Abdelwahed and Chorfi *Boundary Value Problems* (2020) 2020:152
https://doi.org/10.1186/s13661-020-01448-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Spectral discretization of time-dependent vorticity–velocity–pressure formulation of the Navier–Stokes equations

Mohamed Abdelwahed[1] and Nejmeddine Chorfi[1*]

*Correspondence:
nchorfi@ksu.edu.sa
[1]Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia

Abstract

In this work, we propose a nonstationary Navier–Stokes problem equipped with an unusual boundary condition. The time discretization of such a problem is based on the backward Euler's scheme. However, the variational formulation deduced from the nonstationary Navier–Stokes equations is discretized using the spectral method. We prove that the time semidiscrete problem and the full spectral discrete one admit at most one solution.

Keywords: Nonstationary Navier–Stokes equations; Vorticity–velocity–pressure formulation; Implicit Euler's scheme; Spectral discretization

1 Introduction

We consider Ω to be an open, bounded, and simply-connected domain of $\mathbb{R}^d$ $(d = 2, 3)$, and $\partial\Omega$ as its Lipschitz-continuous connected boundary. Let $[0, T]$ be an interval in $\mathbb{R}$ where T is a positive real number. We denote $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$ according to the dimension, and let $\mathbf{n}$ be the unit outward normal vector to Ω on the boundary $\partial\Omega$. We are interested in this paper in the following time-dependent Navier–Stokes system:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) - \nu\Delta\mathbf{u}(\mathbf{x}, t) + (\mathbf{u}\cdot\nabla\mathbf{u})(\mathbf{x}, t) + \nabla P(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) & \text{in } \Omega\times[0,\mathrm{T}], \\ \operatorname{div}\mathbf{u}(\mathbf{x}, t) = 0 & \text{in } \Omega\times[0,\mathrm{T}], \\ \mathbf{u}(\mathbf{x}, t)\cdot\mathbf{n}(\mathbf{x}) = 0 & \text{on } \partial\Omega\times[0,\mathrm{T}], \\ \tau(\mathbf{curl u})(\mathbf{x}, t) = \mathbf{0} & \text{on } \partial\Omega\times[0,\mathrm{T}], \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\mathbf{u}$ and P are the unknowns denoting the velocity and pressure of the fluid, $\mathbf{f}$ represents the density of the body forces, and ν is the viscosity that we suppose to be a positive constant. The operator τ defines the boundary value of **curlu** in dimension $d = 2$ and the boundary tangential components of **curlu** in dimension $d = 3$. This problem with nonstandard boundary conditions was studied for the first time in the pioneering paper [1] where the domain Ω is assumed to be convex in both dimension $d = 2$ and $d = 3$. The

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

basic idea in [2, 3] consists in introducing the vorticity $\omega = \mathbf{curl u}$ as a new unknown (see also [4–6]) and the fact that the convection term can be written as:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \omega \times \mathbf{u} + \frac{1}{2}\mathbf{grad}|\mathbf{u}|^2.$$

Consequently, problem (1) is fully equivalent to the following system:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \nu(\mathbf{curl}\omega)(\mathbf{x},t) + (\omega \times \mathbf{u})(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) & \text{in } \Omega \times [0,\mathrm{T}], \\ \operatorname{div} \mathbf{u}(\mathbf{x},t) = 0 & \text{in } \Omega \times [0,\mathrm{T}], \\ \omega(\mathbf{x},t) = \mathbf{curlu}(\mathbf{x},t) & \text{in } \Omega \times [0,\mathrm{T}], \\ \mathbf{u}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = 0 & \text{on } \partial\Omega \times [0,\mathrm{T}], \\ \tau(\omega)(\mathbf{x},t) = \mathbf{0} & \text{on } \partial\Omega \times [0,\mathrm{T}], \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{2}$$

The dynamical pressure p is defined by the formula

$$p = P + \frac{1}{2}|\mathbf{u}|^2.$$

We assume that the initial velocity and vorticity satisfy the following conditions:

$$\operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega \quad \text{and} \quad \omega(\mathbf{x},0) = \omega_0 = \mathbf{curlu}_0 \quad \text{in } \Omega. \tag{3}$$

This type of problem has been handled in the stationary case by the finite element method [3, 7] and by the spectral method [8, 9]. However, the nonstationary case has been considered for a posteriori analysis of the finite discretization [10, 11] and also for a spectral discretization of the linear Stokes problem [12]. Relying on the equivalent variational formulation of problem (2), we propose a semidiscrete scheme based on the backward Euler method. In dimension $d = 2$, we show that the time semidiscrete problem admits a solution with no restriction on the regularity of the domain, while in dimension $d = 3$, we assume that the domain Ω has either a $\mathcal{C}^{1,1}$ boundary or is a polyhedron with no reentrant corners. Note, however, that this result of existence is established only for a sufficiently large viscosity in dimension $d = 3$. For such a problem, we also study the uniqueness of the solution. We prove the existence of a discrete solution of the full spectral discrete problem.

The paper is organized as follows:

- In Sect. 2, we present the variational formulation and the analysis of the model.
- Sect. 3 is devoted to the time semidiscrete problem and to the proof of existence and uniqueness results.
- In Sect. 4 we prove the well posedness of the full spectral discrete problem.

2 The variational formulation

To write the variational formulation of problem (2), we suggest the following Sobolev spaces:

$$W^{m,p}(\Omega) = \left\{\phi \in L^p(\Omega); \partial^\alpha \phi \in L^p(\Omega), \forall|\alpha| \le m\right\},$$

each of which is a Banach space equipped with the following norm and seminorm:

$$\|\phi\|_{m,p,\Omega} = \Big(\sum_{|\alpha|\le m}\int_\Omega \big|\partial^\alpha\phi(\mathbf{x})\big|^p\Big)^{\frac{1}{p}} \quad \text{and} \quad |\phi|_{m,p,\Omega} = \Big(\sum_{|\alpha|=m}\int_\Omega \big|\partial^\alpha\phi(\mathbf{x})\big|^p\Big)^{\frac{1}{p}}.$$

If $p = 2$, $W^{m,2}(\Omega) = H^m(\Omega)$ is a Hilbert space equipped with the scalar product

$$(\phi,\psi)_{m,\Omega} = \Big(\sum_{|\alpha|\le m}\big(\partial^\alpha\phi,\partial^\alpha\psi\big)^2\Big)^{\frac{1}{2}}.$$

We denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ scalar product. We note that $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ which have a null integral on Ω, and $\mathcal{D}(\Omega)$ is the space of indefinitely differentiable functions with a compact support in Ω. We consider the domain $H(\mathrm{div},\Omega)$ of the div operator,

$$H(\mathrm{div},\Omega) = \big\{\boldsymbol{\varphi}\in L^2(\Omega)^d; \mathrm{div}\,\boldsymbol{\varphi}\in L^2(\Omega)\big\},$$

provided with the norm

$$\|\boldsymbol{\varphi}\|_{H(\mathrm{div},\Omega)} = \big(\|\boldsymbol{\varphi}\|^2_{L^2(\Omega)^d} + \|\,\mathrm{div}\,\boldsymbol{\varphi}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}.$$

We remind (see [13, Chap. I, Sect. 2]) that the normal trace operator is defined from $H(\mathrm{div},\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$ such that for any scalar function χ in $H(\mathrm{div},\Omega)$,

$$\langle\boldsymbol{\varphi}\cdot\mathbf{n},\chi\rangle = \int_\Omega \mathrm{div}\,\boldsymbol{\varphi}(\mathbf{x})\chi(\mathbf{x})\,d\mathbf{x} + \int_\Omega \boldsymbol{\varphi}(\mathbf{x})\cdot\nabla\chi(\mathbf{x})\,d\mathbf{x},$$

where $\langle\cdot,\cdot\rangle$ is the duality product between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. This allows us to introduce the kernel of the normal trace operator in $H(\mathrm{div},\Omega)$,

$$H_0(\mathrm{div},\Omega) = \{\boldsymbol{\varphi}\in H(\mathrm{div},\Omega); \boldsymbol{\varphi}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Furthermore, we consider the domain $H(\mathbf{curl},\Omega)$ of the **curl** operator,

$$H(\mathbf{curl},\Omega) = \big\{\boldsymbol{\varphi}\in L^2(\Omega)^d; \mathbf{curl}\boldsymbol{\varphi}\in L^2(\Omega)^{\frac{d(d-1)}{2}}\big\},$$

equipped with the norm

$$\|\boldsymbol{\varphi}\|_{H(\mathbf{curl},\Omega)} = \big(\|\boldsymbol{\varphi}\|^2_{L^2(\Omega)^d+\|\mathbf{curl}\boldsymbol{\varphi}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}}}\big)^{\frac{1}{2}}.$$

In dimension $d = 3$, the tangential trace operator τ is defined from $H(\mathbf{curl},\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)^3$, for any vector field $\boldsymbol{\chi}\in H(\mathbf{curl},\Omega)$, by

$$\langle\boldsymbol{\varphi}\times\mathbf{n},\boldsymbol{\chi}\rangle = \int_\Omega \boldsymbol{\varphi}(\mathbf{x})\cdot\mathbf{curl}\boldsymbol{\chi}(\mathbf{x})\,d\mathbf{x} - \int_\Omega \mathbf{curl}\boldsymbol{\varphi}(\mathbf{x})\cdot\boldsymbol{\chi}(\mathbf{x})\,d\mathbf{x}.$$

Then we introduce the kernel of the tangential operator in $H(\mathbf{curl},\Omega)$,

$$H_0(\mathbf{curl},\Omega) = \{\boldsymbol{\varphi} \in H(\mathbf{curl},\Omega); \varphi \times \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Remark 1 In dimension $d = 2$, the space $H(\mathbf{curl},\Omega)$, respectively $H_0(\mathbf{curl},\Omega)$, is equal to the space $H^1(\Omega)$, respectively to $H_0^1(\Omega)$.

In order to handle some problems depending on time, we consider the following spaces.

Let Z be a separable Banach space. We introduce $\mathcal{C}^m(0,T;Z)$, the set of time $\mathcal{C}^m$ class functions with values in Z; $\mathcal{C}^m(0,T;Z)$ is a Banach space provided with the norm

$$\|\boldsymbol{\varphi}\|_{\mathcal{C}^m(0,T;Z)} = \sup_{0\leq t\leq T} \sum_{i=0}^{m} \left\|\partial_t^i \boldsymbol{\varphi}\right\|_Z,$$

where $\partial_t^i \boldsymbol{\varphi}$ is the partial derivative of order i in time of the function $\boldsymbol{\varphi}$. Consider also the spaces

$$L^p(0,T;Z) = \left\{\boldsymbol{\varphi} \text{ measurable on }]0,T[\text{ such that } \int_0^T \left\|\boldsymbol{\varphi}(t,\cdot)\right\|_Z^p dt < \infty\right\}$$

and

$$H^s(0,T;Z) = \left\{\boldsymbol{\varphi} \in L^2(0,T;Z); \partial^m \boldsymbol{\varphi} \in L^2(0,T;Z); m \leq s\right\}.$$

Then $L^p(0,T;Z)$ is a Banach space equipped with the norm

$$\|\boldsymbol{\varphi}\|_{L^p(0,T;Z)} = \begin{cases} (\int_0^T \|\boldsymbol{\varphi}(t,\cdot)\|_Z^p \, dt)^{\frac{1}{p}} & \text{for } 1 \leq p < +\infty, \\ \sup_{0\leq t\leq T} \|\boldsymbol{\varphi}(t,\cdot)\|_Z & \text{for } p = +\infty, \end{cases}$$

and $H^s(0,T;Z)$ is a Hilbert space when it is equipped with the following scalar product:

$$(\boldsymbol{\varphi},\psi)_{H^s(0,T;Z)} = \left((\boldsymbol{\varphi},\psi)^2_{L^2(0,T;Z)} + \sum_{k=0}^{s} \left(\partial^k \boldsymbol{\varphi}, \partial^k \psi\right)^2_{L^2(0,T;Z)}\right)^{\frac{1}{2}}.$$

Finally, we define also $\mathcal{L}(Z)$ to be the Banach space of the linear and continuous functions from Z to $\mathbb{R}$ provided with the norm

$$\forall L \in \mathcal{L}(Z), \quad \|L\|_{\mathcal{L}(Z)} = \sup_{x\in Z/\{0\}} \frac{|L(x)|}{\|x\|_Z}.$$

We assume that the data $\mathbf{f}$ belongs to the space $L^2(0,T;H_0(\mathrm{div},\Omega)')$, where $H_0(\mathrm{div},\Omega)'$ is the dual space of $H_0(\mathrm{div},\Omega)$ (see [14] for more details about the space $H_0(\mathrm{div},\Omega)'$). Consider the following variational formulation:

Find $(\omega,\mathbf{u},p) \in L^2(0,T;H_0(\mathbf{curl},\Omega)) \times L^2(0,T;H_0(\mathrm{div},\Omega)) \times L^2(0,T;L_0^2(\Omega))$ such that

$$\begin{cases} \forall \mathbf{v} \in H_0(\mathrm{div},\Omega), & (\frac{\partial \mathbf{u}}{\partial t},\mathbf{v}) + a(\omega,\mathbf{u};\mathbf{v}) + N(\omega,\mathbf{u};\mathbf{v}) + b(\mathbf{v},p) = \prec \mathbf{f},\mathbf{v} \succ, \\ \forall q \in L_0^2(\Omega), & b(\mathbf{u},q) = 0, \\ \forall \boldsymbol{\vartheta} \in H_0(\mathbf{curl},\Omega), & c(\omega,\mathbf{u};\boldsymbol{\vartheta}) = 0, \end{cases} \tag{4}$$

where $\prec \cdot, \cdot \succ$ is the duality product between $H_0(\mathrm{div}, \Omega)'$ and $H_0(\mathrm{div}, \Omega)$; $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$, and $c(\cdot, \cdot; \cdot)$ are bilinear forms defined as follows:

$$a(\omega, \mathbf{u}; \mathbf{v}) = \nu \int_\Omega \mathbf{curl}(\omega)(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}, \qquad b(\mathbf{u}, q) = -\int_\Omega \mathrm{div}\, \mathbf{u}(\mathbf{x}, t) q(\mathbf{x})\, d\mathbf{x}$$
$$c(\omega, \mathbf{u}; \boldsymbol{\vartheta}) = \int_\Omega \omega(\mathbf{x}, t) \cdot \boldsymbol{\vartheta}(\mathbf{x})\, d\mathbf{x} - \int_\Omega \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{curl}\boldsymbol{\vartheta}(\mathbf{x})\, d\mathbf{x}.$$

On the other hand, the trilinear form $N(\cdot, \cdot; \cdot)$ is defined by

$$N(\omega, \mathbf{u}; \mathbf{v}) = \int_\Omega (\omega \times \mathbf{u})(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}. \tag{5}$$

By the same arguments used in our work about the stationary Stokes problem (see [9, Proposition 2.1]), we deduce the following

Proposition 1 $(\omega, \mathbf{u}, p) \in L^2(0, T; H_0(\mathbf{curl}, \Omega)) \times L^2(0, T; H_0(\mathrm{div}, \Omega)) \times L^2(0, T; L^2_0(\Omega))$ *is solution of problem* (2) *such that* $\omega \times \mathbf{u}$ *belongs to* $L^2(0, T; L^2(\Omega)^d)$ *if and only if it is a solution of problem* (4) *in the sense of distributions*.

To prove that problem (4) has a solution, we need to define

$$\mathrm{V} = \left\{\boldsymbol{\varphi} \in H_0(\mathrm{div}, \Omega); \forall q \in L^2_0(\Omega), b(\boldsymbol{\varphi}, q) = 0\right\},$$

which is the kernel of the bilinear form $b(\cdot, \cdot)$ and is the space of divergence-free functions in $H_0(\mathrm{div}, \Omega)$, since $H_0(\mathrm{div}, \Omega)$ is included in $L^2_0(\Omega)$. We also consider

$$\begin{aligned}\mathrm{U} &= \left\{(\boldsymbol{\vartheta}, \boldsymbol{\varphi}) \in H_0(\mathbf{curl}, \Omega) \times \mathrm{V}; \forall \boldsymbol{\psi} \in H_0(\mathbf{curl}, \Omega), c(\boldsymbol{\vartheta}, \boldsymbol{\varphi}; \boldsymbol{\psi}) = 0\right\} \\ &= \left\{(\boldsymbol{\vartheta}, \boldsymbol{\varphi}) \in H_0(\mathbf{curl}, \Omega) \times \mathrm{V}; \boldsymbol{\varphi} = \mathbf{curl}\boldsymbol{\vartheta}\right\},\end{aligned}$$

the kernel of the bilinear form $c(\cdot, \cdot; \cdot)$. We note that V and U are Hilbert spaces, since the bilinear forms $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ are continuous.

So we observe that if $(\omega, \mathbf{u}, p)$ is a solution of problem (4), then $(\omega, \mathbf{u})$ is a solution of the following reduced problem:

Find $(\omega, \mathbf{u}) \in L^2(0, T; \mathrm{U})$ such that

$$\forall \mathbf{v} \in \mathrm{V}, \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + a(\omega, \mathbf{u}; \mathbf{v}) + N(\omega, \mathbf{u}; \mathbf{v}) = \prec \mathbf{f}, \mathbf{v} \succ . \tag{6}$$

The main difficulty consists in proving the existence of a solution of problem (6). Moreover, for the three-dimensional case, in order to give a sense to the nonlinear term $N(\omega, \mathbf{u}; \mathbf{v})$, we need to assume that the spaces $H_0(\mathrm{div}, \Omega) \cap H(\mathbf{curl}, \Omega)$ and $H(\mathrm{div}, \Omega) \cap H_0(\mathbf{curl}, \Omega)$ are compactly embedded in $H^1(\Omega)$ see ([15], Theorem 2.17). Nonetheless, the proof is much simpler in dimension $d = 2$.

Assumption 1 In dimension $d = 3$, we assume that the domain Ω has a $\mathcal{C}^{1,1}$ boundary or is convex.

The spaces $L^2_0(\Omega)$ and $H(\mathrm{div},\Omega)$ verify a uniform inf-sup condition (see, for instance, [16] or [13, Chap. I, Corollary 2.4]):

There exists a constant $\beta_* > 0$ such that

$$\forall q \in L^2_0(\Omega), \qquad \sup_{\varphi \in H_0(\mathrm{div},\Omega)} \frac{b(\varphi,q)}{\|\varphi\|_{H(\mathrm{div},\Omega)}} \geq \beta_* \|q\|_{L^2(\Omega)}. \tag{7}$$

When Assumption 1 is satisfied, the arguments for the proof of the existence of a solution of problems (6) and (4) are exactly the same as for [17, Chap. III, Theorem 1.1], see also [18, Chap. V] and [8]. We prefer to omit this proof since it is beyond the aim of this paper.

3 The time semidiscrete problem

In this section we study the discretization in time of problem (4) using the implicit Euler's method. We consider a partition of the interval $[0,T]$ into subintervals $[t_{n-1},t_n]$, for $1 \leq n \leq M$, where M is a positive integer and $0 = t_0 < t_1 < \cdots < t_N = T$; $h = (h_1,h_2,\ldots,h_n)$ stands for the step of the partition where $h_n = t_n - t_{n-1}$, and $|h| = \max_{1\leq n\leq M} h_n$.

For any data $\mathbf{f} \in L^2(0,T;(H_0(\mathrm{div},\Omega))')$, $\mathbf{u}_0 \in H_0(\mathrm{div},\Omega)$, and $\omega_0 \in H_0(\mathbf{curl},\Omega)$ satisfying condition (3), we consider the following problem:

Find $(\omega^n)_{0\leq n\leq M} \in (H_0(\mathbf{curl},\Omega))^{M+1}$, $(\mathbf{u}^n)_{0\leq n\leq M} \in (H_0(\mathrm{div},\Omega))^{M+1}$ and $(p^n)_{1\leq n\leq M} \in (L^2_0(\Omega))^M$ such that

$$\omega^0 = \omega_0 \quad \text{and} \quad \mathbf{u}^0 = \mathbf{u}_0 \quad \text{in } \Omega \tag{8}$$

and for all $1 \leq n \leq M$,

$$\begin{aligned} &\forall \mathbf{v} \in H_0(\mathrm{div},\Omega), \quad \mathrm{A}\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) + h_n N\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) + h_n b\big(\mathbf{v},p^n\big) = \mathrm{L}(\mathbf{v}), \\ &\forall q \in L^2_0(\Omega), \quad b\big(\mathbf{u}^n,q\big) = 0, \\ &\forall \boldsymbol{\vartheta} \in H_0(\mathbf{curl},\Omega), \quad c\big(\omega^n,\mathbf{u}^n;\boldsymbol{\vartheta}\big) = 0, \end{aligned} \tag{9}$$

where $\mathbf{f}^n = \mathbf{f}(\cdot,t_n)$,

$$\mathrm{A}\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) = \big(\mathbf{u}^n,\mathbf{v}\big) + h_n a\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big),$$

and

$$\mathrm{L}(\mathbf{v}) = \big(\mathbf{u}^{n-1},\mathbf{v}\big) + h_n \prec \mathbf{f}^n,\mathbf{v} \succ.$$

Then, if $(\omega^n,\mathbf{u}^n,p^n)$ is a solution of problem (8)–(9), $(\omega^n,\mathbf{u}^n)$ belongs to U and is a solution of the following reduced problem:

$$\forall \mathbf{v} \in \mathrm{V}, \mathrm{A}\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) + h_n N\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) = \mathrm{L}(\mathbf{v}). \tag{10}$$

The main difficulty now consists in showing that problem (10) admits a solution. We recall here the two properties (positivity and inf-sup condition) related to the bilinear form $\mathrm{A}(\cdot,\cdot;\cdot)$ proved in [12, Lemma 1]:

There exists a constant $\beta > 0$ such that

$$\forall \mathbf{v} \in \mathrm{V}\backslash\{0\}, \qquad \sup_{(\omega^n,\mathbf{u}^n)\in \mathrm{U}} \mathrm{A}\big(\omega^n,\mathbf{u}^n;\mathbf{v}\big) > 0 \tag{11}$$

and

$$\forall\big(\omega^n,\mathbf{u}^n\big) \in \mathrm{U}, \quad \sup_{\mathbf{v}\in \mathrm{V}} \frac{\mathrm{A}(\omega^n,\mathbf{u}^n;\mathbf{v})}{\|\mathbf{v}\|_{L^2(\Omega)^d}} \geq \beta\big(\big\|\omega^n\big\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \big\|\mathbf{u}^n\big\|_{L^2(\Omega)^d}\big). \tag{12}$$

Let now study the properties of non linear term $N(\cdot,\cdot;\cdot)$.

Lemma 1 *Under Assumption* 1, *we have that*

1) $N(\cdot,\cdot;\cdot)$ *is continuous, that is,*

$$\begin{aligned}&\forall(\omega,\mathbf{u}) \in \mathrm{U}, \forall\mathbf{v} \in \mathrm{V},\\ &N(\omega,\mathbf{u};\mathbf{v}) \leq M\|\omega\|_{H(\mathbf{curl},\Omega)}\|\mathbf{u}\|_{H(\mathrm{div},\Omega)}\|\mathbf{v}\|_{H(\mathrm{div},\Omega)},\end{aligned} \tag{13}$$

where M *is a positive constant only depending on the domain* Ω;

2) $N(\cdot,\cdot;\cdot)$ *is antisymmetric, that is,*

$$\forall(\omega,\mathbf{u}) \in \mathrm{U}, \forall\mathbf{v} \in \mathrm{V}, \quad N(\omega,\mathbf{u};\mathbf{v}) = -N(\omega,\mathbf{v};\mathbf{u}). \tag{14}$$

Proof 1) For any couple $(\omega,\mathbf{u}) \in \mathrm{U}$, we have $\omega \in H(\mathrm{div},\Omega) \cap H_0(\mathbf{curl},\Omega)$ and $\mathbf{u} \in H_0(\mathrm{div},\Omega) \cap H(\mathbf{curl},\Omega)$. From the definition of the nonlinear term $N(\cdot,\cdot;\cdot)$ in (5) and using Hölder's inequality, we derive for $\frac{1}{s} + \frac{1}{r} = \frac{1}{2}$,

$$\begin{aligned}&\forall(\omega,\mathbf{u}) \in \mathrm{U}, \forall\mathbf{v} \in \mathrm{V},\\ &N(\omega,\mathbf{u};\mathbf{v}) \leq M\|\omega\|_{L^r(\Omega)^{\frac{d(d-1)}{2}}}\|\mathbf{u}\|_{L^s(\Omega)^d}\|\mathbf{v}\|_{L^2(\Omega)^d}.\end{aligned}$$

• In dimension 2, $H_0(\mathbf{curl},\Omega) = H_0^1(\Omega)$ is compactly included in the space $L^p(\Omega)$ for any $p < +\infty$, while $H_0(\mathrm{div},\Omega) \cap H(\mathbf{curl},\Omega)$ is compactly embedded in $L^3(\Omega)^2$ since it is included in $H^{\frac{1}{2}}(\Omega)^2$, with $r = 6$ and $s = 3$ (see [19]).

• In dimension 3, if Assumption 1 holds, using the Sobolev embedding theorem, the space U is compactly included in $L^r(\Omega)^3 \times L^s(\Omega)^3$ for $r = 4$ and $s = 4$.

2) Since $\mathbf{u} \in \mathrm{V}$ and equality (15) holds, we easily deduce the antisymmetry property (14). Equality (15) is proved by using Green formula:

$$\int_\Omega (\mathbf{curl u} \times \mathbf{u}) \cdot \mathbf{v}\,d\mathbf{x} = -\int_\Omega (\mathbf{curl u} \times \mathbf{v}) \cdot \mathbf{u}\,d\mathbf{x} - \int_\Omega (\mathbf{u} \cdot \mathbf{v})\,\mathrm{div}\,\mathbf{u}\,d\mathbf{x}. \tag{15}$$

□

Proposition 2 *Assume that the data* $\mathbf{f}$ *belongs to* $L^2(0,t;H_0(\mathrm{div},\Omega)')$ *and that the initial vorticity–velocity* $(\omega_0,\mathbf{u}_0)$ *belongs to* $H_0(\mathbf{curl},\Omega) \times H_0(\mathrm{div},\Omega)$ *and satisfies* (3). *Knowing* $\mathbf{u}^{n-1}$ *at each time step* n, *problem* (10) *has a solution* $(\omega^n,\mathbf{u}^n)$ *in* U *for dimension* $d = 2$. *This solution satisfies for* $n \geq 1$:

$$\sum_{j=1}^n \big\|\omega^j\big\|^2_{H(\mathbf{curl},\Omega)} + \big\|\mathbf{u}^n\big\|^2_{L^2(\Omega)^d}$$

$$\leq \frac{c}{\nu}\left(\|\mathbf{u}_0\|^2_{L^2(\Omega)^d} + \sum_{j=1}^{n} h_j \|\mathbf{f}^j\|^2_{(H_0(\mathrm{div},\Omega))'}\right), \tag{16}$$

where c is a positive constant independent of n.

Proof We consider the function ϖ defined on U into its dual space by

$$\begin{aligned}
&\forall(\omega^n, \mathbf{u}^n) \in \mathrm{U}, \forall(\theta, \mathbf{v}) \in \mathrm{U},\\
&\langle \varpi(\omega^n, \mathbf{u}^n), (\theta, \mathbf{v})\rangle = A(\omega^n, \mathbf{u}^n; \mathbf{v} + \mathbf{curl}\theta) + h_n N(\omega^n, \mathbf{u}^n; \mathbf{v} + \mathbf{curl}\theta)\\
&\qquad - L(\mathbf{v} + \mathbf{curl}\theta).
\end{aligned}$$

We derive from the continuity of the trilinear form N in (13) that the function ϖ is continuous on the space U. Having the following inequality proved in [12]:

$$A(\omega, \mathbf{u}; \mathbf{u} + \mathbf{curl}\omega) \geq \frac{\nu}{2}\|\omega\|^2_{H(\mathbf{curl},\Omega)} + \frac{\nu}{2c_0^2}\|\mathbf{u}\|^2_{L^2(\Omega)^2}, \tag{17}$$

and the fact that $N(\omega, \mathbf{u}; \mathbf{curl}\omega) = 0$ in dimension 2, we obtain

$$\begin{aligned}
\langle \varpi(\omega^n, \mathbf{u}^n), (\theta, \mathbf{v})\rangle \geq{}& \frac{\nu}{2}\|\omega^n\|^2_{H(\mathbf{curl},\Omega)} + \frac{\nu}{2c_0^2}\|\mathbf{u}^n\|^2_{L^2(\Omega)^2}\\
&- \|L\|_{\mathcal{L}(H_0(\mathrm{div},\Omega))}\left(\|\omega^n\|^2_{H(\mathbf{curl},\Omega)} + \|\mathbf{u}^n\|^2_{L^2(\Omega)^2}\right),
\end{aligned}$$

where c_0 is the smallest constant such that

$$\forall\varphi \in \mathrm{V}; \quad \|\varphi\|_{L^2(\Omega)^2} \leq c_0\|\mathbf{curl}\varphi\|_{L^2(\Omega)}.$$

Then if we consider the sphere with radius

$$r = \frac{2\sqrt{2}\max(1, c_0^2)}{\nu}\|L\|_{\mathcal{L}(H_0(\mathrm{div},\Omega))},$$

the duality product $\langle \varpi(\omega^n, \mathbf{u}^n), (\theta, \mathbf{v})\rangle$ is nonnegative on this sphere. The space U is a separable Hilbert space since it is included in $L^2(\Omega) \times L^2(\Omega)^2$. Then there exists an increasing sequence $(\mathrm{U_k})_k$ of finite-dimensional subspaces $\mathrm{U_k}$ of U such that $\bigcup_k(\mathrm{U_k})$ is dense in U. Using Brouwer's fixed point theorem [13, Chap. IV, Corollary 1.1], there exists $(\omega_k^n, \mathbf{u}_k^n) \in \mathrm{U_k}$ such that

$$\begin{aligned}
&\forall(\theta_k, \mathbf{v}_k) \in \mathrm{U_k}, \quad \langle \varpi(\omega_k^n, \mathbf{u}_k^n), (\theta_k, \mathbf{v}_k)\rangle = 0\\
&\text{and} \quad \left(\|\omega_k^n\|^2_{H(\mathbf{curl},\Omega)} + \|\mathbf{u}_k^n\|^2_{H(\mathrm{div},\Omega)}\right) \leq r^2.
\end{aligned} \tag{18}$$

Since the sequence $(\omega_k^n, \mathbf{u}_k^n)$ is bounded by r, there exists a subsequence, still denoted by $(\omega_k^n, \mathbf{u}_k^n)$, that converges weakly to $(\omega^n, \mathbf{u}^n)$. Now, using the fact that the two operators $\forall(\theta, \mathbf{v}) \in \mathrm{U}$: $(\omega, \mathbf{u}) \to \omega \times \mathbf{v}$ and $(\omega, \mathbf{u}) \to \theta \times \mathbf{u}$ are compact on U, we conclude in passing to the limit in equation (18) that

$$\forall(\theta_k, \mathbf{v}_k) \in \mathrm{U_k}, \quad \langle \varpi(\omega^n, \mathbf{u}^n), (\theta_k, \mathbf{v}_k)\rangle = 0.$$

Using the density of $\bigcup_k(\mathrm{U_k})$ in U yields

$$\forall(\theta,\mathbf{v})\in \mathrm{U},\quad \left\langle \varpi\left(\omega^n,\mathbf{u}^n\right),(\theta,\mathbf{v})\right\rangle = 0.$$

Then we conclude that $(\omega^n,\mathbf{u}^n)$ is a solution of problem (10) for any $w=\theta+\mathbf{curl}\mathbf{v}\in \mathrm{V}$. Moreover, since the solution $(\omega^n,\mathbf{u}^n)$ is bounded by r, we obtain (16) by the same proof as for [12, Corollary 1]. □

In dimension $d=3$, the existence of the solution of problem (10) is obtained for a large enough viscosity ν with respect to the norm of the operator L.

Proposition 3 *Assume that the data* $\mathbf{f}$ *belongs to* $L^2(0,T;H_0(\mathrm{div},\Omega)')$ *and that the initial vorticity–velocity* $(\omega_0,\mathbf{u}_0)$ *belongs to* $H_0(\mathbf{curl},\Omega)\times H_0(\mathrm{div},\Omega)$ *and satisfies* (3). *If Assumption* 1 *holds and there exists a constant* c^* *such that*

$$c^*\nu^{-2}\|L\|_{\mathcal{L}(H_0(\mathrm{div},\Omega))}<1, \tag{19}$$

then, knowing $\mathbf{u}^{n-1}$ *at each time step* n, *problem* (10) *has a solution* $(\omega^n,\mathbf{u}^n)\in\mathrm{U}$. *This solution satisfies*

$$\sum_{j=1}^{n}\left\|\omega^j\right\|^2_{H(\mathbf{curl},\Omega)}+\left\|\mathbf{u}^n\right\|^2_{L^2(\Omega)^d}\le\frac{c}{\nu}\left(\|\mathbf{u}_0\|^2_{L^2(\Omega)^d}+\sum_{j=1}^{n}h_j\left\|\mathbf{f}^j\right\|^2_{(H_0(\mathrm{div},\Omega))'}\right), \tag{20}$$

where c is a positive constant independent of n.

Proof We consider an iterative sequence $((\omega_k^n,\mathbf{u}_k^n))_k$ such that $(\omega_0^n,\mathbf{u}_0^n)=(0,0)$ and $(\omega_k^n,\mathbf{u}_k^n)$ is the solution in U of the following problem:

$$\forall\mathbf{v}\in\mathrm{V},\quad A\left(\omega_k^n,\mathbf{u}_k^n;\mathbf{v}\right)=L(\mathbf{v})-h_nN\left(\omega_{k-1}^n,\mathbf{u}_{k-1}^n;\mathbf{v}\right). \tag{21}$$

From the properties of the bilinear form $A(\cdot,\cdot;\cdot)$ in (11)–(12) and the continuity of the trilinear form $N(\cdot,\cdot;\cdot)$ in (13), problem (21) has a unique solution; see [9]. Let

$$r=\frac{\nu}{4M\sqrt{2}\max(1,c_0^2)}, \tag{22}$$

where M is the continuity constant of $N(\cdot,\cdot;\cdot)$. By induction on k and by an appropriate choice of c^* in (19), the sequence $((\omega_k^n,\mathbf{u}_k^n))_k$ is bounded by r:

$$\left(\left\|\omega_k^n\right\|^2_{H(\mathbf{curl},\Omega)}+\left\|\mathbf{u}_k^n\right\|^2_{H(\mathrm{div},\Omega)}\right)\le r^2. \tag{23}$$

We have for any $k\ge 2$,

$$\begin{aligned}A\left(\omega_k^n-\omega_{k-1}^n,\mathbf{u}_k^n-\mathbf{u}_{k-1}^n;\mathbf{v}\right)&=N\left(\omega_{k-2}^n,\mathbf{u}_{k-2}^n;\mathbf{v}\right)-N\left(\omega_{k-1}^n,\mathbf{u}_{k-1}^n;\mathbf{v}\right)\\&=-N\left(\omega_{k-1}^n-\omega_{k-2}^n,\mathbf{u}_{k-2}^n;\mathbf{v}\right)-N\left(\omega_{k-1}^n,\mathbf{u}_{k-1}^n-\mathbf{u}_{k-2}^n;\mathbf{v}\right).\end{aligned}$$

Thanks to (17), (13), and (23), together with (22), we obtain that

$$\begin{aligned}&\big(\|\omega_k^n-\omega_{k-1}^n\|^2_{H(\mathbf{curl},\Omega)}+\|\mathbf{u}_k^n-u_{k-1}^n\|^2_{H(\mathrm{div},\Omega)}\big)\\&\quad\le\frac{1}{8}\big(\|\omega_{k-1}^n-\omega_{k-2}^n\|^2_{H(\mathbf{curl},\Omega)}+\|\mathbf{u}_{k-1}^n-u_{k-2}^n\|^2_{H(\mathrm{div},\Omega)}\big).\end{aligned}$$

The sequence $((\omega_k^n,\mathbf{u}_k^n))_k$ is a Cauchy sequence in U, so it converges to a pair $(\omega^n,\mathbf{u}^n)$. By passing to the limit in (21), it is readily checked that $(\omega^n,\mathbf{u}^n)$ is a solution of problem (10). Moreover, the solution $(\omega^n,\mathbf{u}^n)$ is bounded by r, then we obtain (20) by the same proof as for [12, Corollary 1]. □

Theorem 1 *Assume that the data* $\mathbf{f}$ *belongs to* $L^2(0,t;(H_0(\mathrm{div},\Omega))')$ *and that the initial vorticity–velocity* $(\omega_0,\mathbf{u}_0)$ *belongs to* $H_0(\mathbf{curl},\Omega)\times H_0(\mathrm{div},\Omega)$ *and satisfies* (3). *In dimension* $d=2$, *for any* $n, 1\le n\le M$, *problem* (8)–(9) *has at most one solution* $(\omega^n,\mathbf{u}^n,p^n)$ *in* $H_0(\mathbf{curl},\Omega)\times H_0(\mathrm{div},\Omega)\times L^2(\Omega)$. *In dimension* $d=3$, *if Assumption* 1 *holds, such that* (19) *is satisfied, problem* (8)–(9) *has at most one solution* $(\omega^n,\mathbf{u}^n,p^n)$ *in* $H_0(\mathbf{curl},\Omega)\times H_0(\mathrm{div},\Omega)\times L^2(\Omega)$.

Proof Let $(\omega_1^n,\mathbf{u}_1^n)$ and $(\omega_2^n,\mathbf{u}_2^n)$ be two solutions of problem (10) such that $\omega_1^j=\omega_2^j$ and $\mathbf{u}_1^j=\mathbf{u}_2^j$ for $0\le j\le(n-1)$. In dimension $d=2$, both $(\omega_1^n,\mathbf{u}_1^n)$ and $(\omega_2^n,\mathbf{u}_2^n)$ satisfy (16). Similarly, in dimension $d=3$, it follows from Assumption 1 and condition (19) that both $(\omega_1^n,\mathbf{u}_1^n)$ and $(\omega_2^n,\mathbf{u}_2^n)$ satisfy (20). On the other hand, the pair $(\omega^n,\mathbf{u}^n)$, with $\omega^n=\omega_1^n-\omega_2^n$ and $\mathbf{u}^n=\mathbf{u}_1^n-\mathbf{u}_2^n$, belongs to U and satisfies:

$$\forall v\in\mathrm{V},\quad A(\omega^n,\mathbf{u}^n;\mathbf{v})=N(\omega_2^n,\mathbf{u}_2^n;\mathbf{v})-N(\omega_1^n,\mathbf{u}_1^n;\mathbf{v})=-N(\omega^n,\mathbf{u}_2^n;\mathbf{v})-N(\omega_1^n,\mathbf{u}^n;\mathbf{v}).$$

By using (17) and taking $\mathbf{v}=\mathbf{u}^n+\mathbf{curl}\,\omega^n$ in the previous line, we have

$$\frac{\nu}{2\max(1,c_0^2)}\big(\|\omega^n\|^2_{H(\mathbf{curl},\Omega)}+\|\mathbf{u}^n\|^2_{L^2(\Omega)^d}\big)\le\big(|N(\omega^n,\mathbf{u}_2^n;\mathbf{v})|+|N(\omega_1^n,\mathbf{u}^n;\mathbf{v})|\big).$$

Using the antisymmetry property of the trilinear form $N(\cdot,\cdot;\cdot)$ (14) gives $N(\omega_1^n,\mathbf{u}^n;\mathbf{u}^n)=0$. Applying the continuity of $N(\cdot,\cdot;\cdot)$ in (13) together with (16) and (20) gives

$$\begin{aligned}&\frac{\nu}{2\max(1,c_0^2)}\big(\|\omega^n\|^2_{H(\mathbf{curl},\Omega)}+\|\mathbf{u}^n\|^2_{L^2(\Omega)^d}\big)\\&\quad\le c\nu^{-1}\|L\|_{\mathcal{L}(H_0(\mathrm{div},\Omega))}\big(\|\omega^n\|^2_{H(\mathbf{curl},\Omega)}+\|\mathbf{u}^n\|^2_{L^2(\Omega)^d}\big).\end{aligned}$$

Thus, if condition (19) holds, we obtain that ω^n and $\mathbf{u}^n$ are equal to zero.

Let $(\omega^n,\mathbf{u}^n)$ be a solution of problem (8)–(10). We consider for any $\mathbf{v}\in H_0(\mathrm{div},\Omega)$,

$$\Upsilon_n(\mathbf{v})=\prec\mathbf{f}^n,\mathbf{v}\succ-a(\omega^n,\mathbf{u}^n;\mathbf{v})-N(\omega^n,\mathbf{u}^n;\mathbf{v})-\frac{1}{h_n}(\mathbf{u}^n-\mathbf{u}^{n-1},\mathbf{v}),$$

a linear and continuous functional which vanishes on the space V. Then, according to the inf-sup condition (7), there exists a unique p^n in $L_0^2(\Omega)$ such that

$$\forall\mathbf{v}\in H_0(\mathrm{div},\Omega),\ \Upsilon_n(\mathbf{v})=-(\mathrm{div}\,\mathbf{v},p^n)$$

and

$$\|p^n\|_{L^2(\Omega)} \le \left(\frac{1}{\beta_*}\right) \sup_{\mathbf{v}\in H(\mathrm{div},\Omega)} \left(\frac{\Upsilon_n(\mathbf{v})}{\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}}\right). \qquad \square$$

4 The full discrete problem

Hereinafter, we suppose that Ω is a square $[-1,1]^2$ in dimension $d = 2$ or a cube $[-1,1]^3$ in dimension $d = 3$. To define the discrete spaces, we follow the same idea proposed by Nédélec in the case of the finite element method (see [20, Sect. 2]). If $d = 2$ (resp. $d = 3$), let $\mathbb{P}_{nm}(\Omega)$ (resp. $\mathbb{P}_{nms}(\Omega)$) be the space of the restriction on Ω of the polynomials of degree n in the x direction and m in the y direction (resp. and s in the z direction). Let $N \ge 2$ be an integer. We consider the spaces:

$$\mathbb{D}_N = H_0(\mathrm{div},\Omega) \cap \begin{cases} \mathbb{P}_{N,N-1}(\Omega) \times \mathbb{P}_{N-1,N}(\Omega) & \text{if } d = 2, \\ \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega) & \text{if } d = 3, \end{cases}$$

which approximates the velocity in $H_0(\mathrm{div},\Omega)$ and

$$\mathbb{C}_N = \begin{cases} H_0^1(\Omega) \cap \mathbb{P}_N(\Omega) & \text{if } d = 2, \\ H_0(\mathbf{curl},\Omega) \cap (\mathbb{P}_{N-1,N,N}(\Omega) \times \mathbb{P}_{N,N-1,N}(\Omega) \times \mathbb{P}_{N,N,N-1}(\Omega)) & \text{if } d = 3, \end{cases}$$

which approximates the vorticity in $H_0(\mathbf{curl},\Omega)$. This space is rather different conforming to the dimension (see Remark 1).

Lastly, for the approximation of pressure in $L_0^2(\Omega)$, we propose the space

$$\mathbb{M}_N = L_0^2(\Omega) \cap \mathbb{P}_{N-1}(\Omega).$$

Having $\xi_0 = -1$ and $\xi_N = 1$, we consider the $N-1$ nodes ξ_j, $1 \le j \le N-1$, roots of the polynomial $L_N{}'$, where L_N is the Legendre polynomial of degree N and the $N+1$ weights ρ_j, $0 \le j \le N$, of the Gauss–Lobatto quadrature formula. Let $\mathbb{P}_n(-1,1)$ the space of restrictions to $]-1,1[$ of polynomials with degree $\le n$, then

$$\forall \phi \in \mathbb{P}_{2N-1}(-1,1), \qquad \int_{-1}^{1} \phi(x)\,dx = \sum_{j=0}^{N} \phi(\xi_j)\rho_j. \tag{24}$$

We remind the important following property, see [21] for its proof:

$$\forall \chi_N \in \mathbb{P}_N(-1,1), \qquad \|\chi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^{N} \chi_N^2(\xi_j)\rho_j \le 3\|\chi_N\|^2_{L^2(-1,1)}. \tag{25}$$

Based on formula (25), we define the following discrete scalar product on $\mathbb{P}_N(\Omega)$: For a continuous functions φ and ψ on $\bar{\Omega}$,

$$(\varphi,\psi)_N = \begin{cases} \sum_{i=0}^{N}\sum_{j=0}^{N} \varphi(\xi_i,\xi_j)\psi(\xi_i,\xi_j)\rho_i\rho_j & \text{if } d = 2, \\ \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} \varphi(\xi_i,\xi_j,\xi_k)\psi(\xi_i,\xi_j,\xi_k)\rho_i\rho_j\rho_k & \text{if } d = 3. \end{cases}$$

Finally, we consider I_N to be the Lagrange interpolating operator at the nodes (ξ_i, ξ_j), $0 \le i, j \le N$, in dimension $d = 2$, and at the nodes (ξ_i, ξ_j, ξ_k), $0 \le i, j, k \le N$, in dimension $d = 3$, with values in the space $\mathbb{P}_N(\Omega)$.

We assume that the data $\mathbf{f}$ is continuous on $\overline{\Omega} \times [0, T]$. We construct the discrete problem from problem (8)–(9) by means of the Galerkin method combined with numerical integration.

If $\mathbf{u}_N^0 = \mathrm{I}_N(\mathbf{u}_0)$, knowing $\mathbf{u}^{n-1}$, we find $(\omega_N^n, \mathbf{u}_N^n, p_N^n)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ such that for $1 \le n \le M$,

$$\begin{aligned} &\forall \mathbf{v}_N \in \mathbb{D}_N, \quad A_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) + h_n N_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) + h_n b_N\big(\mathbf{v}_N, p_N^n\big) = L_N(\nu_N), \\ &\forall q_N \in \mathbb{M}_N, \quad b_N\big(\mathbf{u}_N^n, q_N\big) = 0, \\ &\forall \boldsymbol{\vartheta}_N \in \mathbb{C}_N, \quad c_N\big(\omega_N^n, \mathbf{u}_N^n; \boldsymbol{\vartheta}_N\big) = 0, \end{aligned} \tag{26}$$

where the bilinear forms $A_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$, and $c_N(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned} &A_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) = \big(\mathbf{u}_N^n, \mathbf{v}_N\big)_N + h_n \nu \big(\mathbf{curl}\,\omega_N^n, \mathbf{v}_N\big)_N, \qquad b_N(\mathbf{v}_N, q_N) = -(\operatorname{div} \mathbf{v}_N, q_N)_N, \\ &\text{and} \quad c_N\big(\omega_N^n, \mathbf{u}_N^n; \boldsymbol{\varphi}_N\big) = \big(\omega_N^n, \boldsymbol{\varphi}_N\big)_N - \big(\mathbf{u}_N^n, \mathbf{curl}\,\boldsymbol{\varphi}_N\big)_N. \end{aligned}$$

By formula (25) combined with Cauchy–Schwarz inequality, we prove that the bilinear forms $A_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$, and $c_N(\cdot,\cdot;\cdot)$ are respectively continuous on $(\mathbb{C}_N \times \mathbb{D}_N) \times \mathbb{D}_N$, $\mathbb{D}_N \times \mathbb{M}_N$, and $(\mathbb{C}_N \times \mathbb{D}_N) \times \mathbb{C}_N$ with constants independent of N. The functional $L_N(\mathbf{v}_N) = (\mathbf{u}_N^{n-1}, \mathbf{v}_N)_N + h_n(\mathrm{I}_N(\mathbf{f}^n), \mathbf{v}_N)_N$ is linear and continuous on $\mathbb{D}_N$. Moreover, as a consequence of the exactness property (24), the bilinear forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ coincide on $\mathbb{D}_N \times \mathbb{M}_N$. While the trilinear form $N_N(\cdot,\cdot;\cdot)$ is defined as follows:

$$N_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) = \big(\omega_N^n \times \mathbf{u}_N^n, \mathbf{v}_N\big)_N.$$

To prove that problem (26) has a solution, we consider the kernel

$$\mathrm{V}_N = \big\{\mathbf{v}_N \in \mathbb{D}_N; \forall q_N \in \mathbb{M}_N, b_N(\mathbf{v}_N, q_N) = 0\big\} = \mathbb{D}_N \cap V. \tag{27}$$

We remark that V_N is the space of divergence-free polynomials in $\mathbb{D}_N$ (if $q_N = \operatorname{div} \mathbf{v}_N$ in (27)).

Let also the kernel

$$\mathrm{U}_N = \big\{(\boldsymbol{\vartheta}_N, \mathbf{v}_N) \in \mathbb{C}_N \times \mathrm{V}_N; \forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, c_N(\boldsymbol{\vartheta}_N, \mathbf{v}_N; \boldsymbol{\varphi}_N) = 0\big\}.$$

We note that the discrete kernel U_N is not contained in U in the general case; see [9, Corollary 3.2]. We consider herein the following reduced discrete problem:

If $\mathbf{u}_N^0 = \mathrm{I}_N(\mathbf{u}_0)$ and knowing $\mathbf{u}^{n-1}$, find $(\omega_N^n, \mathbf{u}_N^n) \in \mathrm{U}_N$ such that for $1 \le n \le M$,

$$\forall \mathbf{v}_N \in \mathrm{V}_N, A_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) + h_n N_N\big(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N\big) = L_N(\mathbf{v}_N). \tag{28}$$

The existence of a solution of problem (28) is proved by the same arguments as for the continuous reduced problem (10), for instance.

Proposition 4 *For any data* $\mathbf{f}$ *continuous on* $\bar{\Omega} \times [0,T]$ *and knowing* $\mathbf{u}_N^{n-1}$ *at each time step* n*, problem* (28) *admits a solution* $(\omega_N^n, \mathbf{u}_N^n)$ *in* U_N*. Moreover, this solution satisfies, for* $1 \le n \le M$*,*

$$\sum_{j=1}^{n} \|\omega_N^j\|^2_{H(\mathbf{curl},\Omega)} + \|\mathbf{u}_N^n\|^2_{L^2(\Omega)^d} \le \frac{3^d c}{2\nu}\left(\|\mathbf{u}_N^0\|^2_{L^2(\Omega)^d} + \sum_{j=1}^{n} h_j \|\mathrm{I}_N(\mathbf{f}^j)\|^2_{L^2(\Omega)^d}\right), \tag{29}$$

where c *is a positive constant independent of* N *and* n.

Proof We introduce the mapping ϖ_N defined from U_N into its dual space by

$$\begin{aligned}&\forall(\omega_N^n, \mathbf{u}_N^n) \in \mathrm{U}_N, \forall(\theta_N, \mathbf{v}_N) \in \mathrm{U}_N,\\ &\langle \varpi_N(\omega_N^n, \mathbf{u}_N^n), (\theta_N, \mathbf{v}_N)\rangle = A_N(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N) + h_n N_N(\omega_N^n, \mathbf{u}_N^n; \mathbf{v}_N) - L_N(\mathbf{v}_N).\end{aligned}$$

We equip U_N with the following norm:

$$\left(\|\omega_N^n\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\mathbf{u}_N^n\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}}.$$

The mapping ϖ_N is continuous since the space U_N has a finite dimension. Next, noting by the same arguments as for Lemma 1 from the property of antisymmetry that $N_N(\omega_N^n, \mathbf{u}_N^n; \mathbf{u}_N^n) = 0$, we have

$$\langle \varpi_N(\omega_N^n, \mathbf{u}_N^n), (\omega_N^n, \mathbf{u}_N^n)\rangle = \nu(\mathbf{curl}\,\omega_N^n, \mathbf{u}_N^n)_N - L_N(\mathbf{u}_N^n).$$

Then using (25) with the definition of U_N gives

$$\langle \varpi_N(\omega_N^n, \mathbf{u}_N^n), (\omega_N^n, \mathbf{u}_N^n)\rangle \ge \|\omega_N^n\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} - 3^{\frac{d}{2}} \|L_N\|_{\mathcal{L}(\mathrm{V}_N)} (\mathbf{u}_N^n, \mathbf{u}_N^n)_N^{\frac{1}{2}}.$$

For any $\mathbf{u}_N^n$ in V_N, it follows from [9, Lemmas 3.4 and 3.5] that there exists τ_N^n in $\mathbb{C}_N$ such that $\mathbf{u}_N^n = \mathbf{curl}\,\tau_N^n$ and

$$\|\tau_N^n\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \le c \|\mathbf{u}_N^n\|_{L^2(\Omega)^d}.$$

Using once more (25) and inserting τ_N^n in the definition of the space U_N, we have

$$(\mathbf{u}_N^n, \mathbf{u}_N^n)_N = (\mathbf{u}_N^n, \mathbf{curl}\,\tau_N^n)_N = (\omega_N^n, \tau_N^n)_N \le 3^d \|\tau_N^n\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \|\omega_N^n\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

Combining all this yields

$$\langle \varpi_N(\omega_N^n, \mathbf{u}_N^n), (\omega_N^n, \mathbf{u}_N^n)\rangle = \nu \|\omega_N^n\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} - c\|L_N\|_{\mathcal{L}(\mathrm{V}_N)} \|\omega_N^n\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

Now letting

$$r_N = \frac{2c \max(1,c)}{\nu} \|L_N\|_{\mathcal{L}(\mathrm{V}_N)},$$

we deduce from (25) that

$$\left(\left\|\omega_N^n\right\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \left\|\mathbf{u}_N^n\right\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}} \le \sqrt{2}\max(1,c)\left\|\omega_N^n\right\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

Then we remark that $\langle \varpi_N(\omega_N^n, \mathbf{u}_N^n), (\omega_N^n, \mathbf{u}_N^n)\rangle$ is nonnegative on the sphere of U_N of radius r_N. So by Brouwer's fixed point theorem (see [13, Chap. IV, Corollary 1.1]), we conclude that problem (28) has a solution $(\omega_N^n, \mathbf{u}_N^n)$ in U_N. Moreover, the solution $(\omega_N^n, \mathbf{u}_N^n)$ is bounded by r_N, so we obtain (29) by the same proof as that of [12, Proposition 5)]. □

The following inf-sup condition is proved in [9, Lemma 3.9]:

There exists a positive constant γ, independent of N, such that the form $b_N(\cdot,\cdot)$ satisfies the inf-sup condition:

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\mathbf{v}_N \in \mathbb{D}_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_{H(\mathrm{div},\Omega)}} \ge \gamma \|q_N\|_{L^2(\Omega)}.$$

The full existence result is deduced from Proposition 4 and the above inf-sup condition. The proof follows the same arguments as for Theorem 1.

Theorem 4.1 *If the data* $\mathbf{f}$ *is continuous on* $\bar{\Omega} \times [0,T]$ *and knowing* $\mathbf{u}_N^{n-1}$ *at each time step* n, *problem* (26) *has a solution* $(\omega_N^n, \mathbf{u}_N^n, p_N^n)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$. *Moreover, the part* $(\omega_N^n, \mathbf{u}_N^n)$ *of this solution satisfies* (29).

Remark 2 Note that the previous existence result still holds when $N_N(\cdot,\cdot;\cdot)$ is replaced by $N(\cdot,\cdot;\cdot)$ in problem (26). This means in practice that a more precise quadrature formula, exact on $\mathbb{P}_{3N-1}(\Omega)$, is used to evaluate the integrals that appear in the treatment of the nonlinear term.

5 Conclusion

This work concerns the numerical analysis of the implicit Euler scheme in time and the spectral discretization in space of the nonstationary vorticity–velocity–pressure formulation of the Navier–Stokes equations. We prove using Brouwer's fixed point theorem that the new discrete formulation has at most one solution. The study of the error, the algorithm solution, and the numerical implementation of these results will be the subject of our forthcoming work.

Acknowledgements
The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No. (RG-1435-026).

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 August 2020 Accepted: 11 September 2020 Published online: 18 September 2020

References

1. Bègue, C., Conca, C., Murat, F., Pironneau, O.: Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression. In: Brezis, H., Lions, J.-L. (eds.) Nonlinear Partial Differential Equations and Their Applications. Collège de France Seminar, vol. IX, pp. 179–264. Longman, Harlow (1988)
2. Dubois, F.: Vorticity–velocity–pressure formulation for the Stokes problem. Math. Methods Appl. Sci. **25**, 1091–1119 (2002)
3. Salmon, S.: Développement numérique de la formulation tourbillon–vitesse–pression pour le problème de Stokes. PhD thesis, Université Pierre et Marie Curie, Paris, France (1999)
4. Dubois, F., Salaün, M., Salmon, S.: Vorticity–velocity–pressure and stream function-vorticity formulations for the Stokes problem. J. Math. Pures Appl. **82**, 1395–1451 (2003)
5. Amara, M., Capatina-Papaghiuc, D., Chacon-Vera, E., Trujillo, D.: Vorticity–velocity–pressure formulation for Navier–Stokes equations. Comput. Vis. Sci. **6**, 47–52 (2004)
6. Amara, M., Capatina-Papaghiuc, D., Chacon-Vera, E., Trujillo, D.: Vorticity–velocity–pressure formulation for the Stokes problem. Math. Comput. **73**, 1673–1697 (2004)
7. Amara, M.D., Chacon-Vera, E., Trujillo, D.: Stabilized finite element method for the Navier–Stokes equations, with nonstandard boundary conditions. J. Sci. Comput. **18**, 4–9 (2005)
8. Azaïez, M., Bernardi, C., Chorfi, N.: Spectral discretization of the vorticity, velocity and pressure formulation of the Navier–Stokes equations. Numer. Math. **104**, 1–26 (2006)
9. Bernardi, C., Chorfi, N.: Spectral discretization of the vorticity, velocity and pressure formulation of the Stokes problem. SIAM J. Numer. Anal. **44**, 826–850 (2006)
10. Bernardi, C., Sayah, T.: A posteriori error analysis of the time dependent Stokes equations with mixed boundary conditions. IMA J. Numer. Anal. **35**, 179–198 (2015)
11. Bernardi, C., Sayah, T.: A posteriori error analysis of the time dependent Navier–Stokes equations with mixed boundary conditions. SeMA J. **69**, 1–23 (1991)
12. Abdelwahed, M., Chorfi, N.: Spectral discretization of the time dependent vorticity velocity pressure formulation of the stokes problem. Math. Methods Appl. Sci., 1–18 (2020)
13. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms. Springer, Berlin (1986)
14. Bernardi, C., Girault, V.: Espaces Duaux des Domaines des Operateurs Divergence et Rotationnel Avec Trace Nulle. Internal Report, Laboratoire Jacques-Louis Lions. Universite Pierre et Marie Curie, Paris (2003)
15. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional nonsmooth domains. Math. Methods Appl. Sci. **21**, 823–864 (1998)
16. Bernardi, C., Hecht, F., Verfurth, R.: Finite element discretization of the three-dimensional Navier–Stokes equations with mixed boundary conditions. Math. Model. Numer. Anal. **3**, 1185–1201 (2009)
17. Temam, R.: Navier–Stokes Equations. Theory and Numerical Analysis. Studies in Mathematics and Its Applications, vol. 2. North-Holland, Amsterdam (1977)
18. Girault, V., Raviart, P.-A.: Finite Element Approximation of the Navier–Stokes Equations. Lecture Notes in Mathematics, vol. 749. Springer, Berlin (1979)
19. Costabel, M.: A remark on the regularity of solutions of Maxwell equations on Lipschitz domains. Math. Methods Appl. Sci. **12**, 365–368 (1990)
20. Nédélec, J.-C.: Mixed finite elements in $\mathbb{R}^3$. Numer. Math. **35**, 315–341 (1980)
21. Bernardi, C., Maday, Y.: Spectral method. In: Ciarlet, P.G., Lions, J.-L. (eds.) Handbook of Numerical Analysis. North-Holland, Amsterdam (1997)

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ▶ springeropen.com