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Existence results for first derivative dependent ϕ -Laplacian boundary value problems

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Abstract

Our main concern in this article is to investigate the existence of solution for the boundary-value problem

$$(\phi(x'(t)))' = g_1(t, x(t), x'(t)), \quad \forall t \in [0, 1],$$

$$\Upsilon_1(x(0), x(1), x'(0)) = 0,$$

$$\Upsilon_2(x(0), x(1), x'(1)) = 0,$$

where $g_1 : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, $\Upsilon_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ are continuous functions, $i = 1, 2$, and $\phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, for $0 < a < \infty$. We obtain the solvability results by imposing some new conditions on the boundary functions. The new conditions allow us to ensure the existence of at least one solution in the sector defined by well ordered functions. These ordered functions do not require one to check the definitions of lower and upper solutions. Moreover, the monotonicity assumptions on the arguments of boundary functions are not required in our case. An application is considered to ensure the applicability of our results.

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1 Introduction

The aim of this article is to study the existence of solution for the following nonlinear boundary value problems (BVPs) with ϕ -Laplacian operator:

$$(\phi(x'(t)))' = g_1(t, x(t), x'(t)), \quad \forall t \in [0, 1], \quad (1)$$

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subject to the the following generalized nonlinear boundary conditions (BCs):

$$\begin{cases} \Upsilon_1(x(0), x(1), x'(0)) = 0, \\ \Upsilon_2(x(0), x(1), x'(1)) = 0, \end{cases} \quad (2)$$

where $g_1 : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is L^1 -Carathéodory function, $\Upsilon_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuous functions, and ϕ satisfies the hypothesis:

(H₁) $\phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, for $0 < a < \infty$.

The solution of (1)–(2) is a continuously differentiable function, $x \in C^1[0, 1]$ with $\|x'\|_\infty < a$, and $(\phi \circ x') \in AC[0, 1]$, which satisfies (1) and (2).

The BVPs (1) with x'' instead of $(\phi(x'(t)))'$ has been studied extensively in the literature employing a lower and upper solutions approach, a topological approach, degree and fixed point index theory, and fixed point and continuation theorems; see for example [1–11], and the references therein. These approaches usually make use of an appropriate Green function, which, for the linear operator $x \mapsto x_1'$, can be found and successfully handled. In recent times, the researchers put their efforts to developing the existence and localization results for the BVPs with a ϕ -Laplacian operator, $x \mapsto (\phi(x'(t)))'$. Such types of BVPs are solved with classical BCs, like periodic, anti-periodic, Dirichlet, Neumann, and mixed cases; see for example [12–27], and the references therein.

The study of ϕ -Laplacian equations is a classical topic having applications in glaciology, population biology, nonlinear flow laws, non-Newtonian mechanics, and combustion theory; see for example [28–31], and the references therein. Moreover, the ϕ -Laplacian operators are involved in some models, e.g., in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, and theory of capillary surfaces [32].

A lower and upper solutions (LUSs) approach is widely investigated to develop the existence and solvability results for classical and fractional order differential equations with ϕ -Laplacian operator; see for example [33–37] and the references therein.

Motivated by the above-mentioned work on ϕ -Laplacian differential equations, we develop an existence criterion for the boundary value problem (BVP), (1)–(2). To the best of our knowledge, (1) is still an untreated problem with BCs (2). The approach we use in our study is simpler than the approaches used in [1, 3, 33, 34]. We prove the existence of solutions of (1)–(2) by using new conditions given on the BCs (2). These conditions allow us to obtain a solution in the sector defined by well-ordered functions. These ordered functions do not require one to check the definitions of LUSs. Moreover, the requirement to impose the monotonicity assumptions on the arguments of the BCs is not necessary in our case. The arguments we use in our study are the Arzelà–Ascoli and Schauder's fixed point theorems.

The rest of the article is organized as follows: in Sect. 2, we present preliminary definitions and auxiliary results, in Sect. 3, we prove the existence of solutions of the problem (1)–(2), in Sect. 4, we consider an example to verify the results of Sect. 3, and in Sect. 5, the conclusion is given.

2 Preliminaries

The following definitions are helpful in understanding the main result.

Definition 1 Let $x \in C^1([0, 1])$ be a solution of (1)–(2), if $\|x'\|_\infty < a$, $(\phi \circ x') \in AC[0, 1]$, and satisfy (1)–(2).

Definition 2 The function $g_1 : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is L^1 -Carathéodory if:

1. $g_1(\cdot, x, y) : [0, 1] \rightarrow \mathbb{R}$ is measurable for all $(x, y) \in \mathbb{R}^2$,
2. $g_1(t, \cdot, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous for a.e. $t \in [0, 1]$,
3. for each compact set $A \subset \mathbb{R}^2$, there is a function $\mu_A \in L^1$, such that

$$|g_1(t, x, y)| \leq \mu_A(t)$$

for a.e. $t \in [0, 1]$ and all $(x, y) \in A$.

The following lemmas are very useful for obtaining our main results.

Lemma 2.1 For each $(k, g) \in C([0, 1]) \times (-a, a)$, we have a unique $d := \mathbf{D}_\phi(k, g)$, such that

$$\int_0^1 \phi^{-1}(k(t) - d) dt = g. \tag{3}$$

Also, $\mathbf{D}_\phi : C([0, 1]) \times (-a, a) \rightarrow \mathbb{R}$ is a continuous function.

Proof First, the uniqueness and existence is shown, then continuity. Let, for $(k, g) \in C([0, 1]) \times (-a, a)$, there exist d_1, d_2 such that

$$\int_0^1 \phi^{-1}(k(t) - d_1) dt = g = \int_0^1 \phi^{-1}(k(t) - d_2) dt. \tag{4}$$

For some $s \in [0, 1]$, we have

$$\int_0^1 \phi^{-1}(k(s) - d_1) dt = g = \int_0^1 \phi^{-1}(k(s) - d_2) dt. \tag{5}$$

Since ϕ^{-1} is injective function, (5) implies that

$$\phi^{-1}(k(s) - d_1) = \phi^{-1}(k(s) - d_2). \tag{6}$$

Equation (6) further implies

$$k(s) - d_1 = k(s) - d_2, \tag{7}$$

hence, $d_1 = d_2$. Now define a function, $\delta : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\delta(s) = \int_0^1 \phi^{-1}(k(t) - s) dt \tag{8}$$

is well defined, decreasing, and continuous. Also

$$\lim_{s \rightarrow \infty} \delta(s) = a, \quad \lim_{s \rightarrow -\infty} \delta(s) = -a. \tag{9}$$

Equation (9) implies the existence of a unique $d = \mathbf{D}_\phi(k, g)$. Now, it remains to show that $\mathbf{D}_\phi : C([0, 1]) \times (-a, a) \rightarrow \mathbb{R}$ is a continuous function.

Let $(k_n, g_n) \in C([0, 1]) \times (-a, a)$ and $(k_n, g_n) \rightarrow (k_0, g_0)$ in $C([0, 1]) \times (-a, a)$. Now let $\mathbf{D}_\phi(k_n, g_n)$ be a subsequence converges to d_0 , then by the application of the dominated convergence theorem, we deduce that $\int_0^1 \phi^{-1}(k_0(t) - d_0) dt = g_0$, which implies $\mathbf{D}_\phi(k_0, g_0) = d_0$. Hence, $\mathbf{D}_\phi : C([0, 1]) \times (-a, a) \rightarrow \mathbb{R}$ is a continuous function. \square

3 Main results

Theorem 1 *Suppose we have the existence of ordered functions $\eta_1, \eta_2 \in C[0, 1]$, such that, $\eta_1(0) < \eta_2(0)$ and $\eta_1(1) < \eta_2(1)$,*

$$\max\{|\eta_2(1) - \eta_1(0)|; |\eta_1(1) - \eta_2(0)|\} < a, \tag{10}$$

$$\Upsilon_1(\eta_1(0), t, w) \geq 0, \quad \forall(t, w) \in [\eta_1(0), \eta_2(0)] \times [-a, a],$$

$$\Upsilon_1(\eta_2(0), t, w) \leq 0, \quad \forall(t, w) \in [\eta_1(0), \eta_2(0)] \times [-a, a],$$

$$\eta_1(1) - \Upsilon_2(x, \eta_1(1), z) \leq 0, \quad \forall x, z \in [\eta_1(0), \eta_2(0)],$$

$$\eta_2(1) - \Upsilon_2(x, \eta_2(1), z) \geq 0, \quad \forall x, z \in [\eta_1(0), \eta_2(0)].$$

Then the problem (1)–(2) has at least one solution, such that $\eta_1(0) - at \leq x(t) \leq \eta_2(0) + at$, for all $t \in [0, 1]$.

Proof Suppose we have the following modified BVP:

$$(\phi(x'(t)))' = G(x)(t), \quad \text{a.e. } t \in [0, 1], \tag{11}$$

with modified BCs

$$x(0) = \mathbf{A}(x), \tag{12}$$

$$x(1) = \mathbf{B}(x), \quad \square$$

where

$$G(x)(t) = g_1(t, \rho(t, x(t)), \varrho(x'(t))),$$

with

$$\rho(t, x(t)) = \max\{\eta_1(0) - at, \min\{x, \eta_2(0) + at\}\},$$

$$\varrho(x'(t)) = \max\{-a, \min\{x, a\}\},$$

and

$$\mathbf{A}(x) = \sigma(x(0) + \Upsilon_1(x(0), x(1), x'(0))),$$

$$\mathbf{B}(x) = \chi\left(\frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1))\right),$$

with

$$\sigma(x) = \max\{\eta_1(0), \min\{x, \eta_2(0)\}\},$$

$$\chi(x) = \max\{\eta_1(1), \min\{x, \eta_2(1)\}\}.$$

For simplicity, we divide the proof in three steps.

Step 1: The solution of the problem (11)–(12) is equivalent to find a fixed points of the operator, $\Omega : C^1[0, 1] \rightarrow C^1[0, 1]$ defined as

$$\Omega(x)(t) = \mathbf{A}(x) + \int_0^t \phi^{-1} \left[\lambda_x + \int_0^s G(x)(x) dx \right] ds, \quad \forall t \in [0, 1]. \tag{13}$$

Firstly, we ensure the existence of $\lambda_x \in \mathbb{R}$. For this, we claim that for each $x \in C^1[0, 1]$, there exists a unique $\lambda_x \in \mathbb{R}$, such that

$$\int_0^1 \left(\phi^{-1} \left[\lambda_x + \int_0^t G(x)(s) ds \right] \right) dt = \mathbf{B}(x) - \mathbf{A}(x). \tag{14}$$

Let

$$k(t) = \int_0^t G(x)(s) ds \in C[0, 1] \tag{15}$$

and

$$g = \mathbf{B}(x) - \mathbf{A}(x). \tag{16}$$

Obviously,

$$|g| \leq |\mathbf{B}(u) - \mathbf{A}(u)| \leq \max\{|\beta(1) - \alpha(0); \alpha(1) - \beta(0)\} < a. \tag{17}$$

Consequently, $\lambda_x \in \mathbb{R}$ exists by Lemma 2.1.

Since $G(x)$ is bounded and continuous on $[0, 1]$, and the integral is a continuous function on $[0, s]$. Furthermore, λ_x exists, ϕ is a homomorphism and its inverse exists. Also, $\phi^{-1}[\lambda_x + \int_0^s G(x)(x) dx]$ is continuous, and its integral exists. Therefore $\Omega(x)$ is continuous on $[0, 1]$. Further, the class $\{\Omega(x) : x \in C^1[0, 1]\}$ is uniformly bounded and equicontinuous. Therefore in view of the Arzelà–Ascoli theorem $\{\Omega(x) : x \in C^1[0, 1]\}$ is relatively compact. Consequently Ω is a compact map. Now the Schauder fixed point theorem guarantees the existence of at least a fixed point since Ω is continuous and compact.

Step 2: If $x(t)$, $t \in [0, 1]$ be a solution of the problem (11)–(12), then it must satisfy

$$\eta_1(0) - at < x(t) < \eta_2(0) + at, \quad \forall t \in [0, 1]. \tag{18}$$

Since $x(0) = \mathbf{A}(x)$,

$$\begin{aligned} x(0) &= \sigma(x(0) + \Upsilon_1(x(0), x(1), x'(0))) \\ &= \max\{\eta_1(0), \min\{x(0) + \Upsilon_1(x(0), x(1), x'(0)), \eta_2(0)\}\} \\ &= \eta_2(0). \end{aligned} \tag{19}$$

Now, we show that

$$\eta_1(0) \leq x(0) + \Upsilon_1(x(0), x(1), x'(0)) \leq \eta_2(0). \tag{20}$$

Suppose on the contrary that

$$\eta_2(0) < x(0) + \Upsilon_1(x(0), x(1), x'(0)),$$

then

$$\eta_2(0) - x(0) < \Upsilon_1(x(0), x(1), x'(0)),$$

$$0 < \Upsilon_1(\eta_2(0), x(1), x'(0)),$$

which implies a contradiction, because $\Upsilon_1(\eta_2(0), x(1), x'(0)) \not\geq 0$. Hence

$$x(0) + \Upsilon_1(x(0), x(1), x'(0)) \leq \eta_2(0). \quad (21)$$

Along the same lines, we can show that

$$\eta_1(0) \leq x(0) + \Upsilon_1(x(0), x(1), x'(0)). \quad (22)$$

Using (21) and (22), we have

$$\eta_1(0) \leq x(0) \leq \eta_2(0). \quad (23)$$

Now using the second boundary condition, $x(1) = \mathbf{B}$, we may show that

$$\eta_1(1) \leq x(1) \leq \eta_2(1). \quad (24)$$

Since $\|x'\|_\infty < a$,

$$x(0) - at < x(t) < x(0) + at, \quad \forall t \in [0, 1]. \quad (25)$$

Using Eqs. (23)–(25), we have

$$\eta_1(0) - at < x(t) < \eta_2(0) + at, \quad \forall t \in [0, 1]. \quad (26)$$

Step3: If $x(t)$ is a solution of the problem (11)–(12), then it must satisfy the BCs

$$\begin{cases} \Upsilon_1(x(0), x(1), x'(0)) = 0, \\ \Upsilon_2(x(0), x(1), x'(1)) = 0. \end{cases} \quad (27)$$

To satisfy the BCs (27), it is sufficient to show that

$$\begin{cases} \eta_1(0) \leq x(0) + \Upsilon_1(x(0), x(1), x'(0)) \leq \eta_2(0), \\ \eta_1(1) \leq \frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1)) \leq \eta_2(1). \end{cases} \quad (28)$$

Suppose on the contrary that

$$\eta_1(0) \geq x(0) + \Upsilon_1(x(0), x(1), x'(0)). \quad (29)$$

Then

$$\begin{aligned} x(0) &= \sigma(x(0) + \Upsilon_1(x(0), x(1), x'(0))) \\ &= \max\{\eta_1(0), \min\{x(0) + \Upsilon_1(x(0), x(1), x'(0)), \eta_2(0)\}\} \\ &= \alpha(0). \end{aligned} \tag{30}$$

Using (29) and (30), we have

$$\Upsilon_1(\eta_1(0), x(1), x'(0)) < 0. \tag{31}$$

This is a contradiction, because $\Upsilon_1(\eta_1(0), x(1), x'(0)) \not\leq 0$. Hence

$$\eta_1(0) \leq x(0) + \Upsilon_1(x(0), x(1), x'(0)). \tag{32}$$

Similarly, we can show that

$$\eta_2(0) \geq x(0) + \Upsilon_1(x(0), x(1), x'(0)). \tag{33}$$

Now, assume on contrary that

$$\eta_2(1) < \frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1)). \tag{34}$$

Then

$$\begin{aligned} x(1) &= \mathbf{B}(x) \\ &= \chi\left(\frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1))\right) \\ &= \max\left\{x(1), \min\left\{\frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1)), \eta_2(1)\right\}\right\} \\ &= \eta_2(1). \end{aligned} \tag{35}$$

Using Eqs. (34) and (35), we have

$$\eta_2(1) - \Upsilon_2(x(0), \eta_2(1), x'(1)) < 0, \tag{36}$$

which implies a contradiction, because $\eta_2(1) - \Upsilon_2(x(0), \eta_2(1), x'(1)) \not\leq 0$. Hence

$$\eta_2(1) \geq \frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1)). \tag{37}$$

Along the same line, we can show that

$$\eta_1(1) \leq \frac{1}{2}x(1) + \frac{1}{2}\Upsilon_2(x(0), x(1), x'(1)). \tag{38}$$

Using Eqs. (32), (33), (37), and (38), the BCs (27) are satisfied.

Hence, $x(t), t \in [0, 1]$ is a solution of the modified BVP (11)–(12) which leads to the solution of the BVP (1)–(2).

4 Example

Consider the following nonlinear BVP:

$$\left(\frac{x'(t)}{1 - |x'(t)|}\right)' = x^3(t) + \frac{x(t)}{\sqrt{t}} + \sin(x'(t)), \quad \text{for a.e. } t \in [0, 1], \tag{39}$$

subject to following nonlinear BCs:

$$\begin{aligned} \Upsilon_1(x(0), x(1), x'(0)) &= x(0)x(1)(x'(0) - 3) = 0, \\ \Upsilon_2(x(0), x(1), x'(1)) &= x(0)x(1)(x'(1) - 5) = 0. \end{aligned} \tag{40}$$

This problem is a particular case of the problem (1)–(2) with

$$\begin{aligned} g_1(t, x, y) &= x^3(t) + \frac{x(t)}{\sqrt{t}} + \sin(y'(t)), \quad t \in [0, 1], \\ \phi(x) &= \frac{x}{\sqrt{1 - |x|}}, \\ \text{and} \\ a &= 1. \end{aligned}$$

Choose

$$\eta_1(t) = 0, \eta_2(t) = \frac{1}{2}, \quad \forall t \in [0, 1],$$

with

$$\eta_1(0) < \eta_2(0), \quad \eta_1(1) < \eta_2(1).$$

Now

$$\begin{aligned} \max\{|\eta_2(1) - \eta_1(0)|, |\eta_1(1) - \eta_2(0)|\} &= \frac{1}{2} < 1 = a, \\ \Upsilon_1(\eta_1(0), t, w) &= \eta_1(0)t(w - 3) = 0, \quad \forall (t, w) \in \left[0, \frac{1}{2}\right] \times [-1, 1], \\ \Upsilon_1(\eta_2(0), t, w) &= \eta_2(0)t(w - 3) \leq 0, \quad \forall (t, w) \in \left[0, \frac{1}{2}\right] \times [-1, 1], \\ \eta_1(1) - \Upsilon_2(x, \eta_1(1), z) &= \eta_1(1) - x\eta_1(1)(z - 5) = 0, \quad \forall (x, z) \in \left[0, \frac{1}{2}\right], \\ \eta_2(1) - \Upsilon_2(x, \eta_2(1), z) &= \eta_2(1) - x\eta_2(1)(z - 5) \geq 0, \quad \forall (x, z) \in \left[0, \frac{1}{2}\right]. \end{aligned}$$

Hence by Theorem 1, the BVP (39)–(40) has at least one solution, satisfying

$$\eta_1(0) - at \leq x(t) \leq \eta_2(0) + at, \quad \forall t \in [0, 1].$$

5 Conclusion

We studied the existence of solutions of ϕ -Laplacian boundary value problems by employing the topological approach and the new conditions on the boundary functions. The conditions on the boundary functions allowed us to obtain a solution in the sector defined by two well-ordered functions which did not require satisfying the differential inequalities to ensure the existence of lower and upper solutions. This way to deal with the ϕ -Laplacian boundary value problems made our approach simpler than the lower and upper solutions approach. Moreover, in our approach the monotonicity assumptions on the arguments of the boundary functions are not required. We considered an example to check the applicability of the developed theoretical results.

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