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Existence of positive solutions for singular Dirichlet boundary value problems with impulse and derivative dependence

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Abstract

In this paper, we present a theorem for some impulsive boundary problems with derivative dependence by the upper and lower solutions method. Using the theorem obtained, we consider the existence of positive solutions of some class of singular impulsive boundary problems.

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Keywords: Impulsive differential equations; Boundary value problem; Positive solutions; Existence; Upper and lower solutions

1 Introduction

In this paper, we consider the existence of positive solutions of the following nonlinear impulsive differential equations:

$$\begin{cases} -u''(t) + |u'(t)|^\mu + K(t)u^{-q} = \lambda u^p, & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = I(u(\frac{1}{2})), \\ \Delta u'|_{t=\frac{1}{2}} = N(u(\frac{1}{2}), u'(\frac{1}{2}-)), \\ u(0) = u(1) = 0, \end{cases} \quad (1.1)$$

where $\Delta u|_{t=\frac{1}{2}} = u(\frac{1}{2}+) - u(\frac{1}{2}-)$, $\Delta u'|_{t=\frac{1}{2}} = u'(\frac{1}{2}+) - u'(\frac{1}{2}-)$, $p \in (0, 1)$, $q \in (0, 1)$, $\mu \in [0, 2]$.

On the one hand, this paper is motivated by [14, 15] in which Ghergu and Rădulescu considered the following elliptic problems:

$$\begin{cases} -\Delta u(x) + q(x)|u'(x)|^\mu + K(x)g(u) = \lambda f(x, u), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.2)$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary and g is singular at $u = 0$, presented the existence of positive solutions via the method of upper and lower solutions.

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On the other hand, in recent years, the impulsive differential equation has been studied extensively for its background it being subject to sudden changes in their states, such as population dynamics, biological systems, optimal control, chemotherapeutic treatment in medicine, mechanical systems with impact, financial systems; see [2, 3, 5, 13, 18, 19, 23, 25, 26, 31–33, 36, 39, 42–44, 47, 50, 51] and the references therein. Many of them are on boundary value problems; see [7, 8, 13, 21, 22, 24, 29, 33, 37, 39–43]. But the impulses lead to some interesting phenomena. In fact, it is well known that the following problem:

$$\begin{cases} -u''(t) = 1, & t \in (0, 1), \\ u(0) = 0, & u(1) = 0, \end{cases}$$

has the unique solution $u(t) = \frac{1}{2}t(1-t)$. Now we consider two corresponding impulsive problems

$$\begin{cases} -u''(t) = 1, & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = I(u(\frac{1}{2})), \\ \Delta u'|_{t=\frac{1}{2}} = N(u'(\frac{1}{2}-)), \\ u(0) = 0 = u(1), \end{cases} \quad (1.3)$$

where

$$I(u) = \frac{1}{4} - u, \quad N(v) = -v,$$

and

$$\begin{cases} -u''(t) = 1, & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = I_1(u(\frac{1}{2})), \\ \Delta u'|_{t=\frac{1}{2}} = N_1(u'(\frac{1}{2}-)), \\ u(0) = 0 = u(1), \end{cases} \quad (1.4)$$

where

$$I_1(u) = -\frac{1}{4} - u, \quad N_1(v) = -v.$$

Let

$$u_c(t) = \begin{cases} ct - \frac{1}{2}t^2, & t \in [0, \frac{1}{2}]; \\ \frac{1}{2}t(1-t), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Obviously, u_c is a solution of problem (1.3) for all $c \in \mathbb{R}$, i.e. problem (1.3) has infinite solutions.

Now we consider the problem (1.4) and assume that problem (1.4) has a solution $u(t)$. It is easy to see that $u(t) = c_0 + c_1t + c_2t^2$ for all $t \in [0, \frac{1}{2}]$. By $u(0) = 0$ and $u''(t) = -1$, we have $u(t) = c_1t - \frac{1}{2}t^2$ and then $u(\frac{1}{2}) = \frac{1}{2}c_1 - \frac{1}{8}$ and $u'(\frac{1}{2}-) = c_1 - \frac{1}{2}$. Hence, by the definition

of I_1 and N_1 , we have $u(\frac{1}{2}+) = u(\frac{1}{2}) + I(u(\frac{1}{2})) = -\frac{1}{4}$ and $u'(\frac{1}{2}+) = u'(\frac{1}{2}-) - u'(\frac{1}{2}-) = 0$. From $-u''(t) \equiv 1$ for all $t \in (\frac{1}{2}, 1]$, we have $u(t) = -\frac{3}{8} + \frac{1}{2}t - \frac{1}{2}t^2$. This contradicts $u(1) = 0$. This means that problem (1.4) has no solution.

The above examples show that it is interesting to consider the existence of solutions for differential equations with impulses. As is well known, the methods to be used for the existence of solutions for boundary value problems mainly include bifurcation theory (see [28, 34, 46]), method of lower and upper solutions (see [10, 11, 14, 15]), topological degree theory (see [16, 20, 48, 49, 53, 54]) and variational methods (see [45]). But the impulses lead to some difficulties in the use of the method of lower and upper solutions for impulsive boundary value problems and some authors worked hard at proving some new theorems for the method of lower and upper solutions. For example, Erbe and Liu [13] considered the following problem:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & 0 < t < 1, t \neq t_1, \\ \Delta u|_{t=t_1} = I(u(t_1)), \\ \Delta u'|_{t=t_1} = N(u'(t_1-)), \\ u(0) = a, & u(1) = b, \end{cases} \quad (1.5)$$

and using the one-side Lipschitz condition

$$f(t, u_1, v) - f(t, u_2, v) \geq -M(u_1 - u_2), \quad \forall (t, v) \in [0, 1] \times \mathbb{R},$$

they got the extremal solutions of problem (1.5) with the help of lower and upper solutions; in [39], under the condition that f has time singularities at $t = 0$ and $t = 1$, the authors proved the existence of a solution to this problem under the assumption that there exist lower and upper functions associated with the problem; in [23], Lee and Liu established the method of lower and upper solutions for the following problem:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & 0 < t < 1, t \neq t_1, \\ \Delta u|_{t=\frac{1}{2}} = I(u(t_1)), \\ \Delta u'|_{t=\frac{1}{2}} = N(u(t_1), u'(t_1-)), \\ u(0) = a, & u(1) = b, \end{cases} \quad (1.6)$$

where f is singular at $t = 0$ and $t = 1$. There are other references for the methods of lower and upper solutions for impulsive differential equations; see [1, 21, 22, 27, 38].

We note that in problem (1.1) the nonlinearity f is dependent on u' and is singular at $u = 0$, and the impulsive function $N(u, v)$ is dependent on u' . And so we cannot use the method of lower and upper solutions for problem (1.5) and problem (1.6) to study problem (1.1) directly. Since problem (1.1) is a special case the following problem:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & 0 < t < 1, t \neq \frac{1}{2}, \\ \Delta u|_{t=\frac{1}{2}} = I(u(\frac{1}{2})), \\ u(0) = a, & u(1) = b, \end{cases} \quad (1.7)$$

and

$$\Delta u'|_{t=\frac{1}{2}} = N\left(u\left(\frac{1}{2}\right), u'\left(\frac{1}{2}-\right)\right), \quad (1.8)$$

if the method of lower and upper solutions for problem (1.7)–(1.8) can be obtained, by constructing suitable lower and upper solutions, we can get the existence of solutions of problem (1.1).

The paper is organized as follows. In Sect. 2, we present a theorem for the upper and lower solutions method for problem (2.1)–(2.2) which is a generalization of problem (1.7)–(1.8). In Sect. 3, we get the existence of positive solutions of problem (1.1) when $K(t) < 0$ on $t \in [0, 1]$. In Sect. 4, we prove the existence of positive solutions of problem (1.1) when $K(t) > 0$ on $t \in [0, 1]$. Some ideas of our proofs come from [12, 14, 15, 30, 35] and [48].

2 The method of lower and upper solutions for impulsive differential equations

To obtain the lower and upper solutions for problem (1.7)–(1.8), we consider the generalization case such as

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & 0 < t < 1, t \neq t_1, \\ \Delta u|_{t=t_1} = I(u(t_1)), \\ u(0) = a \geq 0, & u(1) = b \geq 0, \end{cases} \quad (2.1)$$

and

$$\Delta u'|_{t=t_1} = N(u(t_1), u'(t_1-)). \quad (2.2)$$

The solution is defined as $u \in PC^1([0, 1], \mathbb{R}) \cap PC^2((0, 1), \mathbb{R})$ satisfying (2.1)–(2.2), where $PC([0, 1], \mathbb{R}) = \{u : [0, 1] \rightarrow \mathbb{R} : u(t) \text{ is continuous at } t \neq t_1, u(t_1 - 0) = \lim_{t \rightarrow t_1^-} u(t) \text{ and } u(t_1 + 0) = \lim_{t \rightarrow t_1^+} u(t) \text{ exist}\}$ (denoted by $PC[0, 1]$ simply), $PC^1([0, 1], \mathbb{R}) = \{u \in PC[0, 1] : u'(t) \text{ is continuous on } [0, t_1) \cup (t_1, 1], \lim_{t \rightarrow t_1^-} u'(t) \text{ and } \lim_{t \rightarrow t_1^+} u'(t) \text{ exist}\}$ with the norm $\|u\|_1 = \max\{\sup_{t \in [0, 1]} |u(t)|, \sup_{t \in [0, 1]} |u'(t)|\}$ (denoted by $PC^1[0, 1]$ simply), and $PC^2((0, 1), \mathbb{R}) = \{u \in PC([0, 1], \mathbb{R}) : u''(t) \text{ is continuous on } (0, t_1) \cup (t_1, 1) \text{ and } \lim_{t \rightarrow t_1^-} u''(t), \lim_{t \rightarrow t_1^+} u''(t) \text{ exist}\}$ (denoted by $PC^2(0, 1)$ simply).

If $\alpha \in PC^1[0, 1] \cap PC^2(0, 1)$ satisfies

$$\begin{cases} -\alpha''(t) \leq f(t, \alpha(t), \alpha'(t)), & t \in (0, 1), t \neq t_1, \\ \Delta \alpha|_{t=t_1} = I(\alpha(t_1)), \\ \Delta \alpha'|_{t=t_1} \geq N(\alpha(t_1), \alpha'(t_1-)), \\ \alpha(0) \leq a, & \alpha(1) \leq b, \end{cases}$$

it is a lower solution of problem (2.1)–(2.2), while the upper solution is defined as $\beta \in PC^1[0, 1] \cap PC^2(0, 1)$ satisfying

$$\begin{cases} -\beta''(t) \geq f(t, \beta(t), \beta'(t)), & t \in (0, 1), t \neq t_1, \\ \Delta\beta|_{t=t_1} = I(\beta(t_1)), \\ \Delta\beta'|_{t=t_1} \leq N(\beta(t_1), \beta'(t_1-)), \\ \beta(0) \geq a, \quad \beta(1) \geq b. \end{cases}$$

Denote

$$D_\alpha^\beta = \{(t, u) \in (0, 1) \times \mathbb{R} | \alpha(t) \leq u \leq \beta(t), t \in (0, 1)\},$$

where $\alpha, \beta \in PC^1([0, 1], \mathbb{R})$ with $\alpha(t) \leq \beta(t)$ for all $t \in [0, 1]$. Then we give the lower and upper solution theorem of problem (2.1)–(2.2).

Now we list a condition for convenience:

(A₀) $I \in C(\mathbb{R}, \mathbb{R})$ and $u + I(u)$ is nondecreasing.

To establish a fundamental theorem of the upper and lower solution method for problem (2.1)–(2.2), we need to use a corresponding theorem for the following singular boundary value problem:

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), & t \in (0, 1), \\ u(0) = a, \quad u(1) = b. \end{cases} \quad (2.3)$$

The definitions of upper and lower solutions of problem (2.3) can be found in [11, 39]. Now we list the following lemma [45].

Lemma 2.1 (see [45]) *Let $u \in H^{2,1}[0, 1]$ and $M > 0$ be a constant such that $|u''(t)| \leq K|u'(t)|^2 + f_0(t)$ with $K > 0$, $f_0 \in L^1[0, 1]$ and $\sup_{t \in [0, 1]} |u(t)| \leq M$. Then there exists a constant $k_0 > 0$ such that*

$$|u|_{H^{2,1}[0, 1]} \leq k_0.$$

Lemma 2.2 *Assume that the function $f(t, u, v)$ is continuous in $(0, 1) \times (0, +\infty) \times \mathbb{R}$. Furthermore, suppose that problem (2.3) has a pair of upper and lower solutions $\beta(t)$ and $\alpha(t)$ satisfying the conditions*

- (1) $\alpha, \beta \in C^1[0, 1] \cap C^2(0, 1)$;
- (2) $0 < \alpha(t) \leq \beta(t)$, $\forall t \in (0, 1)$;
- (3) *there exist a nonnegative function $h \in L^1[0, 1]$ and a positive constant $K > 0$ such that*

$$|f(t, u, v)| \leq h(t) + K|v|^2, \quad \forall (t, u) \in D_\alpha^\beta, v \in \mathbb{R}. \quad (2.4)$$

Then problem (2.3) has a solution $u(t)$ belonging to $C^1[0, 1]$ and satisfying

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [0, 1].$$

Proof Let $\bar{f}: (0, 1) \times \mathbb{R} \times \mathbb{R}$ be defined by

$$\bar{f}(t, u, v) = \begin{cases} f(t, u, v), & \text{if } \alpha(t) \leq u \leq \beta(t); \\ f(t, \alpha(t), v) + \frac{1}{\alpha(t)}(|f(t, \alpha(t), 0)| + 1) \min\{\alpha(t), \alpha(t) - u\}, & \text{if } u < \alpha(t); \\ f(t, \beta(t), v) - \frac{1}{\beta(t)}(|f(t, \beta(t), 0)| + 1) \min\{\beta(t), u - \beta(t)\}, & \text{if } u > \beta(t). \end{cases}$$

Now we consider the following problem:

$$\begin{cases} -u''(t) = \bar{f}(t, u(t), u'(t)), & t \in (0, 1), \\ u(0) = a, & u(1) = b. \end{cases} \quad (2.5)$$

First, we show that the solution of problem (2.5) is the solution of problem (2.3).

Now we prove that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in (0, 1)$. In fact, if there is a $t_0 \in (0, 1)$ with $u(t_0) < \alpha(t_0)$, let $t_* = \inf\{t < t_0 | u(s) < \alpha(s), \forall s \in [t, t_0]\}$ and $t^* = \sup\{t > t_0 | u(s) < \alpha(s), \forall s \in [t_0, t]\}$. Since $\alpha(0) \leq a = u(0)$, $\alpha(1) \leq b = u(1)$, we have $u(t) < \alpha(t)$, $\forall t \in (t_*, t^*)$ and $u(t_*) = \alpha(t_*)$ and $u(t^*) = \alpha(t^*)$. Suppose that $t' \in (t_*, t^*)$ satisfies $u(t') - \alpha(t') = \min_{t \in [t_*, t^*]} (u(t) - \alpha(t)) < 0$. Then $\alpha'(t') = u'(t')$. Hence,

$$\begin{aligned} 0 &\geq -(u(t') - \alpha(t'))'' \\ &\geq f(t', \alpha(t'), u'(t')) + \frac{1}{\alpha(t')}(|f(t', \alpha(t'), 0)| + 1) \min\{\alpha(t'), \alpha(t') - u(t')\} \\ &\quad - f(t', \alpha(t'), \alpha'(t')) \\ &= f(t', \alpha(t'), \alpha'(t')) + \frac{1}{\alpha(t')}(|f(t', \alpha(t'), 0)| + 1) \min\{\alpha(t'), \alpha(t') - u(t')\} \\ &\quad - f(t', \alpha(t'), \alpha'(t')) \\ &> 0. \end{aligned}$$

This is a contradiction. Thus, $0 < \alpha(t) \leq u(t)$ for all $t \in [0, 1]$.

A same argument shows that $u(t) \leq \beta(t)$ for all $t \in [0, 1]$.

Hence, $\forall t \in (0, 1)$, we have $\alpha(t) \leq u(t) \leq \beta(t)$, which

$$\bar{f}(t, u(t), u'(t)) = f(t, u(t), u'(t)), \quad t \in (0, 1),$$

that is, $u(t)$ is a solution of problem (2.3).

Next, we show that problem (2.5) has at least one solution.

It follows from $h \in L[0, 1]$, (2.4) and the definition of $\bar{f}$ that

$$|\bar{f}(t, u, v)| \leq 2h(t) + 1 + \|\alpha\|_1 + \|\beta\|_1 + K|v|^2, \quad t \in [0, 1], (u, v) \in \mathbb{R}^2.$$

Hence, for $u \in C^1[0, 1]$, the A can be defined well as

$$(Au)(t) = \psi(t) + \int_0^1 G(t, s) \bar{f}(s, u(s), u'(s)) ds, \quad t \in [0, 1],$$

where $\psi(t) = a + t(b - a)$ and

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1; \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to see that the fixed point of A is the solution of problem (2.5). Moreover, since

$$|(Au)''(t)| \leq 2h(t) + 1 + \|\alpha\|_1 + \|\beta\|_1 + K|u'(t)|^2, \quad t \in [0, 1]$$

and $\bar{f}$ is continuous, the Arzela–Ascoli theorem and the dominated convergence theorem guarantee that $A : C^1[0, 1] \rightarrow C^1[0, 1]$ is compact.

For $u \in C^1[0, 1]$ and $\lambda \in (0, 1)$, if $u = \lambda Au$, we claim that

$$0 \leq u(t) \leq 2 \max_{t \in [0, 1]} \beta(t), \quad \forall t \in [0, 1]. \quad (2.6)$$

(1) Suppose that there exists a $t_1 \in (0, 1)$ such that $u(t_1) < 0$. Then there exists a $t'_1 \in (0, 1)$ such that $u'(t'_1) = \min_{t \in [0, 1]} u(t) < 0$ and $u'(t'_1) = 0$, which together with $u = \lambda Au$ implies that

$$\begin{aligned} 0 &\geq -u''(t'_1) \\ &= \lambda \bar{f}(t'_1, u(t'_1), u'(t'_1)) \\ &= \lambda \left(f(t'_1, \alpha(t'_1), 0) + \frac{1}{\alpha(t'_1)} (|f(t'_1, \alpha(t'_1), 0)| + 1) \min\{\alpha(t'_1), \alpha(t'_1) - u(t'_1)\} \right) \\ &= \lambda (f(t'_1, \alpha(t'_1), 0) + |f(t'_1, \alpha(t'_1), 0)| + 1) \\ &> 0. \end{aligned}$$

This is a contradiction.

(2) Suppose that there exists a $t_2 \in (0, 1)$ such that $u(t_2) > 2 \max_{t \in [0, 1]} \beta(t)$. Then there exists a $t'_2 \in (0, 1)$ such that $u'(t'_2) = \max_{t \in [0, 1]} u(t) > 2 \max_{t \in [0, 1]} \beta(t)$ and $u'(t'_2) = 0$, which together with $u = \lambda Au$ implies that

$$\begin{aligned} 0 &\leq -u''(t'_2) \\ &= \lambda \bar{f}(t'_2, u(t'_2), u'(t'_2)) \\ &= \lambda (f(t'_2, \beta(t'_2), 0) - \frac{1}{\beta(t'_2)} (|f(t'_2, \beta(t'_2), 0)| + 1) \min\{\beta(t'_2), u(t'_2) - \beta(t'_2)\}) \\ &= -\lambda \\ &< 0. \end{aligned}$$

This is a contradiction also.

Then, if $u = \lambda Au$, from (2.6) and

$$|u''(t)| \leq 2h(t) + 1 + \|\alpha\|_1 + \|\beta\|_1 + K|u'(t)|^2,$$

Lemma 2.1 means that there is a $k_0 > 0$ such that $|u|_{H^{2,1}[0,1]} \leq k_0$. The interpolation inequality lemma guarantees that there exists an $R > 0$ such that

$$\|u\|_1 < R.$$

The Leray–Schauder fixed point theorem guarantees that A has at least one fixed point u in $C^1[0, 1]$.

Consequently, problem (2.3) has at least one solution $u \in C^1[0, 1]$ such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, 1]$. $\square$

Remark 2.3 The idea of this lemma comes from [4, 6, 9, 10, 17, 52].

Remark 2.4 If $f(t, u, v) = u^{-\frac{1}{3}} + u^2 + v^2$, let $\alpha(t) = \sin \pi t$, $t \in [0, 1]$ and $\beta(t) \equiv 2$, $t \in [0, 1]$ and $D_\alpha^\beta = \{(t, u) \in (0, 1) \times \mathbb{R} | \alpha(t) \leq u \leq \beta(t)\}$. Then

$$|f(t, u, v)| \leq \frac{1}{(\sin \pi t)^{\frac{1}{3}}} + 4 + v^2 = h(t) + v^2.$$

Obviously, $h \in L^1[0, 1]$. Hence, (2.4) is true.

Lemma 2.5 Suppose that $\alpha, \beta \in PC^1[0, 1] \cap PC^2(0, 1)$ are, respectively, lower and upper solutions for problem (2.1)–(2.2) with $\alpha(t) \leq \beta(t)$ and $\alpha(t_1) = \beta(t_1)$. Moreover, u is a solution of problem (2.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, 1]$. Then u satisfies (2.2), i.e. $u(t)$ is a solution of problem (2.1)–(2.2) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 1]$.

Proof Since $\alpha(t_1) = \beta(t_1)$ and $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, 1]$, we have $u(t_1) = \alpha(t_1) = \beta(t_1)$ and $\alpha'(t_1-) \geq u'(t_1-)$, $\alpha'(t_1+) \leq u'(t_1+)$ and $\beta'(t_1-) \leq u'(t_1-)$, $\beta'(t_1+) \geq u'(t_1+)$. Thus,

$$\begin{aligned} \Delta u'|_{t=t_1} &= u'(t_1+) - u'(t_1-) \\ &\geq \alpha'(t_1+) - \alpha'(t_1-) \\ &\geq N(\alpha(t_1), \alpha'(t_1-)) \\ &\geq N(u(t_1), u'(t_1-)) \end{aligned}$$

and

$$\begin{aligned} \Delta u'|_{t=t_1} &= u'(t_1+) - u'(t_1-) \\ &\leq \beta'(t_1+) - \beta'(t_1-) \\ &\leq N(\beta(t_1), \beta'(t_1-)) \\ &\leq N(u(t_1), u'(t_1-)), \end{aligned}$$

that is, $u(t)$ satisfies (2.2). Consequently, $u(t)$ is a solution of problem (2.1)–(2.2). $\square$

Theorem 2.6 Assume (A_0) and also assume

- (1) $f(t, u, v)$ is continuous in $u, v \in \mathbb{R}$, $t \neq t_1$. Furthermore, $\lim_{t \rightarrow t_1-} f(t, u, v) = f(t_1, u, v)$ and $\lim_{t \rightarrow t_1+} f(t, u, v)$ exists.

- (2) $N(u, v) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is nondecreasing in v .
- (3) α and β are, respectively, lower and upper solutions of problem (2.1)–(2.2) such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, 1]$.
- (4) There exists a function $h \in L^1[0, 1]$ such that $|f(t, u, v)| \leq h(t) + K|v|^2$ for all $(t, u) \in D_\alpha^\beta$ and $v \in \mathbb{R}$.

Then problem (2.1)–(2.2) has at least one solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, 1]$.

Proof Let $C \in [\alpha(t_1), \beta(t_1)]$. Condition (3) implies that α and β are, respectively, lower and upper solutions of the following problem:

$$\begin{cases} -u''(t) = f(t, u, u'), & t \in (0, t_1), \\ u(0) = a, & u(t_1) = C. \end{cases} \quad (2.7)$$

Thus by Lemma 2.2, problem (2.7) has a solution $v \in C^1[0, t_1] \cap C^2(0, t_1)$ satisfying

$$\alpha(t) \leq v(t) \leq \beta(t), \quad \forall t \in [0, t_1]. \quad (2.8)$$

By (A_0) , we get

$$\alpha(t_1+) = \alpha(t_1) + I(\alpha(t_1)) \leq \beta(t_1) + I(\beta(t_1)) = \beta(t_1+).$$

Define

$$\tilde{\alpha}(t) = \begin{cases} \alpha(t_1+), & t = t_1, \\ \alpha(t), & t \in (t_1, 1] \end{cases}$$

and

$$\tilde{\beta}(t) = \begin{cases} \beta(t_1+), & t = t_1, \\ \beta(t), & t \in (t_1, 1]. \end{cases}$$

It is easy to see that $\tilde{\alpha}$ and $\tilde{\beta}$ are lower and upper solutions of the following problem:

$$\begin{cases} -u''(t) = f(t, u, u'), & t \in (t_1, 1), \\ u(t_1) = C + I(C), & u(1) = b. \end{cases} \quad (2.9)$$

Thus again by Lemma 2.2, problem (2.9) has a solution $w \in C^1[t_1, 1] \cap C^2(t_1, 1)$ satisfying

$$\tilde{\alpha}(t) \leq w(t) \leq \tilde{\beta}(t), \quad \forall t \in [t_1, 1]. \quad (2.10)$$

Define

$$u(t) = \begin{cases} v(t), & t \in [0, t_1], \\ w(t), & t \in (t_1, 1], \end{cases}$$

which guarantees that

$$\Delta u|_{t=t_1} = u(t_1+) - u(t_1) = w(t_1) - v(t_1) = I(C) = I(v(t_1)) = I(u(t_1)).$$

From (2.8)–(2.10), u is a solution of (2.1) satisfying $u(t_1) = C$ and $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, 1]$.

Consequently, for each $C \in [\alpha(t_1), \beta(t_1)]$, there exists a solution u_C of problem (2.1) satisfying $u_C(t_1) = C$.

We now show that one of the solutions u_C satisfies condition (2.2). Let $X(C) = \{u | u \text{ is a solution of problem (2.1) such that } u(t_1) = C \text{ and } \alpha(t) \leq u(t) \leq \beta(t) \text{ for all } t \in [0, 1]\}$. Then, for each $C \in [\alpha(t_1), \beta(t_1)]$, $X(C) \neq \emptyset$. There are two cases: (A) $\alpha(t_1) = \beta(t_1) = C$; (B) $\alpha(t_1) < \beta(t_1)$.

First we consider the case (A). Since $\alpha(t_1) = \beta(t_1) = C$, by Lemma 2.3, $u_C(t)$ is a solution of problem (2.1)–(2.2).

Now we consider the case (B). Since $\alpha(t_1) < \beta(t_1)$, we define $S = \{\bar{C} \in [\alpha(t_1), \beta(t_1)) : C \in (\bar{C}, \beta(t_1)) \text{ implies } \Delta u'_C|_{t=t_1} < N(u_C(t_1), u'_C(t_1-))\}$, for all $u_C \in X(C)$. There are two cases also: (B₁) $S \neq \emptyset$; (B₂) $S = \emptyset$.

Now we consider the case (B₁). Let $C^* = \inf S$, then $\alpha(t_1) \leq C^* < \beta(t_1)$.

If $C^* > \alpha(t_1)$, by the definition of C^* , we can choose sequences $C_n \in (\alpha(t_1), C^*)$ and $u_{C_n} \in X(C_n)$ such that $C_n \rightarrow C^*$ and

$$\Delta u'_{C_n} \geq N(u_{C_n}(t_1), u'_{C_n}(t_1-)). \quad (2.11)$$

Obviously $u_{C_n}(t)$ satisfies

$$u_{C_n}(t) = \int_0^{t_1} G_1(t, s) f(s, u_{C_n}(s), u'_{C_n}(s)) ds + a + \frac{C_n - a}{t_1} t, \quad t \in [0, t_1], \quad (2.12)$$

and

$$\begin{aligned} u_{C_n}(t) &= \int_{t_1}^1 G_2(t, s) f(s, u_{C_n}(s), u'_{C_n}(s)) ds + b \\ &\quad + \frac{b - I(C_n) - C_n}{1 - t_1} (1 - t), \quad t \in (t_1, 1], \end{aligned} \quad (2.13)$$

where $G_1(t, s)$ and $G_2(t, s)$ are the Green's functions of linear homogeneous problem with Dirichlet boundary condition corresponding to (2.7) and (2.9). From $\alpha(t) \leq u_{C_n}(t) \leq \beta(t)$, $t \in [0, 1]$, Lemma 2.1 together with condition (4) of Theorem 2.6 guarantees that there is a $k_0 > 0$ such that

$$|u_{C_n}|_{H^{2,p}(0,1)} \leq k_0.$$

Now the Embedding theorem implies that there is an $R > 0$ such that $|u'_{C_n}(t)| \leq R$, $\forall t \in [0, 1]$. Condition (4) implies that

$$|f(x, u_{C_n}(t), u'_{C_n}(t))| \leq h(t) + KR^2$$

also. Since $h \in L^1[0, 1]$, the functions belonging to $\{u_{C_n}(t)\}$ and the functions belonging to $\{u'_{C_n}(t)\}$ are equicontinuous on $[0, t_1]$ and $(t_1, 1]$, respectively.

Therefore, without loss of generality, we assume that

$$u_{C_n}(t) \rightrightarrows u_0(t), \quad u'_{C_n}(t) \rightrightarrows u'_0(t), \quad \text{as } n \rightarrow +\infty.$$

Letting $n \rightarrow +\infty$ in (2.12) and (2.13), we have

$$u_0(t) = \int_0^{t_1} G_1(t, s) f(s, u_0(s), u'_0(s)) ds + a + \frac{C^* - a}{t_1} t, \quad t \in [0, t_1],$$

and

$$u_0(t) = \int_{t_1}^1 G_2(t, s) f(s, u_0(s), u'_0(s)) ds + b + \frac{b - I(C^*) - C^*}{1 - t_1} (1 - t), \quad t \in (t_1, 1].$$

Obviously, u_0 is a solution of problem (2.1) satisfying $u_0(t_1) = C^*$ and $\alpha(t) \leq u_1(t) \leq \beta(t)$ on $[0, 1]$. And from (2.12) and (2.13), we get

$$\begin{aligned} \Delta u'_{C_n}|_{t=t_1} &= u'_{C_n}(t_1+) - u'_{C_n}(t_1-) \\ &= \frac{b - I(C_n) - C_n}{1 - t_1} + \frac{1}{1 - t_1} \int_{t_1}^1 (1 - s) f(s, u_{C_n}(s), u'_{C_n}(s)) ds \\ &\quad - \left(\frac{C_n - t_1}{t_1} - \int_0^{t_1} s f(s, u_{C_n}(s), u'_{C_n}(s)) ds \right). \end{aligned}$$

Letting $n \rightarrow +\infty$, we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \Delta u'_{C_n}|_{t=t_1} &= \frac{b - I(C^*) - C^*}{1 - t_1} + \frac{1}{1 - t_1} \int_{t_1}^1 (1 - s) f(s, u_0(s), u'_0(s)) ds \\ &\quad - \left(\frac{C^* - t_1}{t_1} - \int_0^{t_1} s f(s, u_0(s), u'_0(s)) ds \right) \\ &= \Delta u'_0|_{t=t_1}. \end{aligned} \quad (2.14)$$

Letting $n \rightarrow +\infty$ in (2.11), we obtain

$$\lim_{n \rightarrow +\infty} \Delta u'_{C_n}|_{t=t_1} \geq \lim_{n \rightarrow +\infty} N(u_{C_n}(t_1), u'_{C_n}(t_1-)) = N(u_0(t_1), u'_0(t_1-)). \quad (2.15)$$

Equations (2.14) and (2.15) indicate

$$\Delta u'_0|_{t=t_1} \geq N(u_0(t_1), u'_0(t_1-)), \quad (2.16)$$

which implies that u_0 is a lower solution of problem (2.1)–(2.2) satisfying $u_0(t_1) = C^*$.

By the definition of C^* , for $D_n \in [C^*, \beta(t_1)]$ with $D_n \rightarrow C^*$ as $n \rightarrow +\infty$, we have

$$\begin{cases} -u''_{D_n}(t) = f(t, u_{D_n}, u'_{D_n}), & t \in (0, t_1), \\ u_{D_n}(0) = a, & u_{D_n}(t_1) = D_n, \end{cases}$$

$$\begin{cases} -u''_{D_n}(t) = f(t, u_{D_n}, u'_{D_n}), & t \in (t_1, 1), \\ u_{D_n}(t_1) = D_n + I(D_n), & u_{D_n}(1) = b, \end{cases}$$

and

$$\Delta u_{D_n}|_{t=t_1} < N(u_{D_n}(t_1), u'_{D_n}(t_1-)).$$

By a similar argument to the construction of u_0 , we obtain u_1 , a solution of problem (2.1) satisfying $u_1(t_1) = C^*$, $u_0(t) \leq u_1(t) \leq \beta(t)$ on $[0, 1]$, $u_{D_n}(t) \rightrightarrows u_1(t)$, $u'_{D_n}(t) \rightrightarrows u'_1(t)$ as a subsequence if necessary and

$$\Delta u'_1|_{t=t_1} \leq N(u_1(t_1), u'_1(t_1-)).$$

Thus u_1 is an upper solution of problem (2.1)–(2.2).

Consequently, u_0 and u_1 are lower and upper solutions of problem (2.1)–(2.2), respectively, satisfying $u_0(t_1) = C^* = u_1(t_1)$ and $u_0(t) \leq u_1(t)$ on $[0, 1]$.

Therefore, by the argument of (A), problem (2.1)–(2.2) has at least one solution between u_0 and u_1 .

If $C^* = \alpha(t_1)$, by the definition of C^* , we can choose sequences $C_n \in (\alpha(t_1), \beta(t_1))$ and $u_{C_n} \in X(C_n)$ such that $C_n \rightarrow \alpha(t_1) +$ as $n \rightarrow +\infty$ and

$$\Delta u'_{C_n} < N(u_{C_n}(t_1), u'_{C_n}(t_1-)). \quad (2.17)$$

By the same argument as for u_1 and $\{u_{D_n}\}$, there is a u_2 , a solution of problem (2.1) satisfying $u_2(t_1) = \alpha(t_1)$, $\alpha(t) \leq u_2(t) \leq \beta(t)$ on $[0, 1]$, and $u_{C_n}(t) \rightrightarrows u_2(t)$, $u'_{C_n}(t) \rightrightarrows u'_2(t)$ as a subsequence if necessary and

$$\Delta u'_2|_{t=t_1} \leq N(u_2(t_1), u'_2(t_1-)).$$

Consequently, α and u_2 are lower and upper solutions of problem (2.1)–(2.2), respectively, satisfying $\alpha(t_1) = C^* = u_2(t_1)$ and $\alpha(t) \leq u_2(t)$ on $[0, 1]$.

Therefore, by the argument of (A), problem (2.1)–(2.2) has at least one solution between α and u_2 .

The above proof shows that problem (2.1)–(2.2) has at least one solution if $S \neq \emptyset$.

Next, we consider (B_2) . Since $S = \emptyset$, we may choose sequences $C_n \in (\alpha(t_1), \beta(t_1))$ and $u_{C_n} \in X(C_n)$ satisfying $C_n \rightarrow \beta(t_1)$ as $n \rightarrow +\infty$ and $\Delta u'_{C_n}|_{t=t_1} \geq N(u_{C_n}(t_1), u'_{C_n}(t_1-))$.

By a similar limit argument to the one before, we obtain u_3 , a solution of problem (2.1) satisfying $u_3(t_1) = \beta(t_1)$, $\alpha(t) \leq u_3(t) \leq \beta(t)$, $t \in [0, 1]$, $u_{C_n}(t) \rightrightarrows u_3(t)$ and $u'_{C_n}(t) \rightrightarrows u'_3(t)$ as a subsequence if necessary and

$$\Delta u'_3|_{t=t_1} \geq N(u_3(t_1), u'_3(t_1-)).$$

Therefore, u_3 and β are lower and upper solutions of problem (2.1)–(2.2). By the proof of (A) also, problem (2.1)–(2.2) has at least one solution between u_3 and β .

Consequently, problem (2.1)–(2.2) has at least one solution if $S = \emptyset$.

The proof is completed. $\square$

Remark 2.7 The idea of our theorem comes from [23].

Remark 2.8 In the problem (1.1), let $f(t, u, v) = \lambda u^p - K(t)u^{-q} - |v|^\mu$. Since $\mu \in [0, 2]$, there is a $C' > 0$ such that

$$|f(t, u, v)| \leq \lambda |u^p| + |K(t)u^{-q}| + C' + |v|^2. \quad (2.18)$$

In Sects. 3 and 4, we will obtain the positive solutions of problem (1.1) by constructing a pair of lower and upper solutions which satisfy Theorem 2.6.

Throughout the paper, we always assume that $\mu \in [0, 2]$, $p \in (0, 1)$, $q \in (0, 1)$.

3 The positive solutions for problem (1.1) when $K(t) < 0$ on $t \in [0, 1]$

In this section, we consider existence of positive solutions for problem (1.1) when $K(t) < 0$ on $t \in [0, 1]$. We have the following theorem.

Theorem 3.1 Assume $K \in C[0, 1]$ with $K(t) < 0$ for all $t \in [0, 1]$ and the following conditions hold:

- (1) $I \in C(\mathbb{R}, \mathbb{R})$ and $u + I(u)$ is nondecreasing, $I(0) = 0$ and $I(u) \geq 0$ for all $u \in [0, +\infty)$;
- (2) $N(u, v) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is nondecreasing in v , $N(0, 0) < 0$ and there is a $m_0 > 0$ such that $N(u, 0) > 0$ for all $u \geq m_0$.

Then problem (1.1) has at least one positive solution u for all $\lambda > 0$.

Proof The proof is divided into three steps.

Step 1. We construct a lower positive solution of problem (1.1).

Since $N(0, 0) < 0$, there exists a $\varepsilon_0 > 0$ such that

$$N(\varepsilon, 0) < 0, \quad \forall 0 < \varepsilon \leq \varepsilon_0. \quad (3.1)$$

From $p \in (0, 1)$, for fixed $\lambda > 0$, there exists a positive constant ε_1 such that

$$\varepsilon \pi^2 \sin \pi t \leq \lambda (\varepsilon \sin \pi t)^p, \quad \forall 0 < \varepsilon \leq \varepsilon_1, t \in [0, 1].$$

Moreover, since $K(t) < 0$ for all $t \in [0, 1]$, there is a $\varepsilon_2 > 0$ such that

$$\begin{aligned} & |\varepsilon \pi \cos \pi t|^\mu + \varepsilon \pi^2 \sin \pi t \\ & \leq \lambda (\varepsilon \sin \pi t)^p - K(t)(\varepsilon \sin \pi t)^{-q}, \quad \forall 0 < \varepsilon \leq \varepsilon_2, t \in [0, 1]. \end{aligned} \quad (3.2)$$

Choose a positive constant $\varepsilon < \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Since $I(0) = 0$, $I(u) + u$ is increasing on $[0, +\infty)$ and $\lim_{u \rightarrow +\infty} (I(u) + u) = +\infty$, there exists a unique $0 < \varepsilon' < \varepsilon$ such that

$$\varepsilon = I(\varepsilon') + \varepsilon'. \quad (3.3)$$

Let

$$\underline{u}(t) = \begin{cases} \varepsilon' \sin \pi t, & t \in [0, \frac{1}{2}]; \\ \varepsilon \sin \pi t, & t \in (\frac{1}{2}, 1]. \end{cases}$$

From (3.1) and (3.3), we have

$$\begin{aligned}\Delta \underline{u}|_{t=\frac{1}{2}} &= \varepsilon \lim_{t \rightarrow \frac{1}{2}+} \sin \pi t - \varepsilon' \sin \pi \frac{1}{2} \\ &= I(\varepsilon') \\ &= I\left(\bar{u}\left(\frac{1}{2}\right)\right)\end{aligned}\quad (3.4)$$

and

$$\begin{aligned}\Delta \underline{u}'|_{t=\frac{1}{2}} &= \varepsilon \pi \lim_{t \rightarrow \frac{1}{2}+} \cos \pi t - \varepsilon' \pi \cos \pi \frac{1}{2} \\ &= 0 \\ &> N(\varepsilon', 0) \\ &= N\left(\varepsilon' \sin \pi \frac{1}{2}, \varepsilon' \pi \cos \pi \frac{1}{2}\right).\end{aligned}\quad (3.5)$$

From (3.2), we have

$$\begin{aligned}-\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t)(\underline{u}(t))^{-q} &= \varepsilon' \pi^2 \sin \pi t + |\varepsilon' \pi \cos \pi t|^\mu + K(t)(\varepsilon' \sin \pi t)^{-q} \\ &< \lambda(\varepsilon' \sin \pi t)^p \\ &= \lambda(\underline{u}(t))^p, \quad t \in \left(0, \frac{1}{2}\right),\end{aligned}\quad (3.6)$$

and

$$\begin{aligned}-\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t)(\underline{u}(t))^{-q} &= \varepsilon \pi^2 \sin \pi t + |\varepsilon \pi \cos \pi t|^\mu + K(t)(\varepsilon \sin \pi t)^{-q} \\ &< \lambda(\varepsilon \sin \pi t)^p \\ &= \lambda(\underline{u}(t))^p, \quad t \in \left(\frac{1}{2}, 1\right).\end{aligned}\quad (3.7)$$

Combining (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{cases} -\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t)(\underline{u}(t))^{-q} \leq \lambda \underline{u}^p(t), & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta \underline{u}|_{t=\frac{1}{2}} = I(\underline{u}(\frac{1}{2})), \\ \Delta \underline{u}'|_{t=\frac{1}{2}} \geq N(\underline{u}(\frac{1}{2}), \underline{u}'(\frac{1}{2}-)), \\ \underline{u}(0) = \underline{u}(1) = 0, \end{cases}\quad (3.8)$$

i.e. $\underline{u}$ is a lower solution of problem (1.1).

Step 2. We construct an upper positive solution of problem (1.1).

Choose a positive constant $C > \sup_{t \in [0,1]} \underline{u}(t)$. Since $p \in (0, 1)$, $q \in (0, 1)$, there exists $k_0 > \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ such that

$$k > \lambda(2kt(1-t) + C)^p - K(t)C^{-q}, \quad \forall k \geq k_0, t \in [0, 1]. \quad (3.9)$$

Let $k > \max\{m_0, k_0\}$ and k' satisfy

$$k' + C = I(k + C) + k + C. \quad (3.10)$$

Since $I(u) \geq 0$ for all $u \in [0, +\infty)$, we have $k' \geq k$. Define

$$\bar{u}(t) = \begin{cases} 4kt(1-t) + C, & t \in [0, \frac{1}{2}]; \\ 4k't(1-t) + C, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Now (3.10) implies that

$$\begin{aligned} \Delta \bar{u}|_{t=\frac{1}{2}} &= \lim_{t \rightarrow \frac{1}{2}+} (4k't(1-t) + C) - \left(4k\frac{1}{2}\left(1 - \frac{1}{2}\right) + C \right) \\ &= k' + C - (k + C) \\ &= I(k + C) \\ &= I\left(u\left(\frac{1}{2}\right)\right). \end{aligned} \quad (3.11)$$

Since $k + C > m_0$ (note $N(k + C, 0) > 0$), we have

$$\begin{aligned} \Delta \bar{u}'|_{t=\frac{1}{2}} &= \lim_{t \rightarrow \frac{1}{2}+} 4k'(1-2t) - 4k\left(1 - 2\frac{1}{2}\right) \\ &= 0 \\ &< N(k + C, 0) \\ &= N\left(\bar{u}\left(\frac{1}{2}\right), \bar{u}'\left(\frac{1}{2}-\right)\right). \end{aligned} \quad (3.12)$$

From (3.9), we have

$$\begin{aligned} -\bar{u}''(t) + |\bar{u}'(t)|^\mu + K(t)\bar{u}^{-q}(t) &= 2k + |4k(1-2t)|^\mu + K(t)\bar{u}^{-q}(t) \\ &\geq 2k + K(t)C^{-q} \\ &\geq \lambda(4kt(1-t) + C)^p \\ &= \lambda(\bar{u}(t))^p, \quad t \in \left(0, \frac{1}{2}\right), \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} -\bar{u}''(t) + |\bar{u}'(t)|^\mu + K(t)\bar{u}^{-q}(t) &= 2k' + |4k'(1-2t)|^\mu + K(t)\bar{u}^{-q}(t) \\ &\geq 2k' + K(t)C^{-q} \\ &\geq \lambda(4k't(1-t) + C)^p \\ &= \lambda(\bar{u}(t))^p, \quad t \in \left(\frac{1}{2}, 1\right). \end{aligned} \quad (3.14)$$

Combining (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{cases} -\bar{u}''(t) + |\bar{u}'(t)|^\mu + K(t)(\bar{u}(t))^{-q} \geq \lambda \bar{u}^p(t), & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta \bar{u}|_{t=\frac{1}{2}} = I(\bar{u}(\frac{1}{2})), \\ \Delta \bar{u}'|_{t=\frac{1}{2}} \leq N(\bar{u}(\frac{1}{2}), \bar{u}'(\frac{1}{2}-)), \\ \bar{u}(0) = \bar{u}(1) = 0, \end{cases} \quad (3.15)$$

i.e. $\bar{u}$ is an upper solution of problem (1.1).

Step 3. We claim that problem (1.1) has at least one positive solution.

Let

$$D_{\underline{u}}^{\bar{u}} = \{(t, u) | t \in (0, 1), \underline{u}(t) \leq u \leq \bar{u}(t)\}$$

and

$$h(t) = \lambda(4k't(1-t) + C)^p - K(t)(\varepsilon' \sin \pi t)^{-q}, \quad t \in [0, 1].$$

Then $h \in L^1[0, 1]$ and from (2.18), we have

$$|f(t, u, v)| \leq h(t) + C' + |v|^2, \quad \forall (t, u) \in D_{\underline{u}}^{\bar{u}}, v \in \mathbb{R}. \quad (3.16)$$

By Theorem 2.6, combining (3.8), (3.15) and (3.16), problem (1.1) has at least one positive solution u with $\underline{u}(t) \leq u(t) \leq \bar{u}(t)$ for all $t \in [0, 1]$.

The proof is completed. $\square$

Remark 3.2 Let $I(u) = u^2$ and $N(u, v) = -1 + u + v$. Obviously, I and N satisfy the conditions of Theorem 3.1.

4 The positive solutions for problem (1.1) when $K(t) > 0$ on $t \in [0, 1]$

In this section, we consider the existence of positive solutions for problem (1.1) when $K(t) > 0$ on $t \in [0, 1]$.

Theorem 4.1 Assume $K \in C[0, 1]$ with $K(t) > 0$ for all $t \in [0, 1]$ and the following conditions hold:

- (1) $I \in C(\mathbb{R}, \mathbb{R})$ and $u + I(u)$ is nondecreasing and $\lim_{u \rightarrow +\infty} I(u) = +\infty$;
- (2) $N(u, v) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is nondecreasing in v , $N(u, 0) \geq 0$ for all $u \in [0, +\infty)$ and $\lim_{u \rightarrow +\infty} \frac{N(u, 0)}{u} = 0$.

Then there is a $\lambda^* > 0$ such that problem (1.1) has at least one positive solution u for all $\lambda \geq \lambda^*$.

Proof The proof is divided into three steps.

Step 1. We construct a lower positive solution of problem (1.1).

Since $|\cos \pi t| \geq \frac{\sqrt{2}}{2}$ for all $t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, $N(u, 0) \geq 0$ for all $u \in [0, +\infty)$ and $\lim_{u \rightarrow +\infty} I(u) = +\infty$, $\lim_{u \rightarrow +\infty} \frac{N(u, 0)}{u} = 0$, there is a $k_1 > 0$ big enough such that

$$K(t) \frac{1}{(k_1 \sin \pi t)^{\frac{2q}{1+q}}} \leq k_1 \frac{(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{2q}{1+q}}} \pi^2 (\cos \pi t)^2, \quad \forall t \in \left(0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right), \quad (4.1)$$

and

$$6I(k_1) > N(I(k_1), 0). \quad (4.2)$$

Let $\gamma(t)$ be a smooth function on $[\frac{1}{3}, 1]$ such that $\gamma(t) = t - \frac{1}{3}$ if $t \in [\frac{1}{3}, \frac{2}{3}]$ and $\gamma(t) = 1 - t$ if $t \in [\frac{5}{6}, 1]$, and $\gamma(t) > 0$ for all $t \in [\frac{2}{3}, \frac{5}{6}]$. Define

$$\underline{u}(t) = \begin{cases} k_1 (\sin \pi t)^{\frac{2}{1+q}}, & t \in [0, \frac{1}{2}]; \\ k_1 (\sin \pi t)^{\frac{2}{1+q}} + 6I(k_1)\gamma(t), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Then, for $t \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} & -\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t)\underline{u}^{-q}(t) \\ &= k_1 \frac{2\pi^2}{1+q} (\sin \pi t)^{\frac{2}{1+q}} + \left(k_1 \pi \frac{2}{1+q} \right)^\mu |(\sin^{\frac{1-q}{1+q}} \pi t) \cos \pi t|^\mu \\ & \quad - k_1 \frac{2(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{2q}{1+q}}} \pi^2 (\cos \pi t)^2 + K(t) \frac{1}{(k_1 \sin \pi t)^{\frac{2q}{1+q}}}, \end{aligned} \quad (4.3)$$

and, for $t \in (\frac{1}{2}, 1]$, we have

$$\begin{aligned} & -\underline{u}''(t) + K(t)\underline{u}^{-q}(t) \\ &= k_1 \frac{2\pi^2}{1+q} (\sin \pi t)^{\frac{2}{1+q}} - 6I(k_1)\gamma''(t) + |\underline{u}'(t)|^\mu \\ & \quad - k_1 \frac{2(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{2q}{1+q}}} \pi^2 (\cos \pi t)^2 + K(t) \frac{1}{(k_1 \sin \pi t + 6I(k_1)\gamma(t))^{\frac{2q}{1+q}}} \\ & \leq k_1 \frac{2\pi^2}{1+q} (\sin \pi t)^{\frac{2}{1+q}} - 6I(k_1)\gamma''(t) \\ & \quad - k_1 \frac{2(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{2q}{1+q}}} \pi^2 (\cos \pi t)^2 + K(t) \frac{1}{(k_1 \sin \pi t)^{\frac{2q}{1+q}}}. \end{aligned} \quad (4.4)$$

Choose $0 < \delta < \frac{1}{6}$ small enough such that

$$\begin{aligned} & \left(k_1 \pi \frac{2}{1+q} \right)^\mu |(\sin^{\frac{1-q}{1+q}} \pi t) \cos \pi t|^\mu \\ & \leq k_1 \frac{(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{q}{1+q}}} \pi^2 (\cos \pi t)^2, \quad \forall t \in (0, \delta], \end{aligned} \quad (4.5)$$

and (note $\gamma''(t) = 0$ for all $t \in [1 - \delta, 1]$)

$$\begin{aligned} & -6I(k_1)\gamma''(t) + \left| k_1 \pi \frac{2}{1+q} (\sin^{\frac{1-q}{1+q}} \pi t) \cos \pi t + 6I(k_1)\gamma'(t) \right|^\mu \\ & \leq k_1 \frac{(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{q}{1+q}}} \pi^2 (\cos \pi t)^2, \quad \forall t \in [1 - \delta, 1]. \end{aligned} \quad (4.6)$$

Since $\min_{t \in [0, \delta] \cup [1-\delta, 1]} |\cos \pi t| \geq \min_{t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]} |\cos \pi t|$ for all $\delta \in (0, \frac{1}{6})$, combining (4.1), (4.5) and (4.6), we have

$$\begin{aligned} & K(t) \frac{1}{(k_1 \sin \pi t)^{\frac{2q}{1+q}}} + \left(k_1 \pi \frac{2}{1+q} \right)^\mu \left| \left(\sin^{\frac{1-q}{1+q}} \pi t \right) \cos \pi t \right|^\mu \\ & \leq k_1 \frac{2(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{q}{1+q}}} \pi^2 (\cos \pi t)^2, \quad \forall t \in (0, \delta], \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} & K(t) \frac{1}{(k_1 \sin \pi t)^{\frac{2q}{1+q}}} - 6I(k_1) \gamma''(t) + \left| k_1 \pi \frac{2}{1+q} \left(\sin^{\frac{1-q}{1+q}} \pi t \right) \cos \pi t + 6I(k_1) \gamma'(t) \right|^\mu \\ & \leq k_1 \frac{2(1-q)}{(1+q)^2} \frac{1}{(\sin \pi t)^{\frac{q}{1+q}}} \pi^2 (\cos \pi t)^2, \quad \forall t \in [1-\delta, 1). \end{aligned} \quad (4.8)$$

And then, combining (4.3), (4.4), (4.7) and (4.8), there is a $\lambda_1 > 0$ such that for all $\lambda \geq \lambda_1$

$$-\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t) \underline{u}^{-q}(t) \leq \lambda (\underline{u}(t))^p, \quad t \in (0, \delta] \cup [1-\delta, 1). \quad (4.9)$$

Since $\min_{t \in [\delta, 1-\delta]} \sin \pi t > 0$, we can choose λ_2 big enough such that, for $\lambda \geq \lambda_2$,

$$-\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t) \underline{u}^{-q}(t) \leq \lambda (\underline{u}(t))^p, \quad t \in [\delta, 1-\delta]. \quad (4.10)$$

Let $\lambda^* = \max\{\lambda_1, \lambda_2\}$. Combining (4.9) and (4.10), we have

$$-\underline{u}''(t) + |\underline{u}'(t)|^\mu + K(t) \underline{u}^{-q}(t) \leq \lambda (\underline{u}(t))^p, \quad t \in \left(0, \frac{1}{2}\right] \cup \left(\frac{1}{2}, 1\right), \quad \forall \lambda \geq \lambda^*. \quad (4.11)$$

By the definition of $\underline{u}$ and (4.2), we have

$$\begin{aligned} \Delta \underline{u}|_{t=\frac{1}{2}} &= \lim_{t \rightarrow \frac{1}{2}+} (k_1 \sin \pi t)^{\frac{2}{1+q}} + 6I(k_1) \gamma(t) - \left(k_1 \sin \pi \frac{1}{2} \right)^{\frac{2}{1+q}} \\ &= I(k_1) \\ &= I\left(\underline{u}\left(\frac{1}{2}\right)\right) \end{aligned} \quad (4.12)$$

and (note (4.2))

$$\begin{aligned} \Delta \underline{u}'|_{t=\frac{1}{2}} &= \lim_{t \rightarrow \frac{1}{2}+} \left((k_1 \sin \pi t)^{\frac{2}{1+q}} \right)' + 6I(k_1) \gamma'(t) - (k_1 \sin \pi t)^{\frac{2}{1+q}}|_{t=\frac{1}{2}}' \\ &= 6I(k_1) \\ &> N(I(k_1), 0) \\ &= N\left(\underline{u}\left(\frac{1}{2}\right), \underline{u}'\left(\frac{1}{2}-\right)\right). \end{aligned} \quad (4.13)$$

From (4.11), (4.12) and (4.13), we have

$$\begin{cases} -\underline{u}''(t) + K(t)(\underline{u}(t))^{-q} \leq \lambda \underline{u}^p(t), & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta \underline{u}|_{t=\frac{1}{2}} = I(\underline{u}(\frac{1}{2})), \\ \Delta \underline{u}'|_{t=\frac{1}{2}} \geq N(\underline{u}(\frac{1}{2}), \underline{u}'(\frac{1}{2}-)), \\ \underline{u}(0) = \underline{u}(1) = 0, \end{cases} \quad (4.14)$$

i.e. $\underline{u}$ is a lower solution of problem (1.1).

Step 2. We construct an upper solution of problem (1.1).

Since $p \in (0, 1)$, for $\lambda \geq \lambda^*$, we can choose $k_2 > k_1$ such that

$$2k > \lambda (kt(1-t))^p, \quad \forall t \in [0, 1], \forall k \geq k_2. \quad (4.15)$$

Let

$$\bar{u}(t) = \begin{cases} k_2 t(1-t), & t \in [0, \frac{1}{2}]; \\ (k_2 + 4I(\frac{1}{4}k_2))t(1-t), & t \in (\frac{1}{2}, 1]. \end{cases}$$

From (4.15), we have

$$-\bar{u}''(t) + |\bar{u}'(t)|^\mu + K(t)(\bar{u}(t))^{-q} \geq 2k_2 \geq \lambda \bar{u}^p(t), \quad t \in \left(0, \frac{1}{2}\right), \quad (4.16)$$

and

$$-\bar{u}''(t) + |\bar{u}'(t)|^\mu + K(t)(\bar{u}(t))^{-q} \geq 2\left(k_2 + 4I\left(\frac{1}{4}k_2\right)\right) \geq \lambda \bar{u}^p(t), \quad t \in \left(\frac{1}{2}, 1\right). \quad (4.17)$$

By the definition of $\bar{u}$ and $N(u, 0) \geq 0$ for all $u \geq 0$, we have

$$\Delta \bar{u}|_{t=\frac{1}{2}} = \bar{u}\left(\frac{1}{2}+\right) - \bar{u}\left(\frac{1}{2}-\right) = I\left(\frac{1}{4}k_2\right) = I\left(\bar{u}\left(\frac{1}{2}\right)\right) \quad (4.18)$$

and

$$\Delta \bar{u}'|_{t=\frac{1}{2}} = \bar{u}'\left(\frac{1}{2}+\right) - \bar{u}'\left(\frac{1}{2}-\right) = 0 \leq N\left(\bar{u}\left(\frac{1}{2}\right), \bar{u}'\left(\frac{1}{2}-\right)\right). \quad (4.19)$$

Combining (4.16), (4.17), (4.18) and (4.19), we have

$$\begin{cases} -\bar{u}''(t) + K(t)(\bar{u}(t))^{-q} \geq \lambda \bar{u}^p(t), & t \in (0, 1), t \neq \frac{1}{2}, \\ \Delta \bar{u}|_{t=\frac{1}{2}} = I(\bar{u}(\frac{1}{2})), \\ \Delta \bar{u}'|_{t=\frac{1}{2}} \leq N(\bar{u}(\frac{1}{2}), \bar{u}'(\frac{1}{2}-)), \\ \bar{u}(0) = \bar{u}(1) = 0, \end{cases} \quad (4.20)$$

which implies that $\bar{u}$ is a upper solution of problem (1.1).

Step 3. We claim that problem (1.1) has at least one positive solution.

Let

$$D_{\underline{u}}^{\overline{u}} = \{(t, u) | \underline{u}(t) \leq u \leq \overline{u}(t), t \in [0, 1]\}$$

and

$$h(t) = \lambda \left(k_1 + 4I \left(\frac{1}{4} k_2 \right) t(1-t) \right)^p + (k_1 \sin \pi t)^{\frac{-2q}{1+q}} + C',$$

where C' is defined in (2.18). Then $h \in L^1[0, 1]$ and

$$|f(t, u, v)| \leq h(t) + |v|^2, \quad \forall (t, u) \in D_{\underline{u}}^{\overline{u}}, v \in \mathbb{R}. \quad (4.21)$$

From (4.14), (4.20) and (4.21), Theorem 2.6 guarantees that problem (1.1) has at least one positive solution u with $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ for all $t \in [0, 1]$.

The proof is completed. $\square$

Remark 4.2 Let $I(u) = u^2$ and $N(u, v) = |u|^{\frac{1}{2}} + v$. Obviously, I and N satisfy the conditions of Theorem 2.6.

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