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Multiplicity results for sublinear elliptic equations with sign-changing potential and general nonlinearity

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Abstract

In this paper, we study the following elliptic boundary value problem:

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and f is allowed to be sign-changing and is of sublinear growth near infinity in u . For both cases that $V \in L^{N/2}(\Omega)$ with $N \geq 3$ and that $V \in C(\Omega, \mathbb{R})$ with $\inf_{\Omega} V(x) > -\infty$, we establish a sequence of nontrivial solutions converging to zero for above equation via a new critical point theorem.

MSC: 35J25; 35J60

Keywords: Semilinear elliptic equations; Boundary value problems; Sublinear; Indefinite sign; Genus

1 Introduction

Consider the elliptic boundary value problem

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $V : \Omega \rightarrow \mathbb{R}$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

The semilinear elliptic equation has found a great deal of interest in the past several decades. With the aid of variational methods, existence and multiplicity of nontrivial solutions for problem (1.1) have been extensively studied under various assumptions on the potential $V(x)$ and the nonlinearity $f(x, u)$; see [1, 2, 5, 10, 11, 13, 15–18, 20, 25, 26, 28–32] and the references therein.

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It is well known that weak solutions to (1.1) correspond to critical points of the energy functional:

$$\Phi(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + V(x)u^2) \, dx - \int_{\Omega} F(x, u) \, dx, \quad (1.2)$$

where here and in the sequel $F(x, t) := \int_0^t f(x, s) \, ds$. For the case of $\inf_{\Omega} V > -\lambda_1(\Omega)$, 0 is a local minimum of Φ , techniques based on the mountain-pass theorem have been well applied; see [1, 17, 26, 30]. When $\inf_{\Omega} V \in (-\infty, -\lambda_1(\Omega))$, 0 is a saddle point rather than a local minimum of Φ . Problem (1.1) is indefinite, and the main obstacle is to establish the boundedness of the Palais–Smale sequence for Φ . Under assumptions that $V \in L^{N/2}(\Omega)$ with $N \geq 3$ and f is superlinear near infinity in u , Li and Willem [15] obtained one nontrivial solution via a local linking method; see also Willem [30] via the linking theorem [26]. Later, this result was improved by Jiang and Tang [11] under a weak superquadratic condition introduced by Costa [5, 6]; see also [18], where local linking and symmetric mountain-pass theorem were also used. In [31], infinitely many solutions of (1.1) was proved by Zhang and Liu by using the variant fountain theorem established in [33]. With the aid of symmetric mountain-pass lemma, this result was improved and generalized by Qin et al. in [25]; see also [13] for similar results. Recently, under some new superquadratic conditions on f , Tang [29] showed that (1.1) has a ground state solution, as well as infinitely many pairs of solutions provided that V satisfies

$$(V) \quad V \in C(\Omega, \mathbb{R}) \text{ and } \inf_{\Omega} V(x) > -\infty.$$

The existence of infinitely many nontrivial solutions was obtained by He and Zou [10] for the case that f asymptotically shows linear growth near infinity. For related topics including the case of an unbounded domain, we refer the reader to [3, 4, 6, 7, 9, 12, 14, 16, 19, 21–24, 27, 28] and the references therein.

In [32], Zhang and Tang, via the variant fountain theorem established in [33], first studied the sublinear case provided that $V \in L^{N/2}(\Omega)$ and f satisfies:

$$(F1) \quad F \in C^1(\Omega \times \mathbb{R}, [0, \infty)), \text{ and there exist constants } \mu \in [1, 2) \text{ and } r_1 > 0 \text{ such that}$$

$$f(x, u)u \leq \mu F(x, u), \quad \forall x \in \Omega, |u| \geq r_1;$$

$$(F2) \quad \lim_{|u| \rightarrow 0} \frac{F(x, u)}{|u|^2} = \infty \text{ uniformly for } x \in \Omega, \text{ and there exist constants } c_1, r_2 > 0 \text{ such that}$$

$$F(x, u) \leq c_1 |u|, \quad \forall x \in \Omega, |u| \leq r_2;$$

$$(F3) \quad \text{there exists a constant } d > 0 \text{ such that}$$

$$\liminf_{|u| \rightarrow \infty} \frac{F(x, u)}{|u|} \geq d \quad \text{uniformly for } x \in \Omega;$$

Specifically, the following theorem was established [32].

Theorem 1.1 ([32]) *Suppose that (F1)–(F3), $V \in L^{N/2}(\Omega)$ with $N \geq 3$ and that $F(x, u)$ is even in u hold. Then problem (1.1) possesses infinitely many nontrivial solutions.*

Inspired by the aforementioned work, in the present paper, we continue to study the sublinear case under the following mild assumptions:

(S1) $F \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, and there exist constants $c > 0$ and $p \in (1, 2)$ such that

$$|f(x, u)| \leq c(1 + |u|^{p-1}), \quad \forall (x, u) \in \Omega \times \mathbb{R};$$

(S2) $F(x, 0) = 0$ for all $x \in \Omega$, and

$$\liminf_{|u| \rightarrow 0} \frac{F(x, u)}{|u|^2} = \infty, \quad \text{a.e. } x \in \Omega;$$

(S3) $F(x, u) = F(x, -u)$, $\forall (x, u) \in \Omega \times \mathbb{R}$.

Our main result reads as follows.

Theorem 1.2 *Suppose that (S1), (S2), (S3) and that $V \in L^{N/2}(\Omega)$ or (V) hold. Then problem (1.1) possesses a sequence of nontrivial solutions converging to zero.*

Remark 1.3 Nontrivial solutions obtained in Theorem 1.1 is different from ones established in [4, 25, 31, 32], since the sequence of critical points corresponding to the least energy does not necessarily converge to zero; see [12, Example 1.3].

Remark 1.4 Compared with Theorem 1.1, the crucial condition (F2) used in [32] is weakened to (S2), and the technical conditions (F1) and (F3) are got rid of in Theorem 1.2. Moreover, the nonlinearity f considered in this paper is allowed to be sign-changing. Hence, Theorem 1.2 extends and complements related results in [16, 25, 31, 32].

Before proceeding to the proof of main result, we give two examples to illustrate our assumptions.

Example 1.5 $F(x, u) = |u|^{3/2}[\ln 3 - \ln(1 + |u|^2)]$.

It is easy to verify that F satisfies (S1)–(S3), moreover, $F(x, u) < 0$ for $|u| > \sqrt{2}$, so (F3) does not hold.

Example 1.6 $F(x, u) = |u|^{1/2}(1 - |u|)$.

Note that $F(x, u) < 0$ for $|u| > 1$ and F satisfies (S1)–(S3), but it does not satisfies (F2) and (F3).

2 Variational setting and proofs of the main results

As in [29], we introduce the variational framework associated with problem (1.1) under (V) which holds also for the case that $V \in L^{N/2}(\Omega)$.

Denote by $\mathcal{A}$ the self-adjoint extension of the operator $-\Delta + V$ with domain $\mathcal{D}(\mathcal{A})$ ($C_0^\infty(\Omega) \subset \mathcal{D}(\mathcal{A}) \subset L^2(\Omega)$). Let $\{\mathcal{E}(\lambda) : -\infty \leq \lambda \leq +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [8, Theorem IV 3.3]). Let $E = \mathcal{D}(|\mathcal{A}|^{1/2})$ and

$$E^- = \mathcal{E}(0-)E, \quad E^0 = [\mathcal{E}(0) - \mathcal{E}(0-)]E, \quad E^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]E. \quad (2.1)$$

For any $u \in E$, it is easy to see that $u = u^- + u^0 + u^+$, where

$$\begin{aligned} u^- &:= \mathcal{E}(0-)u \in E^-, & u^0 &:= [\mathcal{E}(0) - \mathcal{E}(0-)]u \in E^0, \\ u^+ &:= [\mathcal{E}(+\infty) - \mathcal{E}(0)]u \in E^+. \end{aligned} \quad (2.2)$$

Define an inner product

$$(u, v) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}, \quad \forall u, v \in E, \quad (2.3)$$

and the corresponding norm

$$\|u, v\| = \| |\mathcal{A}|^{1/2}u \|_2 + \|u^0\|_2, \quad \forall u \in E, \quad (2.4)$$

where $(\cdot, \cdot)_{L^2}$ denotes the inner product of $L^2(\Omega)$, $\|\cdot\|_s$ stands for the usual $L^s(\Omega)$ norm. Then $E \subset H_0^1(\Omega)$ is a Hilbert space. Clearly, $C_0^\infty(\Omega)$ is dense in E .

The following lemma was established in [29, Lemmas 2.4, 2.5, Remark 2.8].

Lemma 2.1 *Let (V) be satisfied. Then, for the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{L^2}$ on E , we have*

$$E^- \perp E^0, \quad E^- \perp E^+, \quad E^0 \perp E^+, \quad \dim(\mathcal{E}(M)E) < +\infty, \quad \forall M \geq 0, \quad (2.5)$$

and

$$E^0 = \text{Ker}(\mathcal{A}), \quad \mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall u \in E \cap \mathcal{D}(\mathcal{A}). \quad (2.6)$$

Moreover, E is compactly embedded in $L^s(\Omega)$ for $1 \leq s < 2^*$, where $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1$ or 2 .

For the case that $V \in L^{N/2}(\Omega)$, spectrum of $\mathcal{A}$ consists of only eigenvalues numbered in $-\infty < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq 0 < \mu_{n+1} \leq \dots \rightarrow +\infty$ (counted with multiplicity) with the corresponding system of eigenfunctions $\{e_n\}$ forming an orthogonal basis in $L^2(\Omega)$; see [8, Theorem VI.1.4] or [31]. In particular,

$$E^- = \text{span}\{e_1, \dots, e_{n^-}\}, \quad E^0 = \text{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\}, \quad E^+ = \overline{\text{span}\{e_{\bar{n}+1}, \dots\}}, \quad (2.7)$$

where

$$n^- := \#\{i : \lambda_i < 0\}, \quad n^0 := \#\{i : \lambda_i = 0\}, \quad \bar{n} := n^- + n^0. \quad (2.8)$$

Moreover, similar to [31, Lemmas 2.1, 2.3], we have the following lemmas.

Lemma 2.2 *Under assumption $V \in L^{N/2}(\Omega)$ with $N \geq 3$, the norm $\|\cdot\|$ in $E = H_0^1(\Omega)$ is equivalent to the usual Sobolev norm $\|\cdot\|_{1,2}$ in $H_0^1(\Omega)$, and E is compactly embedded in $L^s(\Omega)$ for $1 \leq s < 2^*$.*

Lemma 2.3 Suppose that $V \in L^{N/2}(\Omega)$ or (V) holds. Then E is compactly embedded in $L^s(\Omega)$ for $1 \leq s < 2^*$, and there exists $\tau_s > 0$ such that

$$\|u\|_s \leq \tau_s \|u\|, \quad \forall u \in E. \quad (2.9)$$

By (S1) we have

$$|F(x, u)| \leq c(|u| + |u|^p), \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (2.10)$$

Under (S1) and assumptions of Lemma 2.3, the functional Φ defined by (1.2) is of class $C^1(E, \mathbb{R})$. Moreover, by virtue of (2.3) and (2.6), one has

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\Omega} F(x, u) dx, \quad \forall u \in E, \quad (2.11)$$

and

$$\langle \Phi'(u), v \rangle = (u^+, v^+) - (u^-, v^-) - \int_{\Omega} f(x, u)v dx, \quad \forall u, v \in E. \quad (2.12)$$

Before presenting the critical point theorem used in this paper, we give some notions.

Let X be a Banach space and $I \in C^1(X, \mathbb{R})$ a functional. A sequence $\{u_n\} \subset X$ is called a (PS) sequence (or $(PS)_c$ sequence) if

$$\{I(u_n)\} \text{ is bounded} \quad (\text{or } I(u_n) \rightarrow c), \quad I'(u_n) \rightarrow 0. \quad (2.13)$$

The functional I is said to satisfy (PS) condition (or $(PS)_c$ condition) if each (PS) sequence (or $(PS)_c$ sequence) has a convergent subsequence.

A subset $A \subset X$ is said to be symmetric if $u \in A$ implies that $-u \in A$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k , we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subset A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

Theorem 2.4 ([12, Theorem 1], [16, Theorem 1.1]) Let X be an infinite dimensional Banach space, and $I \in C^1(X, \mathbb{R})$ satisfies (H1) and (H2) below.

(H1) I is even, bounded from below, $I(0) = 0$ and I satisfies the (PS) condition;

(H2) for each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then there exists a sequence of critical points $\{u_k\}$ such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.

Remark 2.5 As we see from the proof of [16, Theorem 1.1], the above theorem holds also if we use the $(PS)_c$ condition with $c \leq 0$ instead of the (PS) condition in (H1).

Under $V \in L^{N/2}(\Omega)$ or (V), it follows from (2.5) or (2.7) that $\dim(E^- \oplus E^0) < \infty$. We choose an orthonormal basis $\{\xi_j\}_{j=1}^{k_0}$ for E^- , an orthonormal basis $\{\xi_j\}_{j=k_0+1}^{l_0}$ for E^0 and an

orthonormal basis $\{\xi_j\}_{j=l_0+1}^\infty$ for E^+ , where $k_0, l_0 \in \mathbb{N}$ and $1 \leq k_0 < l_0 < \infty$. Then $\{\xi_j\}_{j=1}^\infty$ is an orthonormal basis of E . Define

$$X_j := \mathbb{R}\xi_j, \quad Y_k := \bigoplus_{j=1}^k X_j, \quad Z_k := \bigoplus_{j=k+1}^\infty X_j, \quad k \in \mathbb{Z}. \quad (2.14)$$

Proof of Theorem 1.2 Consider the truncated functional

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \left(\|u^-\|^2 + \|u^0\|_2^2 + 2 \int_\Omega F(x, u) \, dx \right) \phi(\|u\|^2), \quad u \in E, \quad (2.15)$$

where $\phi: \mathbb{R}^+ \rightarrow [0, 1]$ is a smooth functional such that $\phi(t) = 1$ for $t \in [0, 1]$, $\phi(t) = 0$ for $t \geq 2$, $\phi'(t) \leq 0$ and $|\phi'(t)| \leq 2$, $\forall t \geq 0$. Obviously, $I \in C^1(E, \mathbb{R})$ and $I(0) = 0$ by (S2). If we can prove that I satisfies (H1) and (H2), using Theorem 2.4, I admits a sequence of critical points $\{u_k\}$ such that $I(u_k) \leq 0$, $u_k \neq 0$ and $u_k \rightarrow 0$ as $k \rightarrow \infty$. So does Φ by the fact $\Phi(u) = I(u)$ for $\|u\| \leq 1$. Obviously, I is even by (S3).

Note that $I(u) = \frac{1}{2} \|u\|^2$ if $\|u\| \geq \sqrt{2}$. Then the functional I is coercive, i.e.

$$I(u) \rightarrow +\infty, \quad \text{as } \|u\| \rightarrow \infty, \quad (2.16)$$

which implies that I is bounded from below and satisfies $(PS)_c$ condition with $c \leq 0$. Indeed, any sequence $\{u_n\} \subset E$ satisfying (2.13) is bounded by (2.16). Passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in E . By Lemma 2.3, $u_n \rightarrow u$ in $L^s(\Omega)$ for $s \in [1, 2^*)$, and $u_n^- \rightarrow u^-$, $u_n^0 \rightarrow u^0$ in E since $\dim(E^- \oplus E^0) < \infty$. It follows from (S1) and (2.9) that

$$\begin{aligned} & \int_\Omega (|f(x, u_n)| + |f(x, u)|) |u_n - u| \, dx \\ & \leq c \int_{\mathbb{R}^N} (|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}) |u_n - u| \, dx \\ & \leq c(\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + c(\|u_n\|_p^{p-1} + \|u\|_p^{p-1}) \|u_n - u\|_p \\ & \leq c\tau_2^2 (\|u_n\| + \|u\|) \|u_n - u\|_2 + c\tau_p^{p-1} (\|u_n\|^{p-1} + \|u\|^{p-1}) \|u_n - u\|_p \\ & = o(1). \end{aligned} \quad (2.17)$$

By (2.15), direct computation shows that

$$\begin{aligned} \langle I'(u), v \rangle &= \left[1 - \left(\|u^-\|^2 + \|u^0\|_2^2 + 2 \int_\Omega F(x, u) \, dx \right) \phi'(\|u\|^2) \right] \langle u, v \rangle \\ &\quad - \left[(u^-, v^-) + (u^0, v)_2 + \int_\Omega f(x, u) v \, dx \right] \phi(\|u\|^2), \quad u, v \in E. \end{aligned} \quad (2.18)$$

It follows from (2.13) and (2.15) that

$$-\left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_\Omega F(x, u_n) \, dx \right) \phi(\|u_n\|^2) = 2I(u_n) - \|u_n\|^2 \leq 0,$$

which implies that

$$\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_\Omega F(x, u_n) \, dx \geq 0. \quad (2.19)$$

Note that $\phi'(t) \leq 0$ and $|\phi'(t)|/2, \phi(t) \leq 1$ for any $t \geq 0$, then, by (2.9), (2.17), (2.18) and (2.19), one has

$$\begin{aligned}
 & \|u_n - u\|^2 \\
 &= \langle I'(u_n) - I'(u), u_n - u \rangle + \left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_{\Omega} F(x, u_n) \, dx \right) \\
 &\quad \times \phi'(\|u_n\|^2)(u_n, u_n - u) - \left(\|u^-\|^2 + \|u^0\|_2^2 + 2 \int_{\Omega} F(x, u) \, dx \right) \phi'(\|u\|^2)(u, u_n - u) \\
 &\quad + \phi(\|u_n\|) \left[\|u_n^- - u^-\|^2 + \|u_n^0 - u^0\|_{L^2}^2 + \int_{\Omega} (f(x, u_n) - f(x, u))(u_n - u) \, dx \right] \\
 &\quad + [\phi(\|u_n\|) - \phi(\|u\|)] \left[(u^-, u_n^- - u^-) + (u^0, u_n^0 - u^0)_{L^2} + \int_{\Omega} f(x, u)(u_n - u) \, dx \right] \\
 &\leq o(1) + \left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_{\Omega} F(x, u_n) \, dx \right) \phi'(\|u_n\|^2)(u_n, u_n - u) \\
 &\quad + \int_{\Omega} (|f(x, u_n)| + |f(x, u)|) |u_n - u| \, dx + 2 \int_{\Omega} |f(x, u)| |u_n - u| \, dx \\
 &= o(1) + \left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_{\Omega} F(x, u_n) \, dx \right) \phi'(\|u_n\|^2) \|u_n - u\|^2 \\
 &\quad + \left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_{\Omega} F(x, u_n) \, dx \right) \phi'(\|u_n\|^2)(u, u_n - u) \\
 &= o(1) + \left(\|u_n^-\|^2 + \|u_n^0\|_2^2 + 2 \int_{\Omega} F(x, u_n) \, dx \right) \phi'(\|u_n\|^2) \|u_n - u\|^2 \\
 &\leq o(1). \tag{2.20}
 \end{aligned}$$

Thus I satisfies the $(PS)_c$ condition with $c \leq 0$.

For any fixed $l_0 + 1 \leq k \in \mathbb{N}$, where $l_0 = \dim(E^- \oplus E^0) < \infty$. By (2.14) and the equivalence of the norms on finite dimensional spaces, there exist constants $c_k, d_k > 0$ such that

$$\|u\|_2 \geq c_k \|u\|, \quad \operatorname{ess\,sup}_{x \in \Omega} |u(x)| := \|u\|_{\infty} \leq d_k \|u\|, \quad \forall u \in Y_k, \tag{2.21}$$

(S2) implies the existence of a constant $r \in (0, 1)$ such that $F(x, u) \geq c_k^{-2} |u|^2$ for all $|u| \leq r$ and a.e. $x \in \Omega$. Then, for any $u \in Y_k$ with $\|u\| = l_k := 2^{-1} \min\{1, r d_k^{-1}\}$, one has

$$\begin{aligned}
 I(u) &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \int_{\Omega} F(x, u) \, dx \\
 &\leq \frac{1}{2} \|u^+\|^2 - \int_{\Omega} F(x, u) \, dx \leq \frac{1}{2} \|u\|^2 - \frac{1}{c_k^2} \|u\|_2^2 \\
 &\leq \frac{1}{2} \|u\|^2 - \frac{c_k^2}{c_k^2} \|u\|^2 = -\frac{1}{2} \|u\|^2 \\
 &= -\frac{1}{2} l_k^2, \tag{2.22}
 \end{aligned}$$

which implies that

$$\{u \in Y_k : \|u\| = l_k\} \subset \left\{ u \in E : I(u) \leq -\frac{1}{2} l_k^2 \right\}. \tag{2.23}$$

Let $A_k := \{u \in E : I(u) \leq -2^{-1}l_k^2\}$. Then

$$\gamma(A_k) \geq \gamma(\{u \in Y_k : \|u\| = l_k\}) \geq k. \quad (2.24)$$

Clearly $A_k \in \Gamma_k$ and $\sup_{u \in A_k} I(u) \leq -2^{-1}l_k^2 < 0$. Then we get the result from Theorem 2.4. $\square$

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Availability of data and materials

This paper focuses on theoretical analysis, not involving experiments and data.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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