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Ground states for infinite lattices with nearest neighbor interaction

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Abstract

Sun and Ma (J. Differ. Equ. 255:2534–2563, 2013) proved the existence of a nonzero T-periodic solution for a class of one-dimensional lattice dynamical systems,

$$\ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}), \quad i \in \mathbb{Z},$$

where q_i denotes the co-ordinate of the ith particle and Φ_i denotes the potential of the interaction between the ith and the $(i+1)$th particle. We extend their results to the case of the least energy of nonzero T-periodic solution under general conditions. Of particular interest is a new and quite general approach. To the best of our knowledge, there is no result for the ground states for one-dimensional lattice dynamical systems.

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Keywords: Ground states; Infinite lattice; Strongly indefinite functional

1 Introduction

In this paper, we are concerned with one-dimensional lattices which arises from the Fermi–Pasta–Ulam (FPU) model [6]; one of the most interesting and physically important features of it is that infinite degrees of freedom consist of a one-dimensional lattice of particles, each interacting with its nearest neighbors by means of a force belonging to a certain class. We now turn to the mathematical formulation of the problem. Let Φ_i be the potential of the interaction between the ith and the $(i+1)$th particle (whose displacement is $q_i - q_{i+1}$), then the equation governing the state of $q_i(t)$ can be written as

$$\ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}), \quad i \in \mathbb{Z}, \tag{1.1}$$

where $q_i(t)$ denotes the state of the ith particle, the state of lattice at time t is represented by a sequence $q(t) = \{q_i(t)\}$, $i \in \mathbb{Z}$. Moreover, $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ is defined by

$$\Phi(q) = \sum_{i\in\mathbb{Z}} \Phi_i(q_i - q_{i+1}).$$

Then infinitely many Eqs. (1.1) can be rewritten in a simple form,

$$\ddot{q} = -\Phi'(q). \tag{1.2}$$

Much of the interest in these lattice systems of the type (1.2) is motivated by Fermi–Pasta–Ulam (FPU) [6] who studied finite lattices with nearest-neighbor interaction by numerical simulations. In the past four decades, a great deal of mathematical effort in lattice system has been made devoted to the study of existence results. Variational methods have attracted considerable attention in the research of lattice system, one of the first results in this direction is due to Friesecke and Wattis [7], who obtained a global existence result for localized travelling waves by means of the technique of concentration-compactness. Smets and Willem [15] also proved the existence of travelling waves with a prescribed speed assumption.

An alternative development to derive periodic solution was developed by Ruf and Srikanth [14], who considered time periodic motions of finite FPU type lattices. Later, Arioli and Gazzola [2] extended to some extent Ruf–Srikanth's result to infinite-dimensional system (1.2). In [4], Arioli and Szulkin firstly extended the result of [3] to the strongly indefinite case, they proved that system (1.2) admits a nonzero T-periodic solution of finite energy for all T in a given range of values and given a bifurcate result under some additional conditions using variational methods. In recent years, there has been increasing attention to this problem (1.2) on the existence of nonconstant periodic motions, multibump periodic motions and ground state travelling waves in infinite lattices; see e.g., [3, 10–14, 16–18, 26, 27].

Recently, Sun and Ma [16] proved the existence of nonzero T-periodic solution by means of the abstract critical point theorem for a strongly indefinite functional developed by Bartsch and Ding [5]. However, to the best of our knowledge, there is no result for ground states for infinite-dimensional system (1.2) with the strongly indefinite case. So another question arises: can the result ([16]) on the existence of a ground state solution, i.e., a nontrivial T-periodic solution for (1.2) with the minimal energy, for (1.2) be obtained? Answering this question constitutes the goal of this paper.

Motivated by the interest shared by the mathematical community in this topic and [16, 18, 20, 27], the main goal of this paper is to investigate the question of existence of ground states for (1.2). Based on the recent work [21–25] and the Non–Nehari manifold method [20–25] which is different from the previous work and generalizes the results, this method has been proven successful, for instance, in solving the Schrodinger equation and the Dirac equation. More precisely, we assume the potentials $\Phi_i : \mathbb{R} \to \mathbb{R}$ to be defined by

$$\Phi_i(x) = -\frac{\alpha_i}{2}x^2 + V_i(x), \tag{1.3}$$

satisfy:

(A0) $\alpha_i \neq 0$ *for all* $i \in \mathbb{Z}$ *and take both signs*;
(A1) $V_i' \in C(\mathbb{R}, \mathbb{R})$, $xV_i'(x) \geq 0$, $\forall x \in \mathbb{R}$ *and* $\lim_{|x|\to\infty} \frac{V_i'(x)}{|x|} = \infty$;
(A2) $V_i'(x) = o(x)$ *as* $|x| \to 0$;

(A3) *there exists a constant* $\eta_0 \in (0,1)$ *such that*

$$\frac{1-\eta^2}{2} x V_i'(x) \geq \int_{\eta x}^{x} V_i'(s)\, ds, \quad \forall \eta \in [0, \eta_0];$$

(A4) *there exists* $m \in \mathbb{N}$ *such that* $\Phi_{i+m} = \Phi_i$.

Now, we are ready to state the main result of this paper.

Theorem 1.1 *Assume that* (A0)–(A4) *hold. Then problem* (1.2) *has a ground state, i.e. a nontrivial solution* $q_0 \in H$ *such that* $J(q_0) = \inf_{\mathcal{M}} J > 0$, *where*

$$\mathcal{M} = \bigl\{ q \in H \setminus \{0\} : J'(q) = 0 \bigr\},$$

H is defined in (2.1), *J is defined in* (2.6). *Moreover, there exists* $T_{\min} > 0$, *where* $T_{\min}$ *only depends on a positive* α_i *and* V_i, *such that the solution obtained above is nonconstant if* $T_{\min} < \pi/\sqrt{\beta}$ *and its period* $T \in (T_{\min}, \pi/\sqrt{\beta})$ *where* $\beta \doteq \inf\{\alpha_i\} > 0$.

The present paper is organized as follows. The variational structure and some properties of the associated functional are established in Sect. 2. We establish some instrumental lemmas involving our main theorem in Sect. 3, finally the proofs of Theorem 1.1 are presented by the Non–Nehari method.

2 Variational structure and preliminaries

Before approaching problem (1.2), we first pose the problem of finding what the natural space is in which it lives. Denote $S^1 = \mathbb{R}/(T\mathbb{Z})$, $H^1(S^1, \mathbb{R})$ is the usual Hilbert space endowed with the norm

$$\|q_i\|_{H^1} = \left(\int_0^T \left(\left|\dot{q}_i(t)\right|^2 + \left|q_i(t)\right|^2 \right) dt \right)^{1/2}.$$

Let

$$\begin{aligned} H = \Biggl\{ & q \in H^1\left(S^1, \mathbb{R}\right)^{\mathbb{Z}} : \\ & \int_0^T q_0(t)\, dt = 0, \sum_i \int_0^T \left[\dot{q}_i^2(t) + \left(q_i(t) - q_{i+1}(t)\right)^2 \right] dt < \infty \Biggr\}, \end{aligned} \tag{2.1}$$

which is endowed with the inner product

$$(q, p) = \sum_i \int_0^T \left[\dot{q}_i(t)\dot{p}_i(t) + \left(q_i(t) - q_{i+1}(t)\right)\left(p_i(t) - p_{i+1}(t)\right) \right] dt. \tag{2.2}$$

We note that $\int_0^T q_0(t)\, dt = 0$ in the definition of H is in order to guarantee (2.2) defining a scalar product. Throughout this paper, let $\|\cdot\|_H$ be the norm induced by (2.1) and $\|\cdot\|_p$ the norm of $L^p(S^1, \mathbb{R})$ for $p \in [1, +\infty]$. Under the assumptions (A0)–(A4), solutions of (1.2) are critical points of the functional J given by

$$J(q) = \frac{1}{2} \int_0^T \left|\dot{q}(t)\right|^2 dt - \int_0^T \Phi\left(q(t)\right) dt. \tag{2.3}$$

We define a self-adjoint linear operator $L : H \to H$ by

$$(Lq,p)_H = \sum_i \int_0^T \bigl[\dot{q}_i(t)\dot{p}_i(t) + \alpha_i\bigl(q_i(t) - q_{i+1}(t)\bigr)\bigl(p_i(t) - p_{i+1}(t)\bigr)\bigr]\,dt, \tag{2.4}$$

and a functional $V : H \to \mathbb{R}$

$$V(q) = \sum_i \int_0^T V_i\bigl(q_i(t) - q_{i+1}(t)\bigr)\,dt; \tag{2.5}$$

combining (2.4) and (2.5), we see that

$$J(q) = \frac{1}{2}(Lq,q)_H - V(q) \tag{2.6}$$

and

$$\bigl\langle J'(q),p\bigr\rangle = (Lq,p)_H - \sum_i \int_0^T V_i'(q_i - q_{i+1})(p_i - p_{i+1})\,dt. \tag{2.7}$$

Lemma 2.1 *Suppose that* (A1) *and* (A2) *are satisfied. Then $V(q)$ is nonnegative, weakly sequentially lower semi-continuous, and $V'(q)$ is weakly sequentially continuous.*

It is not difficult to verify the above lemma by means of Sobolev's embedding theorem, the proof will be omitted.

Lemma 2.2 *Assume that* (A2) *and* (A4) *hold. Then $J \in C^1(H,\mathbb{R})$.*

Proof The proof of the argument is analogous to that in [16], we omit its proof process. □

Next, we study the spectrum of the linear operator L (we denote it by $\sigma(L)$) in order to establish a variational setting for system (1.2). For simplicity, we denote

$$I = \{i \in \mathbb{Z} | \alpha_i > 0\}, \qquad H^- = \{q \in H | q \equiv \text{const.}, q_i - q_{i+1} = 0, i \in I\}, \qquad H^+ = \bigl(H^-\bigr)^{\perp}.$$

Using Lemma 2.2 in [16], we may define a new scalar product $(\cdot,\cdot)$ on H with corresponding norm $\|\cdot\|$ such that $(Lq,q)_H = -\|q\|_H^2$ for $q \in H^-$, and $(Lq,q)_H = \|q\|_H^2$ for $q \in H^+$. Indeed

$$(q,p) = \bigl(Lq^+,p^+\bigr) - \bigl(Lq^-,p^-\bigr) \quad \text{for } q = q^- + q^+, p = p^- + p^+ \in H^- \oplus H^+.$$

It is easy to verify that the norm $\|\cdot\|$ is equivalent to the standard norm $\|\cdot\|_H$ in H as $\sigma(L) \subset \mathbb{R} \setminus (-\lambda,\lambda)$, then we can derive the decomposition $H = H^- \oplus H^+$ with respect to $(\cdot,\cdot)$. Moreover, $J(q)$ can be rewritten as a simple form

$$J(q) := \frac{1}{2}\bigl(\bigl\|q^+\bigr\|^2 - \bigl\|q^-\bigr\|^2\bigr) - V(q) \tag{2.8}$$

on H, where $q = q^- + q^+ \in H^- \oplus H^+$. Let $P^- : H \to H^-$ and $P^+ : H \to H^+$ be the orthogonal projections, then the spaces H^- and H^+ are $\mathbb{Z}$-invariant because they are L-invariant. Indeed, spectral theory asserts that the projectors P^-, P^+ commute with any operator which commutes with L, especially, they commute with the $\mathbb{Z}$-action.

Lemma 2.3 ([16]) *If $\{q^{(n)}\}$ is a bounded sequence in H, then passing to a subsequence, there exists $q \in H$ such that $q^{(n)} \rightharpoonup q$ in H. Moreover, we have $q_i^{(n)} \rightharpoonup q_i$ in H^1 and $q_i^{(n)} \to q_i$ in L^∞ for all $i \in \mathbb{Z}$.*

3 Proof of the result

Let W be a real Hilbert space with $W = W^- \oplus W^+$ and $W^- \perp W^+$. For a functional $\psi \in C^1(W,\mathbb{R})$, ψ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in W one has $\psi(u) \le \liminf_{n\to\infty} \psi(u_n)$, and ψ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \psi'(u_n), v\rangle = \langle \psi'(u_n), v\rangle$ for each $v \in W$.

Lemma 3.1 ([8]) *Let W be a real Hilbert space, $W = W^- \oplus W^+$ and $W^- \perp W^+$, and $\psi \in C^1(X,\mathbb{R})$ of the form*

$$\psi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \psi(u), \quad u = u^- + u^+ \in W^- \oplus W^+.$$

Suppose that the following assumptions hold:

(A1) *$\psi \in C^1(W,\mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*

(A2) *ψ' is weakly sequentially continuous;*

(A3) *there exist $r > \rho > 0$, $e \in W^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \psi\left(S_\rho^+\right) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \left\{u \in X^+ : \|u\| = \rho\right\}, \qquad Q = \left\{v + se : v \in X^-, s \ge 0, \|v + se\| \le r\right\}.$$

Then, for some $c \in [\kappa, \sup \varphi(Q)]$, there exists a sequence $\{u_n\} \subset W$ satisfying

$$\psi(u_n) \to c, \quad \left\|\psi'(u_n)\right\|\left(1 + \|u_n\|\right) \to 0.$$

Lemma 3.2 *Assume that* (A0)–(A4) *are satisfied, then, for any $q \in H$,*

$$\begin{aligned} J(q) \ge{}& J\left(\mu q^+\right) + \frac{\mu^2\|q^-\|^2}{2} + \frac{1-\mu^2}{2}\left\langle J'(q), q\right\rangle + \mu^2\left\langle J'(q), q^-\right\rangle \\ &- \mu^2 \sum_i \int_{\mu|q_i^+ - q_{i+1}^+| > \eta_0|q_i - q_{i+1}|} V_i'(q_i - q_{i+1})\left(q_i^+ - q_{i+1}^+\right) dt, \quad \forall \mu \ge 0. \end{aligned} \tag{3.1}$$

Proof Fix $x, y \in \mathbb{R}$. Let

$$g(r) = \frac{1+r^2}{2} V_i'(x)x - r^2 V_i'(x)y + V_i(ry) - V_i(x).$$

If $xy \le 0$, using the assumption (A1), we have

$$\begin{aligned} g(r) &= \frac{1+r^2}{2} V_i'(x)x - r^2 V_i'(x)y + V_i(ry) - V_i(x) \\ &\ge \frac{1+r^2}{2} V_i'(x)x - V_i(x), \quad \forall r \ge 0. \end{aligned} \tag{3.2}$$

If $xy \geq 0$, let $\eta = ry/x$, using the assumption (A3), we have

$$\begin{aligned} g(r) &= \frac{1+r^2}{2} V_i'(x)x - r^2 V_i'(x)y + V_i(ry) - V_i(x) \\ &= \frac{1+r^2-2\eta r}{2} V_i'(x)x - \int_{\eta x}^{x} V_i'(s)\,ds \\ &= \frac{(\eta - r)^2}{2} V_i'(x)x + \frac{1-\eta^2}{2} V_i'(x)x - \int_{\eta x}^{x} V_i'(s)\,ds \\ &\geq \frac{1-\eta^2}{2} V_i'(x)x - \int_{\eta x}^{x} V_i'(s)\,ds \\ &\geq 0, \quad r \geq 0, ry/x \leq \eta_0. \end{aligned} \tag{3.3}$$

Based on the above two arguments, we obtain

$$\frac{1+r^2}{2} V_i'(x)x - r^2 V_i'(x)y + V_i(ry) - V_i(x) \geq 0, \quad r \geq 0, |ry| \leq \eta_0 |x|. \tag{3.4}$$

Taking the assumption (A3) into consideration, we get

$$\begin{aligned} &J(q) - J(rq^+) \\ &= \frac{1}{2}\big[(Lq,q) - \big(L(rq^+, rq^+)\big)\big] + \sum_i \int_0^T \big[V_i\big(r\big(q_i^+ - q_{i+1}^+\big)\big) - V_i(q_i - q_{i+1})\big]\,dt \\ &= \frac{1}{2}\big[\big(1-r^2\big)(Lq,q) + r^2\big(Lq,q^-\big)\big] + \sum_i \int_0^T \big[V_i\big(r\big(q_i^+ - q_{i+1}^+\big)\big) - V_i(q_i - q_{i+1})\big]\,dt \\ &= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}(Lq,q) + r^2\big(Lq,q^-\big) \\ &\quad + \sum_i \int_0^T \big[V_i\big(r\big(q_i^+ - q_{i+1}^+\big)\big) - V_i(q_i - q_{i+1})\big]\,dt \\ &= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\big(J'(q),q\big) + r^2\big(J'(q),q^-\big) \\ &\quad + \sum_i \int_0^T \bigg[\frac{1-r^2}{2} V_i'(q_i - q_{i+1})(q_i - q_{i+1}) + r^2 V_i'(q_i - q_{i+1})\big(q_i^- - q_{i+1}^-\big)\bigg]\,dt \\ &\quad + \sum_i \int_0^T \big[V_i\big(r\big(q_i^+ - q_{i+1}^+\big) - V_i(q_i - q_{i+1})\big]\,dt \\ &= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\big(J'(q),q\big) + r^2\big(J'(q),q^-\big) \\ &\quad + \sum_i \int_0^T \bigg[\frac{1+r^2}{2} V_i'(q_i - q_{i+1})(q_i - q_{i+1}) - r^2 V_i'(q_i - q_{i+1})\big(q_i^- - q_{i+1}^-\big)\bigg]\,dt \\ &\quad + \sum_i \int_0^T \big[V_i\big(r\big(q_i^+ - q_{i+1}^+ - V_i(q_i - q_{i+1})\big]\,dt \\ &= \frac{r^2}{2}\|q^-\|^2 + \frac{1-r^2}{2}\big(J'(q),q\big) + r^2\big(J'(q),q^-\big) \end{aligned}$$

$$\begin{aligned}
&+\sum_i \int_{\mu|q_i^+-q_{i+1}^+|\le\eta_0|q_i-q_{i+1}|}\biggl[\frac{1+r^2}{2}V_i'(q_i-q_{i+1})(q_i-q_{i+1})\\
&-r^2V_i'(q_i-q_{i+1})\bigl(q_i^- - q_{i+1}^-\bigr)\biggr]\,dt\\
&+\sum_i \int_{\mu|q_i^+-q_{i+1}^+|\le\eta_0|q_i-q_{i+1}|}\bigl[V_i\bigl(r\bigl(q_i^+-q_{i+1}^+\bigr)-V_i(q_i-q_{i+1})\bigr]\,dt\\
&+\sum_i \int_{\mu|q_i^+-q_{i+1}^+|>\eta_0|q_i-q_{i+1}|}\biggl[\frac{1+r^2}{2}V_i'(q_i-q_{i+1})(q_i-q_{i+1})\\
&-r^2V_i'(q_i-q_{i+1})\bigl(q_i^- - q_{i+1}^-\bigr)\biggr]\,dt\\
&+\sum_i \int_{\mu|q_i^+-q_{i+1}^+|>\eta_0|q_i-q_{i+1}|}\bigl[V_i\bigl(r\bigl(q_i^+-q_{i+1}^+\bigr)-V_i(q_i-q_{i+1})\bigr]\,dt\\
\ge{}&\frac{r^2}{2}\bigl\|q^-\bigr\|^2+\frac{1-r^2}{2}\bigl(J'(q),q\bigr)+r^2\bigl(J'(q),q^-\bigr)\\
&-r^2\sum_i\int_{\mu|q_i^+-q_{i+1}^+|>\eta_0|q_i-q_{i+1}|}V_i'(q_i-q_{i+1})\bigl(q_i^+-q_{i+1}^+\bigr)\,dt,\quad r\ge 0. \qquad\square
\end{aligned}$$

Lemma 3.3 *Assume that* (A0)–(A4) *are satisfied. Then there is a constant* $\rho>0$ *such that* $\kappa:=\inf J(S_\rho^+)>0$, *where* $S_\rho^+=\partial B_\rho\cap H^+$.

Lemma 3.3 can be proved in the same way as [19].

Lemma 3.4 *Suppose that* (A0)–(A4) *are satisfied. Let* $e\in E^+$ *with* $\|e\|=1$. *Then there is a constant* $r_0>0$ *such that* $\sup J(\partial Q)\le 0$, *where*

$$Q=\bigl\{q=se+q^-:q^-\in H^-,s\ge 0,\|q\|\le r_0\bigr\}.$$

Proof From (A1) we have $V_i(x)\ge 0$ for all x and i, so we get $J(q)\le 0$ for any $q\in H^-$. Next, it remains to show that $J(q)\to-\infty$ as $q\in H^-\oplus\mathbb{R}e$, $\|q\|\to\infty$. The proof is by contradiction, assume that, for some sequence $\{q^{(n)}\}\subset H^-\oplus\mathbb{R}e$ with $\|q^{(n)}\|\to\infty$, there exists $M>0$ such that $J(q^{(n)})\ge -M$ for all $n\in\mathbb{N}$. Denote $h^{(n)}=q^{(n)}/\|q^{(n)}\|=h^{(n)^-}+s_ne$, obviously $\|h^{(n)}\|=1$. Passing to a subsequence, we may suppose that $h^{(n)}\rightharpoonup h$ in H, thus $h^{(n)}\to v$ a.e. on $\mathbb{R}$, $h^{(n)^-}\rightharpoonup h^-$ in H, $s_n\to\bar{s}$ and

$$-\frac{M}{\|q^{(n)}\|^2}\le\frac{J(q^{(n)})}{\|q^{(n)}\|^2}=\frac{s_n^2}{2}-\frac{1}{2}\bigl\|h^{(n)^-}\bigr\|^2-\sum_i\int_0^T\frac{V_i(q_i^{(n)}-q_{i+1}^{(n)})}{\|q^{(n)}\|^2}\,dt. \tag{3.5}$$

If $\bar{s}=0$, thanks to (3.5), it follows that

$$0\le\frac{1}{2}\bigl\|h^{(n)^-}\bigr\|^2+\sum_i\int_0^T\frac{V_i(q_i^{(n)}-q_{i+1}^{(n)})}{\|q^{(n)}\|^2}\,dt\le\frac{s_n^2}{2}+\frac{M}{\|q^{(n)}\|^2}\to 0,$$

which leads to $\|h^{(n)^-}\|\to 0$, and so $1=\|h^{(n)}\|\to 0$, a contradiction.

If $\bar{s} \neq 0$, then $h \neq 0$, combining (3.5), (A1) and Fatou's lemma, we see that

$$\begin{aligned}
0 &\le \limsup_{n\to\infty}\left[\frac{s_n^2}{2} - \frac{1}{2}\|h^{(n)-}\|^2 - \sum_i \int_0^T \frac{V_i(q_i^{(n)} - q_{i+1}^{(n)})}{\|q^{(n)}\|^2}\,dt\right] \\
&= \limsup_{n\to\infty}\left[\frac{s_n^2}{2} - \frac{1}{2}\|h^{(n)-}\|^2 - \sum_i \int_0^T \frac{V_i(q_i^{(n)} - q_{i+1}^{(n)})}{|q_i^{(n)} - q_{i+1}^{(n)}|^2}\big(h_i^{(n)} - h_{i+1}^{(n)}\big)^2\,dt\right] \\
&\le \frac{1}{2}\lim_{n\to\infty} s_n^2 - \liminf_{n\to\infty}\sum_i \int_0^T \frac{V_i(q_i^{(n)} - q_{i+1}^{(n)})}{|q_i^{(n)} - q_{i+1}^{(n)}|^2}\big(h_i^{(n)} - h_{i+1}^{(n)}\big)^2\,dt \\
&\le \frac{\bar{s}^2}{2} - \sum_i \int_0^T \liminf_{n\to\infty} \frac{V_i(q_i^{(n)} - q_{i+1}^{(n)})}{|q_i^{(n)} - q_{i+1}^{(n)}|^2}\big(h_i^{(n)} - h_{i+1}^{(n)}\big)^2\,dt \\
&= -\infty,
\end{aligned}$$

this leads to a contradiction. Hence the Lemma 3.4 are proved. □

Lemma 3.5 *Assume that* (A0)–(A4) *are satisfied. Then there exist a constant $c \ge \kappa$ and a sequence $\{q^{(n)}\} \subset H$ satisfying*

$$J\big(q^{(n)}\big) \to c, \qquad \big\|J'\big(q^{(n)}\big)\big\|\big(1 + \big\|q^{(n)}\big\|\big) \to 0. \tag{3.6}$$

Proof Lemma 3.5 is a direct corollary of Lemmas 2.1, 2.2 and 3.2. □

Lemma 3.6 *Suppose that* (A0)–(A4) *are satisfied. Then any sequence $\{q^{(n)}\} \subset H$ satisfying*

$$J\big(q^{(n)}\big) \to c, \qquad \big\langle J'\big(q^{(n)}\big), \big(q^{(n)}\big)^{\pm}\big\rangle \to 0 \tag{3.7}$$

is bounded in H.

Proof We prove boundedness of $\{q^{(n)}\}$ by negation, suppose that $\|q^{(n)}\| \to \infty$. Let $h^{(n)} = q^{(n)}/\|q^{(n)}\|$, it is easy to show that $\|h^{(n)}\| = 1$ and there exists a constant C_1 such that $\|h^{(n)}\|_2 \le C_1$. Passing to a subsequence, we may assume that $(h^{(n)}) \rightharpoonup h$ in H, $h_i^{(n)} \rightharpoonup h_i$ in $L^\infty(S^1, \mathbb{R})$. Based on the concentration-compactness principle of Lions [9] (see also Lemma 1 in [1]), we will divide our proof into two cases: either $((h^{(n)})^+)_n$ is vanishing or it is nonvanishing.

Now, we assume that $((h^{(n)})^+)_n$ is vanishing, that is,

$$\lim_{n\to\infty}\sup_i \big\|\big(h_i^{(n)}\big)^+ - \big(h_{i+1}^{(n)}\big)^+\big\|_\infty = 0.$$

Fix $R = [2(1+c)^{1/2}]$. It follows from (A1) and (A2) that

$$\big|V_i(x)\big| \le \frac{x^2}{4(RC_1)^2}, \qquad \forall i \in \mathbb{Z}, |x| \le \eta. \tag{3.8}$$

According to the theorem of Lions [9], it follows that $\|(h_i^{(n)})^+ - (h_{i+1}^{(n)})^+\|_\infty \le \eta/R$, where n is sufficiently large. Hence,

$$\limsup_{n\to\infty}\sum_i \int_0^T V_i\big(h_i^{(n)} - h_{i+1}^{(n)}\big) \le \frac{1}{4C_1^2}\lim_{n\to\infty}\big\|h^{(n)}\big\|_2^2 \le \frac{1}{4}. \tag{3.9}$$

Using (A1) and (A2), for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|V_i'(x)\right| \le \varepsilon|x| + C_\varepsilon|x|^{p-1} \tag{3.10}$$

and

$$\left|V_i(x)\right| \le \varepsilon|x|^2 + C_\varepsilon|x|^p \tag{3.11}$$

for any $x \in \mathbb{R}$ and $i \in \mathbb{Z}$, where $p > 2$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \sum_i \int_{R|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} V_i'(q_i-q_{i+1})\left|h_i^+ - h_{i+1}^+\right| dt \\
&\quad\le \lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \sum_i \int_{R|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} \left(\varepsilon|q_i-q_{i+1}| + C_\varepsilon|q_i-q_{i+1}|^{p-1}\right)\left|h_i^+ - h_{i+1}^+\right| dt \\
&\quad\le \lim_{n\to\infty} \frac{R^2}{\|q^{(n)}\|} \\
&\qquad\times \sum_i \int_{R|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} \left(\varepsilon R\eta_0^{-1}\left|h_i^+ - h_{i+1}^+\right|^2 + C_\varepsilon R^{p-1}\eta_0^{1-p}\left|h_i^+ - h_{i+1}^+\right|^p\right) dt \\
&\quad\le \lim_{n\to\infty} \frac{\varepsilon R^3\eta_0^{-1}\|h^{(n)}\|_2^2 + C_\varepsilon R^{p+1}\eta_0^{1-p}\|h^{(n)}\|_p^p}{\|q^{(n)}\|} = 0.
\end{aligned} \tag{3.12}$$

Let $\mu_n = R/\|q^{(n)}\|$, it follows from (3.7), (3.9), (3.12) and Lemma 3.2 that

$$\begin{aligned}
&c + o(1) \\
&\quad= J\left(q^{(n)}\right) \\
&\quad\ge J\left(\mu_n\left(q^{(n)}\right)^+\right) + \frac{\mu_n^2\|(q^{(n)})^-\|^2}{2} + \frac{1-\mu_n^2}{2}\left\langle J'\left(q^{(n)}\right), \left(q^{(n)}\right)\right\rangle + \mu_n^2\left\langle J'\left(q^{(n)}\right), \left(q^{(n)}\right)^-\right\rangle \\
&\qquad- \mu_n^2 \sum_i \int_{\mu_n|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} V_i'(q_i-q_{i+1})\left|h_i^+ - h_{i+1}^+\right| dt \\
&\quad= J\left(R\left(h^{(n)}\right)^+\right) + \frac{R^2\|(h^{(n)})^-\|}{2} + \left(\frac{1}{2} - \frac{R^2}{2\|q^{(n)})\|^2}\right)\left\langle J'\left(q^{(n)}, q^{(n)}\right)\right\rangle \\
&\qquad+ \frac{R^2}{\|q^{(n)})\|^2}\left\langle J'\left(q^{(n)}\right)\right), \left(q^{(n)}\right)^-)\right\rangle \\
&\qquad- \frac{R^2}{\|q^{(n)})\|} \sum_i \int_{R|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} V_i'(q_i-q_{i+1})\left|h_i^+ - h_{i+1}^+\right| dt \\
&\quad= \frac{R^2}{2}\left(\left\|\left(h^{(n)}\right)^+\right\|^2 + \left\|\left(h^{(n)}\right)^-\right\|^2\right) \\
&\qquad- \frac{R^2}{\|q^{(n)})\|} \sum_i \int_{R|h_i^+-h_{i+1}^+|>\eta_0|q_i-q_{i+1}|} V_i'(q_i-q_{i+1})\left|h_i^+ - h_{i+1}^+\right| dt \\
&\qquad- \sum_i \int_0^T V_i\left(R\left(h_i^+ - h_{i+1}^+\right)\right) dt + \left(\frac{1}{2} - \frac{R^2}{2\|q^{(n)})\|^2}\right)\left\langle J'\left(q^{(n)}, q^{(n)}\right)\right\rangle \\
&\qquad+ \frac{R^2}{\|q^{(n)})\|^2}\left\langle J'\left(q^{(n)}\right)\right), \left(q^{(n)}\right)^-)\right\rangle
\end{aligned}$$

$$\geq \frac{R^2}{2} - \sum_i \int_0^T V_i\bigl(R\bigl(h_i^+ - h_{i+1}^+\bigr)\bigr)\,dt$$

$$- \frac{R^2}{\|q^{(n)})\|} \sum_i \int_{R|h_i^+ - h_{i+1}^+|>\eta_0|q_i - q_{i+1}|} V_i'(q_i - q_{i+1})\bigl|h_i^+ - h_{i+1}^+\bigr|\,dt + o(1)$$

$$\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > c + \frac{3}{4} + o(1),$$

this leads to a contradiction.

Next, suppose that $((h^{(n)})^+)_n$ is nonvanishing, that is, there exist $\rho_0 > 0$ and $(i_n) \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty} \bigl\| \bigl(h_{i_n}^{(n)}\bigr)^+ - \bigl(h_{i_n+1}^{(n)}\bigr)^+ \bigr\| \geq \rho_0. \tag{3.13}$$

Since J and J' are $\mathbb{Z}$ and S^1-translation invariant, up to a subsequence, we may assume that $i_n = j \in \{1, 2, \ldots, m\}$. Noticing that $(h_{i_n}^{(n)})^+ - (h_{i_n+1}^{(n)})^+ \to h_i^+ - h_{i+1}^+$ in L^∞ and (3.13), thus $h_j - h_{j+1} \neq 0$. Set $|l_j| = \|q^{(n)}\| \|\tilde{l}_j\| \to \infty$, where $l_j := q_j^{(n)} - q_{j+1}^{(n)}$ and $\tilde{l}_j := h_j^{(n)} - h_{j+1}^{(n)}$, from (A1) and Fatou's lemma we have

$$\int_0^T \frac{V_j(l_j)}{l_j^2} \tilde{l}_j^2 \to +\infty,$$

which indicates that

$$0 \leq \frac{J(q^{(n)})}{\|q^{(n)}\|^2} \leq \frac{1}{2}\bigl\|h^+\bigr\|^2 - \frac{1}{2}\bigl\|h^-\bigr\|^2 - \int_0^T \frac{V_j(l_j)}{l_j^2} \tilde{l}_j^2 \to -\infty,$$

as $n \to \infty$, this contradiction implies that $\{q^{(n)}\}$ is bounded in H. □

Lemma 3.7 *The* $(C)_c$ *sequence* $(q^{(n)})$ *obtained above is bounded, and up to a translation of indices,* $q^{(n)} \rightharpoonup q \neq 0$ *in* H *and* $J'(q) = 0$.

Proof Using Lemma 3.6, we can now derive that $(q^{(n)})$ is bounded. Since $c \geq \kappa$, where κ is defined in Lemma 3.3, we can proceed in the same way as in the proof of Lemma 4.3 in [16] to obtain $q^{(n)} \rightharpoonup q \neq 0$ in H up to a translation of indices, and $J'(q) = 0$. We omit the details. □

Proof of Theorem 1.1 Lemma 3.7 shows that $\mathcal{M}$ is not an empty set. To obtain the ground state solution, we denote $c_0 = \inf_{\mathcal{M}} J$. By Lemma 2.3, one has $J(u) \geq J(0) = 0$ for all $u \in \mathcal{M}$. Thus $c_0 \geq 0$. Let $(q^{(n)}) \in \mathcal{M}$ be such that $J(q^{(n)}) \to c_0$. Then $\langle J'(q^{(n)}), p\rangle = 0$ for any $p \in H$. According to the proof of Lemma 3.6, we can certify that $(q^{(n)})$ is bounded in H, so it demonstrates that $q_0 \in \mathcal{M}$ such that $J(q_0) = c_0 = \inf_{\mathcal{M}} J$ by a standard argument.

Finally, we show that the solution obtained above is nonconstant for some suitable T. One can proceed in the same way as in the proof of Theorem 1.1 in [16] to prove that there exists $T_{\min} > 0$ which depends on $\alpha_i : \alpha_i > 0$ and corresponding V_i such that if $T_{\min} < T$, then the solution obtained above is nonconstant, we omit its proof. Since the conditions on $T_{\min}$ and $\pi/\sqrt{\beta}$ are independent of each other, there are potentials for which this inequality $T_{\min} < \pi/\sqrt{\beta}$ is satisfied. Therefore when the coefficients α_i take both signs, our method guarantees the existence of a nonconstant solution for system (1.2) only if $T_{\min} < \pi/\sqrt{\beta}$. □

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