of $G(t)$, we get

$$
\begin{aligned}
G'(t) &= -k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' - a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' + \varepsilon\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\quad + \varepsilon\int_{\mathbb{B}} w(w_{tt} - k_2\Delta_{\mathbb{B}} w_t)\frac{dx_1}{x_1}\,dx' \\
&= -k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' - a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' + \varepsilon\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\quad + \varepsilon\int_{\mathbb{B}} b|w|^p \frac{dx_1}{x_1}\,dx' - \varepsilon\int_{\mathbb{B}} aww_t|u_t|^{m-2} \frac{dx_1}{x_1}\,dx' \\
&\quad - \varepsilon\int_{\mathbb{B}} w\Delta_{\mathbb{B}}^2 w \frac{dx_1}{x_1}\,dx'
\end{aligned}
$$

$$
\begin{aligned}
&= -k_2\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} + \varepsilon\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \varepsilon\Bigg(\frac{p}{2}\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad + \frac{p}{2}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - pE(t)\Bigg) - \varepsilon\int_{\mathbb{B}} aww_t|w_t|^{m-2}\frac{dx_1}{x_1}\,dx' \\
&\quad - \varepsilon\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&= -k_2\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} + \varepsilon\left(\frac{p}{2}+1\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad + \varepsilon\left(\frac{p}{2}-1\right)\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \varepsilon pE(t) \\
&\quad - \varepsilon\int_{\mathbb{B}} aww_t|w_t|^{m-2}\frac{dx_1}{x_1}\,dx'.
\end{aligned}
\tag{3.37}
$$

Using Lemma 2.2, we obtain

$$
\begin{aligned}
G'(t) &\le \left[\varepsilon\left(\frac{p}{2}+1\right)c_\star^2 - k_2\right]\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&\quad + \left(\frac{p}{2}-1\right)\varepsilon\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \varepsilon pE(t) \\
&\quad - \varepsilon a\int_{\mathbb{B}} ww_t|u_t|^{m-2}\frac{dx_1}{x_1}\,dx'.
\end{aligned}
\tag{3.38}
$$

Then, we will show that from the estimate of the last term in (3.38), by Young's inequality and the proof of Lemma 3.3, we obtain

$$
\left|\int_{\mathbb{B}} ww_t|u_t|^{m-2}\frac{dx_1}{x_1}\,dx'\right| \le \theta\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} + c(\theta)\|w\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})}, \tag{3.39}
$$

$$
\|w\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \le C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \tag{3.40}
$$

Then by exploiting (3.38)–(3.40), we arrive at

$$
\begin{aligned}
G'(t) &\le \left[\varepsilon\left(\frac{p}{2}+1\right)c_\star^2 - k_2\right]\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + (\varepsilon\theta-1)a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&\quad + \varepsilon\left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad - \varepsilon pE(t) \\
&\le \left[\varepsilon\left(\frac{p}{2}+1\right)c_\star^2 - k_2\right]\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + (\varepsilon\theta-1)a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&\quad + \varepsilon\left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]E(t) - \varepsilon pE(t) \\
&= \left[\varepsilon\left(\frac{p}{2}+1\right)c_\star^2 - k_2\right]\|\nabla_{\mathbb{B}} w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + (\varepsilon\theta-1)a\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&\quad - \varepsilon\left\{p - \left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]\right\}E(t).
\end{aligned}
\tag{3.41}
$$

Choose ε so small that $\varepsilon(\frac{p}{2}+1)c_\star^2 - k_2 \leq 0$, $\varepsilon\theta - 1 \leq 0$. And choose suitable θ such that $p - [ac(\theta)C_0^m(\frac{2p}{p-2}E(0))^{\frac{m-2}{2}} + (\frac{p}{2}-1)] > 0$. Then from the above inequality, we obtain

$$G'(t) \leq -\varepsilon\left\{p - \left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]\right\}E(t). \tag{3.42}$$

Then, by the relation between $E(t)$ and $G(t)$, we get

$$G'(t) \leq -\varepsilon\alpha_1\left\{p - \left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]\right\}G(t). \tag{3.43}$$

We take ε small enough such that

$$G(0) = E(0) + \varepsilon\left(\int_{\mathbb{B}} w_0 w_1 \frac{dx_1}{x_1}\,dx' + \frac{k_2}{2}\|\nabla_{\mathbb{B}} w_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right) > 0.$$

Integrating (3.43), we obtain

$$G(t) \leq G(0)e^{-kt}, \quad t \geq 0,$$

where

$$k = \varepsilon\alpha_1\left\{p - \left[ac(\theta)C_0^m\left(\frac{2p}{p-2}E(0)\right)^{\frac{m-2}{2}} + \left(\frac{p}{2}-1\right)\right]\right\} > 0.$$

By using (3.33) again, we get

$$E(t) \leq Ke^{-kt}, \quad t \geq 0,$$

where $K = \alpha_2 G(0)$. This completes the proof. □

4 Finite time blow-up of solution

In this section, we show that the solution of problem (1.7)–(1.9) blows up in finite time if $p > m$ and $E(0) < d$. For this purpose, we first give the following lemma which will be used later.

Lemma 4.1 *Suppose that $2 < p < p^\star$, $E(0) < d$, $w_1 \in L_2^{\frac{n}{2}}(\mathbb{B})$. Let $w_0 \in V$, then we have*

$$w(t) \in V, \quad \forall t \in [0,T), \tag{4.1}$$

$$d < \frac{p-2}{2p}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}, \quad \forall t \in [0,T). \tag{4.2}$$

Proof Let $w_0 \in V$, we have to prove that $w(t) \in V$ for all $t \in [0,T)$. We argue by contradiction. Assume that there exists $t_0 \in [0,T)$ such that $w(t_0) \notin V$. This implies that

$$\|\Delta_{\mathbb{B}} w(t_0)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq b\|w(t_0)\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}.$$

By the continuity of $w(t)$, there exists at least one $\bar{t} \in (0, t_0]$ such that

$$\left\| \Delta_{\mathbb{B}} w(\bar{t}) \right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = b \left\| w(\bar{t}) \right\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}.$$

Let

$$\tilde{t} = \inf\left\{ \bar{t} \in (0, t_0] : \left\| \Delta_{\mathbb{B}} w(\bar{t}) \right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = b \left\| w(\bar{t}) \right\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \right\}.$$

In particular, the regularity of $w(t)$ implies that $\tilde{t} \in (0, t_0]$. Thus, we know

$$\left\| \Delta_{\mathbb{B}} w(\tilde{t}) \right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = b \left\| w(\tilde{t}) \right\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}$$

and $w(t) \in V$ for all $t \in [0, \tilde{t})$. We have two cases to consider.

First case: $\|\Delta_{\mathbb{B}} w(\tilde{t})\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$.

In this case, by the continuity of $w(t)$, we have

$$\lim_{t \to \tilde{t}^-} \left\| \Delta_{\mathbb{B}} w(t) \right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0. \tag{4.3}$$

On the other hand, the fact that $w(t) \in V$ for all $t \in [0, \tilde{t})$ implies that $\|\Delta_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ and

$$\left\| \Delta_{\mathbb{B}} w(t) \right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < b \left\| w(t) \right\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}, \quad t \in [0, \tilde{t}). \tag{4.4}$$

By Lemma 2.4, we get

$$\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \le C_0^p \|\Delta_{\mathbb{B}} w\|^p_{L_2^{\frac{n}{2}}(\mathbb{B})}, \quad t \in [0, \tilde{t}). \tag{4.5}$$

Then, by (4.4), (4.5), we have

$$\lim_{t \to \tilde{t}^-} \left\| \Delta_{\mathbb{B}} w(t) \right\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \left(\frac{1}{bC_0^p} \right)^{\frac{1}{p-2}}.$$

This contradicts (4.3).

Second case: $\|\Delta_{\mathbb{B}} w(\tilde{t})\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$.

In this case, by recalling (3.8), we know that $J(w(\tilde{t})) \ge d$. Thus, $E(\tilde{t}) \ge d$, which contradicts the fact that $E(t) \le E(0) < d$. Hence, in either case we conclude that $w(t) \in V$ for all $t \in [0, T)$. Since

$$J(\lambda w) = \frac{1}{2}\lambda^2 \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{b}{p}\lambda^p \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})},$$

we obtain

$$\frac{d}{d\lambda} J(\lambda w) = \lambda \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - b\lambda^{p-1} \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}$$

and

$$\frac{d^2}{d\lambda^2}J(\lambda w) = \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - b(p-1)\lambda^{p-2}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}.$$

Let $\frac{d}{d\lambda}J(\lambda w) = 0$, which implies

$$\bar{\lambda}_1 = 0, \bar{\lambda}_2 = \left(\frac{\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{b\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}}\right)^{\frac{1}{p-2}}.$$

An elementary calculation shows

$$\frac{d^2}{d\lambda^2}J(\bar{\lambda}_1 w) > 0, \qquad \frac{d^2}{d\lambda^2}J(\bar{\lambda}_2 w) < 0.$$

So we have

$$\sup_{\lambda\geq 0} J(\lambda w) = J(\bar{\lambda}_2 w) = \frac{p-2}{2p}\frac{(\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})})^{\frac{p}{p-2}}}{(b\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})})^{\frac{2}{p-2}}}.$$

By $I(u) < 0$, we have

$$\begin{aligned} d &\leq \sup_{\lambda\geq 0} J(\lambda w) = J(\bar{\lambda}_2 w) = \frac{p-2}{2p}\frac{(\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})})^{\frac{p}{p-2}}}{(b\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})})^{\frac{2}{p-2}}} \\ &< \frac{p-2}{2p}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \end{aligned} \tag{4.6}$$

□

Lemma 4.2 *Let* $2 < p < p^\star$. *Then there exists a positive constant* C *depending only on* $\mathbb{B}$ *such that*

$$\|w\|^s_{L_p^{\frac{n}{p}}(\mathbb{B})} \leq C\left(\left\|\Delta_{\mathbb{B}} w(t)\right\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\right), \quad \text{with } 2 \leq s \leq p, \tag{4.7}$$

for any $w \in \tilde{\mathcal{H}}^{2,\frac{n}{2}}_{2,0}(\mathbb{B})$.

Proof If $\|w\|_{L_p^{\frac{n}{p}}(\mathbb{B})} \leq 1$, then $\|w\|^s_{L_p^{\frac{n}{p}}(\mathbb{B})} \leq \|w\|^2_{L_p^{\frac{n}{p}}(\mathbb{B})} \leq C_0^2\|\Delta_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$ by Lemma 2.4. If $\|w\|_{L_p^{\frac{n}{p}}(\mathbb{B})} > 1$, then $\|w\|^s_{L_p^{\frac{n}{p}}(\mathbb{B})} \leq \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}$. Therefore (4.7) follows. □

Now we introduce the following auxiliary function:

$$H(t) = d_1 - E(t), \quad t \geq 0, \tag{4.8}$$

where $d_1 = \frac{E(0)+d}{2} > 0$.

From Lemma 4.1 and Lemma 4.2, we obtain the following corollary.

Corollary 4.1 *Let the assumption of Lemma 4.2 hold. Then we have*

$$\|w\|^s_{L_p^{\frac{n}{p}}(\mathbb{B})} \le C\big(\big|H(t)\big| + \|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\big), \quad \text{with } 2 \le s \le p, \tag{4.9}$$

for any $w \in \tilde{\mathcal{H}}^{2,\frac{n}{2}}_{2,0}(\mathbb{B})$.

Theorem 4.1 *Suppose that* $2 < p < p^\star$ *and* $p > m \ge 2$, $w_1 \in L_2^{\frac{n}{2}}(\mathbb{B})$, $w_0 \in V$. *If one of the following is satisfied*:

(1) $0 \le E(0) < d$ *and* $\|\Delta_{\mathbb{B}} w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \lambda_1$;

(2) $E(0) < 0$,

then the local solution w of problem (1.7)–(1.9) *blows up in finite time*; *that is, the maximum existence time* $T_{\max}$ *of w is finite and*

$$\lim_{T\to T_{\max}^-}\big[\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + \|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\big] = +\infty.$$

Moreover, the lifespan $T_{\max}$ *is estimated by* $0 < T_{\max} \le \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}$, *here* $L(0)$ *and* Γ *are given in* (4.30) *and* (4.36) *respectively.* α *is a constant given in* (4.26).

Proof (1) For $0 \le E(0) < d$, from (4.8), it follows that

$$H'(t) = -E'(t) = k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' + a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' \ge 0. \tag{4.10}$$

Thus, we have

$$H(t) \ge H(0) = d_1 - E(0) > 0, \qquad t \ge 0. \tag{4.11}$$

Let

$$A(t) = \int_{\mathbb{B}} w(t)w_t(t)\frac{dx_1}{x_1}\,dx'. \tag{4.12}$$

By differentiating (4.12) and using (1.7), (4.8), we obtain

$$\begin{aligned}
&A'(t)\\
&\quad= \int_{\mathbb{B}} |w_t|^2\frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} ww_{tt}\frac{dx_1}{x_1}\,dx'\\
&\quad= \int_{\mathbb{B}} |w_t|^2\frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} w\big[-\Delta^2_{\mathbb{B}} w + k_2\Delta_{\mathbb{B}} w_t - aw_t|w_t|^{m-2}\\
&\qquad + bw|w|^{p-2}\big]\frac{dx_1}{x_1}\,dx'\\
&\quad= \|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - k_2\int_{\mathbb{B}} \nabla_{\mathbb{B}} w\cdot\nabla_{\mathbb{B}} w_t\frac{dx_1}{x_1}\,dx'\\
&\qquad + b\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} - a\int_{\mathbb{B}} ww_t|u_t|^{m-2}\frac{dx_1}{x_1}\,dx'
\end{aligned}$$

$$
\begin{aligned}
&= \left(1+\frac{p}{2}\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \left(\frac{p}{2}-1\right)\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad - a\int_{\mathbb{B}} w w_t |w_t|^{m-2}\frac{dx_1}{x_1}\,dx' - k_2\int_{\mathbb{B}} \nabla_{\mathbb{B}} w\cdot\nabla_{\mathbb{B}} w_t \frac{dx_1}{x_1}\,dx' \\
&\quad + pH(t) - pd_1. \qquad (4.13)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left(\frac{p}{2}-1\right)\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - pd_1 \\
&= \left(\frac{p}{2}-1\right)\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \left(\frac{p}{2}-1\right)\lambda_1^2\frac{\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\lambda_2^2} - pd_1 \\
&\geq c_1\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + c_2, \qquad (4.14)
\end{aligned}
$$

where λ_2 is given in Lemma 3.2, $c_1 = (\frac{p}{2}-1)\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}$ and $c_2 = (\frac{p}{2}-1)\lambda_1^2 - pd_1$. By Lemma 3.2(ii), we have $c_1 > 0$, and by (3.7), we see that

$$
\begin{aligned}
c_2 &= \left(\frac{p}{2}-1\right)\lambda_1^2 - pd_1 \\
&= \left(\frac{p}{2}-1\right)\lambda_1^2 - \frac{p(d+E(0))}{2} \\
&= pd - \frac{p(d+E(0))}{2} \\
&= \frac{p(d-E(0))}{2} > 0. \qquad (4.15)
\end{aligned}
$$

Thus, by (4.13)–(4.15), we arrive at

$$
\begin{aligned}
A'(t) &> \left(1+\frac{p}{2}\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + c_1\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - a\int_{\mathbb{B}} w w_t|w_t|^{m-2}\frac{dx_1}{x_1}\,dx' \\
&\quad - k_2\int_{\mathbb{B}} \nabla_{\mathbb{B}} w\cdot\nabla_{\mathbb{B}} w_t\frac{dx_1}{x_1}\,dx' + pH(t). \qquad (4.16)
\end{aligned}
$$

We estimate the right-hand side of the above equation. By Hölder's inequality and the inequality $\|w\|_{L_m^{\frac{n}{m}}(\mathbb{B})} \leq C\|w\|_{L_p^{\frac{n}{p}}(\mathbb{B})}$, we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{B}} w w_t|w_t|^{m-2}\frac{dx_1}{x_1}\,dx'\right| &\leq \|w\|_{L_m^{\frac{n}{m}}(\mathbb{B})}\|w_t\|^{m-1}_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&\leq C\|w\|_{L_p^{\frac{n}{p}}(\mathbb{B})}\|w_t\|^{m-1}_{L_m^{\frac{n}{m}}(\mathbb{B})} \\
&= C\|w\|^{1-\frac{p}{m}}_{L_p^{\frac{n}{p}}(\mathbb{B})}\|w\|^{\frac{p}{m}}_{L_p^{\frac{n}{p}}(\mathbb{B})}\|w_t\|^{m-1}_{L_m^{\frac{n}{m}}(\mathbb{B})}. \qquad (4.17)
\end{aligned}
$$

Note that from (4.8) and (4.2) we get

$$\begin{aligned} H(t) &= d_1 - E(t) \\ &< d - \frac{1}{2}\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{2}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{b}{p}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\ &< \frac{p-2}{2p}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{2}\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{2}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\quad + \frac{b}{p}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\ &\le \frac{b}{p}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}. \end{aligned} \tag{4.18}$$

Thus, by (4.11) and (4.18), we see that

$$0 < H(0) \le H(t) < \frac{b}{p}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}, \quad t \ge 0. \tag{4.19}$$

Then, using (4.19), we have from (4.17) that

$$\left|\int_{\mathbb{B}} w w_t |u_t|^{m-2}\frac{dx_1}{x_1}\,dx'\right| \le C\left(\frac{p}{b}H(t)\right)^{\frac{1}{p}(1-\frac{p}{m})} \|w\|^{\frac{p}{m}}_{L_p^{\frac{n}{p}}(\mathbb{B})} \|w_t\|^{m-1}_{L_m^{\frac{n}{m}}(\mathbb{B})}. \tag{4.20}$$

Hence, by Young's inequality and (4.10), we obtain

$$\begin{aligned} &a\left|\int_{\mathbb{B}} w w_t |w_t|^{m-2}\frac{dx_1}{x_1}\,dx'\right| \\ &\quad\le c_3 H(t)^{-\alpha^*}\left(\frac{a\theta^m}{m}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + \frac{a(m-1)}{m}\theta^{-m/(m-1)}\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})}\right) \\ &\quad\le c_4 H(t)^{-\alpha^*}\left(\theta^m\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + a\theta^{-m/(m-1)}\|w_t\|^m_{L_m^{\frac{n}{m}}(\mathbb{B})}\right), \end{aligned} \tag{4.21}$$

where $c_3 = C(\frac{p}{b})^{\frac{1}{p}-\frac{1}{m}}$, $\alpha^* = \frac{1}{m} - \frac{1}{p} > 0$, $\theta > 0$, and $c_4 = c_3 \max\{\frac{a}{m}, \frac{m-1}{m}\}$.

Letting $0 < \alpha < \alpha^*$ and by (4.19), we see that

$$\begin{aligned} &a\left|\int_{\mathbb{B}} w w_t |w_t|^{m-2}\frac{dx_1}{x_1}\,dx'\right| \\ &\quad\le c_4\left[\theta^m H(0)^{-\alpha^*}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + \theta^{-m/(m-1)}H(t)^{-\alpha^*}H'(t)\right] \\ &\quad\le c_4\left[\theta^m H(0)^{-\alpha^*}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + \theta^{-m/(m-1)}H(0)^{\alpha-\alpha^*}H(t)^{-\alpha}H'(t)\right]. \end{aligned} \tag{4.22}$$

Using Young's inequality again, we obtain

$$\begin{aligned} &\int_{\mathbb{B}} \nabla_{\mathbb{B}} w \cdot \nabla_{\mathbb{B}} w_t \frac{dx_1}{x_1}\,dx' \\ &\quad= -\int_{\mathbb{B}} \Delta_{\mathbb{B}} w \cdot w_t \frac{dx_1}{x_1}\,dx' \end{aligned}$$

$$\begin{aligned}&\leq \left|\int_{\mathbb{B}} \Delta_{\mathbb{B}} w \cdot w_t \frac{dx_1}{x_1}\, dx'\right| \\ &\leq \frac{1}{2}\bigl(\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr).\end{aligned} \tag{4.23}$$

Then (4.16) becomes

$$\begin{aligned}&A'(t) \\ &\quad > \left(1+\frac{p}{2}\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + c_1\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - c_4\theta^m H(0)^{-\alpha^*}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\ &\qquad - c_4\theta^{-m/(m-1)} H(0)^{\alpha-\alpha^*} H(t)^{-\alpha} H'(t) - \frac{k_2}{2}\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\qquad - \frac{k_2}{2}\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + pH(t).\end{aligned} \tag{4.24}$$

Now, we define

$$L(t) = H^{1-\alpha}(t) + \varepsilon A(t), \quad t \geq 0, \tag{4.25}$$

where ε is small to be specified later and

$$0 < \alpha \leq \frac{p-2}{2p}. \tag{4.26}$$

By differentiating (4.25), by Lemma 2.2 and (4.24), we see that

$$\begin{aligned}L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon A'(t) \\ &> (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\biggl[\left(1+\frac{p}{2}-\frac{k_2}{2}\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\quad + \left(c_1 - \frac{k_2}{2}\right)\|\Delta_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - c_4\theta^m H(0)^{-\alpha^*}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\ &\quad - c_4\theta^{-m/(m-1)} H(0)^{\alpha-\alpha^*} H(t)^{-\alpha} H'(t) + pH(t)\biggr].\end{aligned} \tag{4.27}$$

Letting

$$a_1 = \min\left\{\frac{p}{2}, c_1 - \frac{k_2}{2}, 1 + \frac{p}{2} - \frac{k_2}{2}\right\} > 0$$

and decomposing $\varepsilon pH(t)$ in (4.27) by

$$\varepsilon pH(t) = 2a_1\varepsilon H(t) + (p-2a_1)\varepsilon H(t).$$

Thus, by (4.8) and (3.3), we obtain

$$\begin{aligned}
L'(t) &> \left(1-\alpha-c_4\varepsilon\theta^{-m/(m-1)}H(0)^{\alpha-\alpha^*}\right)H^{-\alpha}(t)H'(t) \\
&\quad + \varepsilon\left(1+\frac{p}{2}-\frac{k_2}{2}\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \varepsilon\left(c_1-\frac{k_2}{2}\right)\|\Delta_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad - c_4\varepsilon\theta^m H(0)^{-\alpha^*}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + 2a_1\varepsilon\left(-\frac{1}{2}\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\right. \\
&\quad \left. - \frac{1}{2}\|\Delta_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{b}{p}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\right) + (p-2a_1)\varepsilon H(t) \\
&= \left(1-\alpha-c_4\varepsilon\theta^{-m/(m-1)}H(0)^{\alpha-\alpha^*}\right)H^{-\alpha}(t)H'(t) \\
&\quad + \varepsilon\left(1+\frac{p}{2}-\frac{k_2}{2}-a_1\right)\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad + \varepsilon\left(c_1-\frac{k_2}{2}-a_1\right)\|\Delta_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\quad + \varepsilon\left[\frac{2a_1b}{p}-c_4\theta^m H(0)^{-\alpha^*}\right]\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\
&\quad + (p-2a_1)\varepsilon H(t).
\end{aligned} \tag{4.28}$$

Now, we choose $\theta > 0$ small such that

$$\frac{2a_1b}{p} - c_4\theta^m H(0)^{-\alpha^*} \geq \frac{a_1b}{2p},$$

and we pick ε small enough so that

$$1-\alpha-c_4\varepsilon\theta^{-m/(m-1)}H(0)^{\alpha-\alpha^*} \geq 0.$$

Then (4.28) becomes

$$L'(t) > c_5\varepsilon\left(\|w_t\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\Delta_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} + H(t)\right), \tag{4.29}$$

here $c_5 = \min\{\frac{a_1b}{2p}, c_1-\frac{k_2}{2}-a_1, 1+\frac{p}{2}-\frac{k_2}{2}-a_1, p-2a_1\}$. Thus $L(t)$ is a nondecreasing function on $t \geq 0$, and we take ε small enough such that

$$L(0) = H^{1-\alpha}(0) + \varepsilon\int_{\mathbb{B}} w_0w_1\frac{dx_1}{x_1}\,dx' > 0. \tag{4.30}$$

Hence, we have

$$L(t) > 0, \quad \forall t \geq 0. \tag{4.31}$$

Next we estimate the second term in (4.25) as follows:

$$\left|\int_{\mathbb{B}} ww_t\frac{dx_1}{x_1}\,dx'\right| \leq \|w\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\|w_t\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq C\|w\|_{L_p^{\frac{n}{p}}(\mathbb{B})}\|w_t\|_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

So we have

$$\left|\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right|^{1/(1-\alpha)} \le C\|w\|^{1/(1-\alpha)}_{L_p^{\frac{n}{p}}(\mathbb{B})} \|w_t\|^{1/(1-\alpha)}_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Again Young's inequality gives

$$\left|\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right|^{1/(1-\alpha)} \le C\bigl[\|w\|^{\mu_1/(1-\alpha)}_{L_p^{\frac{n}{p}}(\mathbb{B})} + \|w_t\|^{\mu_2/(1-\alpha)}_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr]. \tag{4.32}$$

We take $\mu_1 = \frac{2(1-\alpha)}{1-2\alpha}$, $\mu_2 = 2(1-\alpha)$ to get $\mu_1/(1-\alpha) = 2/(1-2\alpha) \le p$ by condition (4.26). Therefore (4.32) becomes

$$\left|\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right|^{1/(1-\alpha)} \le C\bigl[\|w\|^{s}_{L_p^{\frac{n}{p}}(\mathbb{B})} + \|w_t\|^{2}_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr], \tag{4.33}$$

where $s = 2/(1-2\alpha) \le p$. By using Corollary 4.1, we obtain

$$\left|\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right|^{1/(1-\alpha)} \le C\bigl[H(t) + \|w\|^{p}_{L_p^{\frac{n}{p}}(\mathbb{B})} + \|w_t\|^{2}_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr], \quad \forall t \ge 0. \tag{4.34}$$

Consequently, we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right)^{1/(1-\alpha)} \\ &\le 2^{\alpha/(1-\alpha)}\left(H(t) + \left|\int_{\mathbb{B}} ww_t \frac{dx_1}{x_1}\, dx'\right|^{1/(1-\alpha)}\right) \\ &\le C\bigl[H(t) + \|\Delta_{\mathbb{B}} w\|^{2}_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^{p}_{L_p^{\frac{n}{p}}(\mathbb{B})} + \|w_t\|^{2}_{L_2^{\frac{n}{2}}(\mathbb{B})}\bigr]. \end{aligned} \tag{4.35}$$

We then combine (4.29) and (4.35) to arrive at

$$L'(t) \ge \Gamma L^{1/(1-\alpha)}(t), \tag{4.36}$$

where Γ is a constant dependent on C, c_3 and ε only (and hence is independent of the solution w). A simple integration of (4.36) over $(0,t)$ then yields

$$L^{\alpha/(1-\alpha)}(t) \ge \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Gamma t\alpha/(1-\alpha)}. \tag{4.37}$$

Since $L(0) > 0$, (4.37) shows that $L(t)$ becomes infinite in a finite time $T_{\max} \le T^* = \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}$.

(2) For $E(0) < 0$, we set

$$H(t) = -E(t),$$

instead of (4.8). Then, applying the same arguments as in part (1), we have the result. □

Theorem 4.2 *Under the assumption of Theorem 4.1, let $w(x,t)$ be a blow-up solution of problem (1.7)–(1.9). Then a lower bound T for the lifespan $t^\star$ of w is given by*

$$T := \int_{\phi(0)}^{+\infty} \frac{ds}{\bar{c}_2 s^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \bar{c}_3} \le t^\star, \tag{4.38}$$

with

$$\phi(0) = \frac{1}{2}\int_{\mathbb{B}} |w_1|^2 \frac{dx_1}{x_1}\,dx' + \frac{1}{2}\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w_0|^2 \frac{dx_1}{x_1}\,dx' + \frac{b}{p}\int_{\mathbb{B}} |w_0|^p \frac{dx_1}{x_1}\,dx',$$

where $1 < \alpha < 2$ and $\bar{c}_2, \bar{c}_3$ are positive constants to be determined later.

Proof Now we want to derive a lower bound for the lifespan $t^\star$ of the blow-up solution. To this end, we introduce the auxiliary function

$$\phi(t) = \frac{1}{2}\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' + \frac{1}{2}\int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' + \frac{b}{p}\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx' \tag{4.39}$$

and compute a value $T > 0$ such that $\phi(t)$ remains bounded for $t \in [0, T]$. Clearly, T is a lower bound for $t^\star$. Differentiating (4.39) and making use of the second Green's formula, we obtain in view of (1.7)

$$\begin{aligned}
\phi'(t) &= \int_{\mathbb{B}} w_t w_{tt} \frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} \Delta_{\mathbb{B}} w \cdot \Delta_{\mathbb{B}} w_t \frac{dx_1}{x_1}\,dx' \\
&\quad + b\int_{\mathbb{B}} |w|^{p-2} w w_t \frac{dx_1}{x_1}\,dx' \\
&= \int_{\mathbb{B}} w_t \bigl[2b|w|^{p-2}w - a w_t |u_t|^{m-2} + k_2 \Delta_{\mathbb{B}} w_t\bigr] \frac{dx_1}{x_1}\,dx' \\
&= 2b\int_{\mathbb{B}} |w|^{p-2} w w_t \frac{dx_1}{x_1}\,dx' - a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' \\
&\quad - k_2 \int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\le 2b\int_{\mathbb{B}} |w_t||w|^{p-1} \frac{dx_1}{x_1}\,dx' - k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx'.
\end{aligned} \tag{4.40}$$

Now we make use of Hölder's inequality to the first term on the right-hand side of (4.40) to obtain

$$\begin{aligned}
\int_{\mathbb{B}} |w_t||w|^{p-1} \frac{dx_1}{x_1}\,dx' &\le \left(\int_{\mathbb{B}} |w|^p \frac{dx_1}{x_1}\,dx'\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{B}} |w_t|^p \frac{dx_1}{x_1}\,dx'\right)^{\frac{1}{p}} \\
&= \|w\|^{p-1}_{L_p^{\frac{n}{p}}} \|w_t\|_{L_p^{\frac{n}{p}}}.
\end{aligned} \tag{4.41}$$

Then

$$\phi'(t) \le 2b\|w\|^{p-1}_{L_p^{\frac{n}{p}}} \|w_t\|_{L_p^{\frac{n}{p}}} - k_2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx'. \tag{4.42}$$

In the rest of the proof, we apply Young's inequality to the first term on the right-hand side of (4.42) with exponents α and $\frac{\alpha}{\alpha-1}$, where $1 < \alpha < 2$ is a constant. Thus we obtain

$$\begin{aligned} 2b\|w\|^{p-1}_{L_p^{\frac{n}{p}}}\|w_t\|_{L_p^{\frac{n}{p}}} &\leq \bar{c}_1\|w\|^{\frac{\alpha(p-1)}{\alpha-1}}_{L_p^{\frac{n}{p}}} + \|w_t\|^{\alpha}_{L_p^{\frac{n}{p}}} \\ &\leq \bar{c}_2\phi(t)^{\frac{\alpha(p-1)}{p(\alpha-1)}} + c_\star^{\alpha}\|\nabla_{\mathbb{B}}w_t\|^{\alpha}_{L_2^{\frac{n}{2}}} \end{aligned} \tag{4.43}$$

with $\bar{c}_1 = (2b)^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha}$ and $\bar{c}_2 = \bar{c}_1(\frac{p}{b})^{\frac{\alpha(p-1)}{p(\alpha-1)}}$. Because of $\alpha < 2$, we can use Young's inequality with exponents $\frac{2}{\alpha}$ and $\frac{2}{2-\alpha}$ to have

$$c_\star^{\alpha}\|\nabla_{\mathbb{B}}w_t\|^{\alpha}_{L_2^{\frac{n}{2}}} \leq k_2\|\nabla_{\mathbb{B}}w_t\|^{2}_{L_2^{\frac{n}{2}}} + \bar{c}_3$$

with $\bar{c}_3 = c_\star^{\frac{2\alpha}{2-\alpha}}(\frac{\alpha}{2k_2})^{\frac{\alpha}{2-\alpha}}\frac{2-\alpha}{2}$. Inserting this in (4.43) yields

$$2b\|w\|^{p-1}_{L_p^{\frac{n}{p}}}\|w_t\|_{L_p^{\frac{n}{p}}} \leq \bar{c}_2\phi(t)^{\frac{\alpha(p-1)}{p(\alpha-1)}} + k_2\|\nabla_{\mathbb{B}}w_t\|^{2}_{L_2^{\frac{n}{2}}} + \bar{c}_3. \tag{4.44}$$

Inequality (4.44) along with (4.42) implies that

$$\phi'(t) \leq \bar{c}_2\phi(t)^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \bar{c}_3. \tag{4.45}$$

Then

$$\frac{d\phi}{\bar{c}_2\phi(t)^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \bar{c}_3} \leq dt. \tag{4.46}$$

Integrating (4.46) from 0 to $t^\star$, we obtain

$$\int_{\phi(0)}^{\phi(t)} \frac{ds}{\bar{c}_2 s^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \bar{c}_3} \leq t^\star. \tag{4.47}$$

Thus, we obtain the desired result. □

In the following theorem, by means of a first order differential inequality technique, we obtain a lower bound for the blow-up time which is different from (4.38).

Theorem 4.3 *Suppose that the conditions of Theorem 4.1 hold. Let $w(x,t)$ be a blow-up solution of problem (1.7)–(1.9). Then a lower bound $\tilde{T}$ for the lifespan $t^\star$ of w is given by*

$$\tilde{T} := \left\{(p-2)b\kappa^{\frac{1}{2}}\left(\psi(0)\right)^{\frac{p-2}{2}}\right\}^{-1} < t^\star, \tag{4.48}$$

with

$$\psi(0) = \int_{\mathbb{B}} |w_1|^2 \frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} |\Delta_{\mathbb{B}}w_0|^2 \frac{dx_1}{x_1}\,dx',$$

where $\kappa = C_0^{2(p-1)}$.

Proof We introduce the auxiliary function

$$\psi(t) = \int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' + \int_{\mathbb{B}} |\Delta_{\mathbb{B}} w|^2 \frac{dx_1}{x_1}\,dx' \tag{4.49}$$

and compute a value $\tilde{T} > 0$ such that $\psi(t)$ remains bounded for $t \in [0, \tilde{T}]$. Clearly $\tilde{T}$ is a lower bound for $t^\star$. Differentiating (4.49) and making use of the second Green's formula, we obtain in view of (1.7)

$$\begin{aligned}
\psi'(t) &= 2\int_{\mathbb{B}} w_t w_{tt} \frac{dx_1}{x_1}\,dx' + 2\int_{\mathbb{B}} \Delta_{\mathbb{B}} w \cdot \Delta_{\mathbb{B}} w_t \frac{dx_1}{x_1}\,dx' \\
&= 2\int_{\mathbb{B}} w_t \big[w_{tt} + \Delta_{\mathbb{B}}^2 w\big] \frac{dx_1}{x_1}\,dx' \\
&= 2b\int_{\mathbb{B}} |w|^{p-2} w w_t \frac{dx_1}{x_1}\,dx' - 2a\int_{\mathbb{B}} |w_t|^m \frac{dx_1}{x_1}\,dx' \\
&\quad - 2k_2 \int_{\mathbb{B}} |\nabla_{\mathbb{B}} w_t|^2 \frac{dx_1}{x_1}\,dx' \\
&\le 2b\int_{\mathbb{B}} |w_t||w|^{p-1} \frac{dx_1}{x_1}\,dx'.
\end{aligned} \tag{4.50}$$

Making use of the Schwarz inequality leads to

$$\psi'(t) \le 2b\left(\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' \int_{\mathbb{B}} |w|^{2(p-1)} \frac{dx_1}{x_1}\,dx'\right)^{\frac{1}{2}}. \tag{4.51}$$

Applying the Poincaré inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{B}} |w|^{2(p-1)} \frac{dx_1}{x_1}\,dx' &= \|w\|^{2(p-1)}_{L^{\frac{n}{2(p-1)}}_{2(p-1)}} \\
&\le C_0^{2(p-1)} \|\Delta_{\mathbb{B}} w\|^{2(p-1)}_{L_2^{\frac{n}{2}}} \\
&\le C_0^{2(p-1)} \big(\psi(t)\big)^{p-1}.
\end{aligned} \tag{4.52}$$

Moreover, we have

$$\int_{\mathbb{B}} |w_t|^2 \frac{dx_1}{x_1}\,dx' < \psi(t). \tag{4.53}$$

From (4.51)–(4.53), we obtain the differential inequality

$$\psi'(t) < 2b\kappa^{\frac{1}{2}} \psi(t)^{\frac{p}{2}}, \tag{4.54}$$

where $\kappa = C_0^{2(p-1)}$, then (4.54) can be rewritten as

$$\big(\psi^{\frac{2-p}{2}}(t)\big)' > -(p-2)b\kappa^{\frac{1}{2}}. \tag{4.55}$$

Integrating (4.55) from 0 to t, we obtain

$$\big(\psi^{\frac{2-p}{2}}(t)\big) > \psi^{\frac{2-p}{2}}(0) - (p-2)b\kappa^{\frac{1}{2}} t. \tag{4.56}$$

Inequality (4.56) shows that $\psi(t)$ remains bounded for

$$t < \tilde{T} := \frac{(\psi(0))^{\frac{2-p}{2}}}{(p-2)b\kappa^{\frac{1}{2}}}. \tag{4.57}$$

From the discussion above in Theorem 3.1 and Theorem 4.1, we immediately obtain a specifying result of the global existence and nonexistence of solutions for problem (1.7)–(1.9) as follows. □

Remark 4.1 Suppose that $2 < p < p^\star$, $w_1 \in L_2^{\frac{n}{2}}(\mathbb{B})$, and $0 < E(0) < d$, then problem (1.7)–(1.9) admits a global weak solution without relation between m and p provided $I(w_0) > 0$ and w satisfies the assumption of Lemma 3.3; problem (1.7)–(1.9) does not admit any global solution provided $p > m \geq 2$, $I(w_0) < 0$, and $\|\Delta_{\mathbb{B}} w_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \lambda_1$.

From the discussion above in Theorem 4.1 and Theorem 4.2, we give the bounds for blow-up time for problem (1.7)–(1.9) under the initial condition $I(w_0) < 0$.

Remark 4.2 Suppose that $2 < p < p^\star$, $p > m \geq 2$ and $E(0) < d$, $w_1 \in L_2^{\frac{n}{2}}(\mathbb{B})$, then problem (1.7)–(1.9) does not admit any global solution provided $I(w_0) < 0$. Furthermore, the corresponding upper and lower bounds of blow-up time $T_{\max}$ are given by the following form:

$$\int_{\phi(0)}^{+\infty} \frac{ds}{\bar{c}_2 s^{\frac{\alpha(p-1)}{p(\alpha-1)}} + \bar{c}_3} \leq T_{\max} \leq \frac{1-\alpha}{\Gamma\alpha[L(0)]^{\alpha/(1-\alpha)}}. \tag{4.58}$$

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