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Well-posedness and behaviors of solutions to an integrable evolution equation

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Abstract

This work is devoted to investigating the local well-posedness for an integrable evolution equation and behaviors of its solutions, which possess blow-up criteria and persistence property. The existence and uniqueness of analytic solutions with analytic initial values are established. The solutions are analytic for both variables, globally in space and locally in time. The effects of coefficients λ and β on the solutions are given.

Keywords: Local well-posedness; Blow-up; Persistence property; Analytic solutions

1 Introduction

We focus on investigating the following Cauchy problem:

$$\begin{cases} w_t - w_{xxt} + \beta(w_x - w_{xxx}) + \lambda(w - w_{xx}) - 16ww_x \\ \quad = 2w_{xx}^2 - 8w_x w_{xx} + 2w_x w_{xxx} - 4ww_{xxx}, \\ w(0,x) = w_0(x). \end{cases} \tag{1.1}$$

Here, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $\lambda \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, w is the fluid velocity, $\beta(w_x - w_{xxx})$ is the dispersive term, $\lambda(w - w_{xx})$ is the dissipative term, $w_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$.

Problem (1.1) is viewed as a member of the integrable model

$$(1 - \partial_x^2)w_t = F(w, w_x, w_{xx}, w_{xxx}),$$

which has been investigated in [24]. The famous integrable Camassa–Holm (CH) equation is

$$(1 - \partial_x^2)w_t + 3ww_x = -\beta w_x + 2w_x w_{xx} + ww_{xxx}, \tag{1.2}$$

which admits peakon solutions and wave breaking mechanisms. By replacing w with $w + k$ in Eq. (1.2), Zhou and Chen [33] establish that a solution w to Eq. (1.2) may be regarded as a perturbation around the coefficient β. The wave breaking phenomena and infinite propagation speed of solutions are investigated. The behaviors of solutions to the CH equation with dissipative term and dispersion term are studied in [25]. The local well-posedness for the Cauchy problem of the CH type equations [6, 15, 20, 26, 28–31], asymptotic stability [17, 22], solitons solutions [14], and regularity of conservative solutions [18] are considered. The readers may refer to [8–10, 18, 20–22] for the related results.

Other two famous integrable models are the Degasperis–Procesi (DP) equation

$$w_t - w_{xxt} + 4ww_x = 3w_x w_{xx} + ww_{xxx}$$

and the Novikov equation

$$w_t - w_{xxt} + 4w^2 w_x = 3ww_x w_{xx} + w^2 w_{xxx}. \tag{1.3}$$

Molinet [23] considers the peakon solutions of the DP equation. The Novikov equation has N-peakon solutions. It is worth noticing that the first explicit 2-peakon solutions of the Novikov equation are investigated in [13]. Cai et al. [2] study the Lipschitz metric of Eq. (1.3) which possesses cubic nonlinearity. Himonas et al. [11] illustrate the construction of 2-peakon solutions and ill-posedness for the Novikov equation. The blow-up criteria of solutions to a Novikov type equation are presented in [7, 32]. The formation of singularities for solutions to problem (1.1) when $\lambda = \beta = 0$ is established (see [27]). The scholars focus much attention on the CH equation and similar equations with weakly dissipative term. It is shown in [16] that some models (i.e., CH equation, DP equation, Novikov equation, and Hunter–Saxton equation) which contain weakly dissipative term can be reduced to their non-dissipative versions by applying an exponentially time-dependent scaling $u(t,x) \to e^{-\lambda t} u(\frac{1-e^{-\lambda t}}{\lambda}, x)$.

To our knowledge, the influence of coefficients and properties of solutions to problem (1.1) have not been considered yet. Our study mainly focuses on investigating the influence of dissipative coefficient λ and dispersive coefficient β on the solutions to problem (1.1). We establish the blow-up criteria and blow-up rate of solutions, which are related to $n = (1-\partial_x^2)w$ and dissipative coefficient λ. Moreover, the persistence properties and analytic properties of solutions are analyzed.

We define

$$E^s_{p,r}(T) = \begin{cases} C([0,T]; B^s_{p,r}(\mathbb{R})) \cap C^1([0,T]; B^{s-1}_{p,r}(\mathbb{R})), & 1 \le r < \infty, \\ L^\infty([0,T]; B^s_{p,\infty}(\mathbb{R})) \cap \mathrm{Lip}([0,T]; B^{s-1}_{p,\infty}(\mathbb{R})), & r = \infty, \end{cases}$$

where $T > 0$, $s \in \mathbb{R}$, $p \in [1,\infty]$, $r \in [1,\infty]$. Problem (1.1) is written as

$$\begin{cases} w_t - 4ww_x = -w_x^2 + P_1(D)[2w_x^2 + 6w^2] + P_2(D)[w_x^2] - \lambda w - \beta w_x, \\ w(0,x) = w_0(x), \end{cases} \tag{1.4}$$

where $P_1(D) = \partial_x(1-\partial_x^2)^{-1}$, $P_2(D) = (1-\partial_x^2)^{-1}$.

Let $n_0 = (1-\partial_x^2)w_0$ and $n = (1-\partial_x^2)w$. Then problem (1.1) is reformulated as

$$\begin{cases} n_t + (2w_x - 4w + \beta)n_x = 2n^2 + (8w_x - 4w)n + 2(w+w_x)^2 - \lambda n, \\ n(0,x) = n_0(x). \end{cases} \tag{1.5}$$

We are in the position to summarize the main results.

Theorem 1.1 *Let* $1 \le p, r \le \infty$, $w_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$. *Then a solution* $w \in E^s_{p,r}(T)$ *to problem* (1.1) *is unique for certain* $T > 0$.

Theorem 1.2 *Let* $1 \le p$, $r \le \infty$, $w_0 \in B^s_{p,r}(\mathbb{R})(\max(\frac{5}{2}, 2 + \frac{1}{p}) < s < 3)$, $t \in [0, T]$. *Then a solution* w *to problem* (1.1) *blows up in finite time if and only if*

$$\int_0^t \left(\|n(\tau)\|_{L^\infty} - \lambda\right) d\tau = +\infty.$$

Theorem 1.3 *Let* $1 \le p, r \le \infty$, $w_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$, $t \in [0, T]$. *Then a solution* w *to problem* (1.1) *blows up in finite time if and only if*

$$\int_0^t \left(\|n(\tau)\|_{L^\infty} - \lambda\right) d\tau = +\infty. \tag{1.6}$$

Theorem 1.4 *Let* $1 \le p$, $r \le \infty$, $w_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$, $n_0 = w_0 - w_{0,xx}$. *Assume that* $n_0(x)$ *satisfies* $n_0(x_0) > \frac{\lambda}{2} + \sqrt{K}$, *where the point* x_0 *is defined by* $n_0(x_0) = \sup_{x\in\mathbb{R}} n_0(x)$, $K = \frac{\lambda^2}{4} + 18\|w_0\|^2_{H^1}$. *Let* $t \in [0, T]$. *Then a solution* w *to problem* (1.1) *blows up in finite time if and only if*

$$\lim_{t\to T^-}\left[\sup_{x\in\mathbb{R}}\left(n(t,x) - \frac{\lambda}{2}\right)\right] = +\infty. \tag{1.7}$$

Theorem 1.5 *Let* $1 \le p, r \le \infty$, $w_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$, $n_0 = w_0 - w_{0,xx}$, $t \in [0, T]$. *Suppose that* $[n_0 + 2w_{0,x} - w_0](x_0) > \frac{\lambda}{4} + \frac{1}{2}\sqrt{K_1}$, *where the point* x_0 *is defined by*

$$[n_0 + 2w_{0,x} - w_0](x_0) = \sup_{x\in\mathbb{R}}[n_0 + 2w_{0,x} - w_0](x),$$

$K_1 = 2(C_4\|w_0\|^2_{H^1} + C_5\|w_0\|_{H^1} + C_6)$ *and* C_4, C_5, C_6 *are certain positive constants. Let* w *be a solution to problem* (1.1). *Then it holds that*

$$\lim_{t\to T^-}\left[\sup_{x\in\mathbb{R}}\left(n(t,x) - \frac{\lambda}{4}\right)(T - t)\right] = \frac{1}{2}.$$

Theorem 1.6 *Assume* $w_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$, $t \in [0, T]$ *and* $\theta \in (0, 1)$. *Let* w_0 *satisfy*

$$|w_0(x)|, |\partial_x w_0(x)|, |\partial_x^2 w_0(x)| \sim O(e^{-\theta x}) \quad \text{as } x \to \infty.$$

Then a solution w *to problem* (1.1) *satisfies*

$$|w(t,x)|, |\partial_x w(t,x)|, |\partial_x^2 w(t,x)| \sim O(e^{-\theta x}) \quad \text{as } x \to \infty$$

uniformly on $[0, T]$.

Theorem 1.7 *Let* w_0 *be analytic on* $\mathbb{R}$ *and* $t \in \mathbb{R}$ *in problem* (1.1). *Then problem* (1.1) *admits a unique analytic solution* w *on* $(-\delta, \delta) \times \mathbb{R}$ *for certain constant* $\delta \in (0, 1]$.

Remark 1.1 We deduce the local well-posedness for problem (1.1) in $B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 1 + \frac{2}{p}))$. For presence of term w_x^2 in (1.4), the regularity index of solutions is $s > \max(\frac{5}{2}, 1 + \frac{2}{p})$, which is different from the regularity index $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ of solutions to the CH equation, DP equation, and Novikov equation.

Remark 1.2 We derive blow-up criterion of solutions in the Besov space in Theorem 1.2. This result is new. From Theorems 1.2, 1.3, and 1.4, we conclude that dissipative coefficient λ is related to blow-up mechanisms of solutions. From Theorem 1.4, we recognize that the blow-up phenomenon of solution w occurs if n is unbounded. From Theorem 1.5, we establish that dissipative coefficient λ is related to the precise blow-up rate of solution w. From Theorem 1.6, we observe that if initial value w_0 with its derivatives exponentially decays at infinity, then the solution w with its derivatives also exponentially decays at infinity. The existence and uniqueness of analytic solution w with analytic initial value are illustrated in Theorem 1.7. The solution w is analytic in both variables, globally in space and locally in time.

Remark 1.3 We extend parts of results in [27]. In the case $\lambda = \beta = 0$ in problem (1.1), the local well-posedness for the Cauchy problem and formation of singularities of solutions are investigated in [27]. However, we mainly focus on the influence of the dispersive term and dissipative term in problem (1.1). Theorems 1.1, 1.4, and 1.5 contain the results in [27] as special cases when $\lambda = \beta = 0$. In addition, for problem (1.1), we also establish blow-up criteria of solutions in the Besov space and persistence property of solutions. The existence and uniqueness of analytic solutions with analytic initial values are also studied (see detailed illustration in Remarks 1.1–1.2).

2 Proof of Theorem 1.1

2.1 Several lemmas

We review several basic facts in the Besov space. One may check [1] for more details.

Lemma 2.1 ([1]) *There exists a couple of smooth functions* $(\chi(\xi), \varphi(\xi))$ *valued in* $[0, 1]$ *such that* χ *is supported in the ball* $B = \{\xi \in \mathbb{R} | |\xi| \leq \frac{4}{3}\}$, φ *is supported in the ring* $C = \{\xi \in \mathbb{R} | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. *Moreover, it satisfies that*

$$\chi(\xi) + \sum_{q \in \mathbb{N}} \varphi\left(2^{-q}\xi\right) = 1, \quad \forall \xi \in \mathbb{R}$$

and

$$\begin{aligned} &\operatorname{supp} \varphi\left(2^{-q}\cdot\right) \cap \operatorname{supp} \varphi\left(2^{-q'}\cdot\right) = \emptyset, \quad \textit{if } \left|q - q'\right| \geq 2, \\ &\operatorname{supp} \chi(\cdot) \cap \operatorname{supp} \varphi\left(2^{-q}\cdot\right) = \emptyset, \quad \textit{if } q \geq 1. \end{aligned}$$

Then, for all $u \in S'(\mathbb{R})$, *the non-homogeneous dyadic blocks are defined as follows. Let*

$$\begin{aligned} &\Delta_q u = 0, \quad \textit{if } q \leq -2, \\ &\Delta_{-1} u = \int_{\mathbb{R}} \chi(\xi)\widehat{u}(\xi)e^{ix\xi}\,d\xi, \quad \textit{if } q = -1, \end{aligned}$$

$$\Delta_q u = \int_{\mathbb{R}} \varphi\big(2^{-q}\xi\big)\widehat{u}(\xi)e^{ix\xi}\,d\xi, \quad \text{if } q \geq 0.$$

Then $u = \sum_{q=-1}^{\infty}\Delta_q u$ is called the non-homogeneous Littlewood–Paley decomposition of u. Assume $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The non-homogeneous Besov space is defined by $B^s_{p,r} = \{f \in S'(\mathbb{R}) \mid \|f\|_{B^s_{p,r}} < \infty\}$, where

$$\|f\|_{B^s_{p,r}} = \begin{cases} (\sum_{j=-1}^{\infty} 2^{jrs}\|\Delta_j f\|^r_{L^p})^{\frac{1}{r}}, & r < \infty, \\ \sup_{j\geq -1} 2^{js}\|\Delta_j f\|_{L^p}, & r = \infty. \end{cases}$$

In addition, $S_j f = \sum_{q=-1}^{j-1}\Delta_q f$.

Lemma 2.2 ([1, 5, 27]) *Assume $s \in \mathbb{R}$, $1 \leq p, r, p_j, r_j \leq \infty$, $j = 1, 2$. Then*

1) *Embedding properties: $B^s_{p_1,r_1} \hookrightarrow B^{s-(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$ for $p_1 \leq p_2$, $r_1 \leq r_2$. $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact if $s_1 \leq s_2$.*

2) *Algebraic properties: For all $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. $B^s_{p,r}$ is an algebra $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{1}{p}$ or $s = \frac{1}{p}$, $r = 1$.*

3) *Morse type estimates:*

(i) *Let $s > 0$ and $f, g \in B^s_{p,r} \cap L^\infty$. Then there exists a positive constant C such that*

$$\|fg\|_{B^s_{p,r}} \leq C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big).$$

(ii) *For $s_1 \leq \frac{1}{p}$, $s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, then*

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

4) *Fatou's lemma: If a sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $f_n \to f$ in $S'(\mathbb{R})$, then it holds that $f \in B^s_{p,r}$ and*

$$\|f\|_{B^s_{p,r}} \leq \lim_{n\to\infty}\inf\|f_n\|_{B^s_{p,r}}.$$

5) *Multiplier properties: Let $m \in \mathbb{R}$. Assume that f is an S^m-multiplier (i.e., $f : \mathbb{R} \to \mathbb{R}$ is smooth and it satisfies that, for all $\alpha \in \mathbb{N}$, there exists a positive constant C_α such that $|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}$). Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

6) *Density: C^∞_c is dense in $B^s_{p,r} \Leftrightarrow 1 \leq p, r < \infty$.*

We present two lemmas which are related to the transport equation

$$\begin{cases} f_t + d\partial_x f = F, \\ f|_{t=0} = f_0, \end{cases} \tag{2.1}$$

where $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ represents a given time-dependent scalar function, $f_0 : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the known data.

Lemma 2.3 ([1]) *Assume* $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$, $p' = \frac{p}{p-1}$. *Suppose that* $s > -\min(\frac{1}{p_1}, \frac{1}{p'})$ *or* $s > -1 - \min(\frac{1}{p_1}, \frac{1}{p'})$ *when* $\partial_x d = 0$. *Then there exists a constant* C_1 *depending only on* p, p_1, r, s *such that the following estimate holds*:

$$\begin{aligned} &\|f\|_{L_t^\infty([0,t];B^s_{p,r})} \\ &\quad \leq e^{C_1\int_0^t Z(\tau)\,d\tau}\left[\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-C_1\int_0^\tau Z(\xi)\,d\xi}\big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau\right], \end{aligned} \tag{2.2}$$

where

$$Z(t) = \begin{cases} \|\partial_x d(t)\|_{B^{\frac{1}{p_1}}_{p_1,\infty}\cap L^\infty}, & s < 1 + \frac{1}{p_1}, \\ \|\partial_x d(t)\|_{B^{s-1}_{p_1,r}}, & s > 1 + \frac{1}{p_1} \text{ or } s = 1 + \frac{1}{p_1}, r = 1. \end{cases}$$

If $f = d$, *then for all* $s > 0$ ($s > -1$ *if* $\partial_x d = 0$), (2.2) *holds with* $Z(t) = \|\partial_x d(t)\|_{L^\infty}$.

We present an existence result for the transport equation with initial value in the Besov space.

Lemma 2.4 ([1]) *Let* p, p_1, r, s *be in the statement of Lemma* 2.3 *and* $f_0 \in B^s_{p,r}$. $F \in L^1([0,T];B^s_{p,r})$, $d \in L^\rho([0,T];B^{-M}_{\infty,\infty})$ *is a time-dependent vector field for some* $\rho > 1$, $M > 0$ *such that if* $s < 1 + \frac{1}{p_1}$, *then* $\partial_x d \in L^1([0,T];B^{\frac{1}{p_1}}_{p_1,\infty} \cap L^\infty)$; *if* $s > 1 + \frac{1}{p_1}$ *or* $s = 1 + \frac{1}{p_1}$, $r = 1$, *then* $\partial_x d \in L^1([0,T];B^{s-1}_{p_1,r})$. *Therefore, problem* (2.1) *has a unique solution* $f \in L^\infty([0,T];B^s_{p,r}) \cap (\cap_{s'<s} C([0,T];B^{s'}_{p,1}))$ *and* (2.2) *holds true. If* $r < \infty$, *it holds that* $f \in C([0,T];B^s_{p,r})$.

Lemma 2.5 ([19]) *Let* $1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $s > \max(\frac{1}{2}, \frac{1}{p})$. $f_0 \in B^{s-1}_{p,r}$, $F \in L^1([0,t];B^{s-1}_{p,r})$, $d \in L^1([0,t];B^{s+1}_{p,r})$. *Then a solution* f *to problem* (2.1) *satisfies* $f \in L^\infty([0,T];B^{s-1}_{p,r})$ *and*

$$\begin{aligned} &\|f\|_{L_t^\infty([0,t];B^{s-1}_{p,r})} \\ &\quad \leq e^{C_1\int_0^t Z(\tau)\,d\tau}\left[\|f_0\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C_1\int_0^\tau Z(\xi)\,d\xi}\big\|F(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right], \end{aligned}$$

where $Z(t) = \int_0^t \|d(\tau)\|_{B^{s+1}_{p,r}}\,d\tau$, *the constant* C_1 *depends only on* s, p, *and* r.

2.2 Proof of Theorem 1.1

We show the framework of proof with $n_0 \in B^s_{p,r} (s > \max(\frac{1}{p}, \frac{1}{2}))$.

Step 1: Set $n^0 = 0$. The smooth functions $(n^i)_{i\in\mathbb{N}} \in C(\mathbb{R}^+;B^\infty_{p,r})$ solve the problem

$$\begin{cases} (\partial_t + (2w^i_x - 4w^i + \beta)\partial_x)n^{i+1} = G(t,x), \\ n^{i+1}(0,x) = n^{i+1}_0(x) = S_{i+1}n_0, \end{cases} \tag{2.3}$$

where

$$G(t,x) = 2\big(n^i\big)^2 + \big(8w^i_x - 4w^i\big)n^i + 2\big(w^i + w^i_x\big)^2 - \lambda n^i. \tag{2.4}$$

Let $S_{i+1}n_0 \in B^\infty_{p,r}$. In view of Lemma 2.4, we establish that $n^{i+1} \in C(\mathbb{R}^+;B^\infty_{p,r})$ to problem (2.3) is global with $i \in \mathbb{N}$.

Step 2: If $s > \max\{1+\frac{1}{p}, 1+\frac{1}{2}\}$ or $s = \max\{1+\frac{1}{p}, 1+\frac{1}{2}\}$, $r=1$, we have

$$\begin{aligned} Z(t) &= \int_0^t \left\| \partial_x \left[2\left(w_x^i - 2w^i + \frac{1}{2}\beta \right) \right](\tau) \right\|_{B^{s-1}_{p,r}} d\tau \\ &= \int_0^t \left\| \partial_x \left[2\left(w_x^i - 2w^i \right) \right](\tau) \right\|_{B^{s-1}_{p,r}} d\tau \\ &\le C_0 \int_0^t \left\| \left(w_x^i - 2w^i \right)(\tau) \right\|_{B^{s}_{p,r}} d\tau \\ &\le C_0 \int_0^t \left(1 + \lambda + \left\| n^i(\tau) \right\|_{B^{s}_{p,r}} \right) d\tau. \end{aligned}$$

Using Lemma 2.3, we arrive at

$$\begin{aligned} \left\| n^{i+1}(t) \right\|_{B^s_{p,r}} \le{}& e^{C_1 \int_0^t \| \partial_x 2(w_x^i - 2w^i + \frac{1}{2}\beta)(\tau) \|_{B^{s-1}_{p,r}} d\tau} \times \Bigg[\|n_0\|_{B^s_{p,r}} \\ &+ \int_0^t e^{-C_1 \int_0^\tau \| \partial_x 2(w_x^i - 2w^i + \frac{1}{2}\beta)(\tau) \|_{B^{s-1}_{p,r}} d\xi} \left\| G(\tau,\cdot) \right\|_{B^s_{p,r}} d\tau \Bigg]. \end{aligned} \tag{2.5}$$

Let $a \lesssim b$ mean $a \le Cb$ for a certain constant $C > 0$. Bearing in mind the embedding property $B^s_{p,r} \hookrightarrow L^\infty (s > \max(\frac{1}{p}, \frac{1}{2}))$, the algebra property in the Besov space and the Morse type estimate (i) in Lemma 2.2 (see [5] for more details), we acquire

$$\begin{aligned} &\left\| 2\left(n^i\right)^2 \right\|_{B^s_{p,r}} \lesssim \left\| n^i \right\|_{L^\infty} \left\| n^i \right\|_{B^s_{p,r}} \lesssim \left\| n^i \right\|^2_{B^s_{p,r}}, \\ &\left\| \left(8w_x^i - 4w^i \right) n^i \right\|_{B^s_{p,r}} \\ &\quad \lesssim \left\| 8w_x^i - 4w^i \right\|_{B^s_{p,r}} \left\| n^i \right\|_{B^s_{p,r}} + \left\| 8w_x^i - 4w^i \right\|_{B^s_{p,r}} \left\| n^i \right\|_{B^s_{p,r}} \\ &\quad \lesssim \left\| n^i \right\|^2_{B^s_{p,r}}, \\ &\left\| 2\left(w^i + w_x^i \right)^2 \right\|_{B^s_{p,r}} \lesssim \left\| w^i + w_x^i \right\|^2_{B^s_{p,r}} \lesssim \left\| n^i \right\|^2_{B^s_{p,r}}, \\ &\left\| \lambda n^i \right\|_{B^s_{p,r}} \lesssim \lambda \left\| n^i \right\|_{B^s_{p,r}}. \end{aligned}$$

Thus, we obtain

$$\left\| G(t) \right\|_{B^s_{p,r}} \lesssim \left\| n^i \right\|_{B^s_{p,r}} \left(1 + \lambda + \left\| n^i(t) \right\|_{B^s_{p,r}} \right). \tag{2.6}$$

It is worth noticing that

$$\left\| \partial_x \left(w_x^i - 2w^i \right)(\tau) \right\|_{B^{s-1}_{p,r}} \lesssim 1 + \lambda + \left\| n^i(\tau) \right\|_{B^s_{p,r}}. \tag{2.7}$$

Combining (2.5) with (2.7), we deduce

$$\begin{aligned} \left\| n^{i+1}(t) \right\|_{B^s_{p,r}} \lesssim{}& e^{C_1 \int_0^t \| \partial_x (w_x^i - 2w^i)(\tau) \|_{B^{s-1}_{p,r}} d\tau} \|n_0\|_{B^s_{p,r}} \\ &+ \int_0^t e^{C_1 \int_\tau^t \| \partial_x (w_x^i - 2w^i)(\xi) \|_{B^{s-1}_{p,r}} d\xi} \left\| G(\tau,\cdot) \right\|_{B^s_{p,r}} d\tau \end{aligned}$$

$$\lesssim e^{C_2\int_0^t(1+\lambda+\|n^i(\tau)\|_{B^s_{p,r}})\,d\tau}\|n_0\|_{B^s_{p,r}} + \int_0^t e^{C_2\int_\tau^t(1+\lambda+\|n^i(\xi)\|_{B^s_{p,r}})\,d\xi}\big\|G(\tau,\cdot)\big\|_{B^s_{p,r}}\,d\tau. \tag{2.8}$$

Plugging (2.6) into (2.8) leads to the inequality

$$\begin{aligned}\big\|n^{i+1}(t)\big\|_{B^s_{p,r}} \le C_2\cdot e^{C_2\int_0^t(1+\lambda+\|n^i(\tau)\|_{B^s_{p,r}})\,d\tau}\bigg[&\|n_0\|_{B^s_{p,r}} \\ &+ \int_0^t e^{-C_2\int_0^\tau(1+\lambda+\|n^i(\xi)\|_{B^s_{p,r}})\,d\xi} \\ &\times\big(1+\lambda+\big\|n^i(\tau)\big\|_{B^s_{p,r}}\big)\big\|n^i(\tau)\big\|_{B^s_{p,r}}\,d\tau\bigg].\end{aligned} \tag{2.9}$$

If $\max\{\frac{1}{p},\frac{1}{2}\} < s < \max\{1+\frac{1}{p},1+\frac{1}{2}\}$, applying the embedding property $B^s_{p,r}\hookrightarrow L^\infty$, we have

$$\begin{aligned}Z(t) &= \int_0^t\bigg\|\partial_x\bigg[2\bigg(w^i_x - 2w^i + \frac{1}{2}\beta\bigg)\bigg](\tau)\bigg\|_{B^{\frac{1}{p}}_{p,\infty}\cap L^\infty}\,d\tau \\ &\lesssim \int_0^t\big\|\big[\partial_x\big(w^i_x-2w^i\big)\big](\tau)\big\|_{B^s_{p,r}}\,d\tau \\ &\lesssim \int_0^t\big\|\big(w^i_x-2w^i\big)(\tau)\big\|_{B^{s+1}_{p,r}}\,d\tau \lesssim \int_0^t\big(1+\lambda+\big\|n^i(\tau)\big\|_{B^s_{p,r}}\big)\,d\tau.\end{aligned}$$

Similarly, we deduce that (2.9) holds true in this case.

Therefore, one can choose certain $T>0$ to satisfy $2C_2^2(1+\lambda+\|n_0\|_{B^s_{p,r}})T<1$ and

$$1+\lambda+\big\|n^i(t)\big\|_{B^s_{p,r}} \le \frac{C_2(1+\lambda+\|n_0\|_{B^s_{p,r}})}{1-2C_2^2(1+\lambda+\|n_0\|_{B^s_{p,r}})t}, \tag{2.10}$$

which combined with (2.9) results in

$$1+\lambda+\big\|n^{i+1}(t)\big\|_{B^s_{p,r}} \le \frac{C_2(1+\lambda+\|n_0\|_{B^s_{p,r}})}{1-2C_2^2(1+\lambda+\|n_0\|_{B^s_{p,r}})t}.$$

We achieve that $(n^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Step 3: Utilizing problem (2.3) gives rise to

$$\begin{aligned}&\big(\partial_t + \big(2w^{i+j}_x - 4w^{i+j}+\beta\big)\partial_x\big)\big(n^{i+j+1}-n^{i+1}\big) \\ &\quad= -\big[2\big(w^{i+j}_x-w^i_x\big)-4\big(w^{i+j}-w^i\big)\big]n^{i+1}_x \\ &\qquad+ 2\big(n^{i+j}+n^i\big)\big(n^{i+j}-n^i\big) + \big(8w^{i+j}_x-4w^{i+j}\big)\big(n^{i+j}-n^i\big) \\ &\qquad+ \big(8\big(w^{i+j}_x-w^i_x\big)-4\big(w^{i+j}-w^i\big)\big)n^i \\ &\qquad+ 2\big(w^{i+j}+v^{i+j}_x+w^i+w^i_x\big)\big(w^{i+j}-w^i+w^{i+j}_x-w^i_x\big) \\ &\qquad- \lambda\big(n^{i+j}-n^i\big).\end{aligned} \tag{2.11}$$

Thanks to Lemma 2.5, we acquire

$$\begin{aligned}
&\left\|n^{i+j+1}-n^{i+1}\right\|_{B^{s-1}_{p,r}}\\
&\le e^{C\int_0^t \|n^{i+j}\|_{B^s_{p,r}}\,d\tau}\Bigg[\left\|n_0^{i+j+1}-n_0^{i+1}\right\|_{B^{s-1}_{p,r}}\\
&\quad + C\int_0^t e^{-C\int_0^\tau \|n^{i+j}\|_{B^s_{p,r}}\,d\xi}\left\|n^{i+j}-n^{i}\right\|_{B^{s-1}_{p,r}}\\
&\quad \times\left(1+\lambda+\left\|n^{i}\right\|_{B^s_{p,r}}+\left\|n^{i+j}\right\|_{B^s_{p,r}}+\left\|n^{i+1}\right\|_{B^s_{p,r}}\right)d\tau\Bigg].
\end{aligned}$$

Since

$$n_0^{i+j+1}-n_0^{i+1}=\sum_{q=i+1}^{i+j}\Delta_q w_0,$$

we can choose a constant $C_1>0$ to satisfy

$$\left\|n^{i+j+1}-n^{i+1}\right\|_{L^\infty([0,T];B^{s-1}_{p,r})}\le C_1 2^{-i}.$$

As a consequence, we derive that $(n^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$.

Step 4: Existence of solutions.

Using the Fatou property in Lemma 2.2 yields that $n\in L^\infty([0,T];B^s_{p,r})$. It is worth noticing that $(n^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$ which converges to a limit function $n\in C([0,T];B^{s-1}_{p,r})$. Making use of an interpolation argument yields that the convergence holds in $C([0,T];B^{s'}_{p,r})$ for all $s'<s$. Sending $i\to\infty$ in (2.3) yields that n is a solution to (2.3). Then the right-hand side of the first equation in (2.3) belongs to $L^\infty([0,T];B^s_{p,r})$. In the case $r<\infty$, taking advantage of Lemma 2.4 gives rise to $n\in C([0,T];B^{s'}_{p,r})$ for all $s'<s$.

Applying (1.5) yields that $n_t\in C([0,T];B^{s-1}_{p,r})$ if $r<\infty$, and $n_t\in L^\infty([0,T];B^{s-1}_{p,r})$ otherwise. Thus, $n\in E^s_{p,r}(T)$. Employing a sequence of viscosity approximate solutions $(n_\varepsilon)_{\varepsilon>0}$ to problem (1.5) which converges uniformly in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$, we achieve the continuity of solution $n\in E^s_{p,r}(T)$.

Step 5: Uniqueness and continuity with respect to initial data.

We assume that n^1 and n^2 are two given solutions to problem (1.5) with initial values $n_0^1,n_0^2\in B^s_{p,r}$. $n^1,n^2\in L^\infty([0,T];B^s_{p,r})\cap C([0,T];B^{s-1}_{p,r})$ and $n^{12}=n^1-n^2$. Then it holds that

$$\begin{cases}(\partial_t+(2w^1_x-4w^1+\beta)\partial_x)n^{12}=-(2w^{12}_x-4w^{12})n^1_x+G_1(t,x),\\ n^{12}(0,x)=n^{12}_0=n^1_0-n^2_0,\end{cases}\tag{2.12}$$

where

$$\begin{aligned}G_1(t,x)&=2\left(n^1+n^2\right)n^{12}+\left(8w^1_x-4w^1\right)n^{12}+\left(8w^{12}_x-4w^{12}\right)n^2\\
&\quad+2\left(w^1+w^1_x+w^2+w^2_x\right)\left(w^{12}+w^{12}_x\right)-\lambda n^{12}.\end{aligned}$$

In view of Lemma 2.5, we deduce

$$\begin{aligned} &e^{-C\int_0^t \|2w_x^1-4w^1\|_{B^{s+1}_{p,r}}\,d\tau}\big\|n^{12}(t)\big\|_{B^{s-1}_{p,r}} \\ &\quad\le \big\|n_0^{12}\big\|_{B^{s-1}_{p,r}} + C\int_0^t e^{-C\int_0^\tau \|2w_x^1-4w^1\|_{B^{s+1}_{p,r}}\,d\xi} \\ &\qquad\times\big(\big\|-\big(2w_x^{12}-4w^{12}\big)n_x^1\big\|_{B^{s-1}_{p,r}} + \big\|G_1(\tau)\big\|_{B^{s-1}_{p,r}}\big)\,d\tau. \end{aligned} \tag{2.13}$$

Taking advantage of the Morse type estimates in Lemma 2.2 and applying $s > \max(\frac{1}{p}, \frac{1}{2})$, we have

$$\begin{aligned} \big\|-\big(2w_x^{12}-4w^{12}\big)n_x^1\big\|_{B^{s-1}_{p,r}} &\lesssim \big\|-\big(2w_x^{12}-4w^{12}\big)\big\|_{B^{s}_{p,r}}\big\|n_x^1\big\|_{B^{s-1}_{p,r}} \\ &\lesssim \big\|n^{12}\big\|_{B^{s-1}_{p,r}}\big\|n^1\big\|_{B^{s}_{p,r}}. \end{aligned}$$

Similarly, we acquire

$$\big\|G_1(t)\big\|_{B^{s-1}_{p,r}} \lesssim \big\|n^{12}\big\|_{B^{s-1}_{p,r}}\big(1+\lambda+\big\|n^1\big\|_{B^{s}_{p,r}}+\big\|n^2\big\|_{B^{s}_{p,r}}\big).$$

Direct computation shows that

$$\begin{aligned} &e^{-C\int_0^t \|n^1\|_{B^{s}_{p,r}}\,d\tau}\big\|n^{12}\big\|_{B^{s-1}_{p,r}} \\ &\quad\le \big\|n_0^{12}\big\|_{B^{s-1}_{p,r}} + C\int_0^t e^{-C\int_0^\tau \|n^1\|_{B^{s}_{p,r}}\,d\xi}\big\|n^{12}\big\|_{B^{s-1}_{p,r}} \\ &\qquad\times\big(1+\lambda+\big\|n^1\big\|_{B^{s}_{p,r}}+\big\|n^2\big\|_{B^{s}_{p,r}}\big)\,d\tau. \end{aligned}$$

Making use of the Gronwall inequality yields

$$e^{-C\int_0^t \|n^1\|_{B^{s}_{p,r}}\,d\tau}\big\|n^{12}\big\|_{B^{s-1}_{p,r}} \le \big\|n_0^{12}\big\|_{B^{s-1}_{p,r}} e^{\int_0^t (1+\lambda+\|n^1\|_{B^{s}_{p,r}}+\|n^2\|_{B^{s}_{p,r}})\,d\tau}.$$

It follows that

$$\big\|n^{12}\big\|_{B^{s-1}_{p,r}} \le \big\|n_0^{12}\big\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|n^1\|_{B^{s}_{p,r}}\,d\tau} e^{\int_0^t (1+\lambda+\|n^1\|_{B^{s}_{p,r}}+\|n^2\|_{B^{s}_{p,r}})\,d\tau}. \tag{2.14}$$

From step 2 in this section, we observe that $\|n^1\|_{B^{s}_{p,r}}$ and $\|n^2\|_{B^{s}_{p,r}}$ are uniformly bounded for all $t \in (0, T]$.

Therefore, $e^{C\int_0^t \|n^1\|_{B^{s}_{p,r}}\,d\tau}$ and $e^{\int_0^t (1+\lambda+\|n^1\|_{B^{s}_{p,r}}+\|n^2\|_{B^{s}_{p,r}})\,d\tau}$ in (2.14) are bounded for all $t \in (0, T]$. In particular, if $n_0^1 = n_0^2$, we have $n_0^{12}(x) = n_0^1 - n_0^2 = 0$ for $x \in \mathbb{R}$. It is deduced from (2.14) that $\|n^{12}\|_{B^{s-1}_{p,r}} \le 0$ for all $t \in (0, T]$. It follows that $n^{12}(t,x) = n^1 - n^2 = 0$ for all $t \in (0, T]$, $x \in \mathbb{R}$.

Thus, we arrive at the desired results.

Remark 2.1 When $p = r = 2$, the Besov space $B^{s}_{p,r}(\mathbb{R})$ coincides with the Sobolev space $H^s(\mathbb{R})$. It is worth noticing that $(1-\partial_x^2)^{-1}$ is an S^{-2} multiplier. Then it holds that

$$\|w\|_{B^{s+2}_{p,r}} = \big\|\big(1-\partial_x^2\big)^{-1}\big(1-\partial_x^2\big)w\big\|_{B^{s+2}_{p,r}} \lesssim \big\|\big(1-\partial_x^2\big)^{-1}n\big\|_{B^{s+2}_{p,r}} \lesssim \|n\|_{B^{s}_{p,r}}.$$

Theorem 1.1 indicates that under the assumption $w_0 \in H^s(\mathbb{R})(s > \frac{5}{2})$, we establish the local well-posedness for problem (1.1) and the solution satisfies $w \in C([0,T];H^s(\mathbb{R})) \cap C^1([0,T];H^{s-1}(\mathbb{R}))$.

Remark 2.2 Let $1 \le p, r \le \infty$ and $w_0 \in B^s_{p,r}(\mathbb{R})(s > \max(\frac{5}{2}, 2 + \frac{1}{p}))$. Then a solution w to problem (1.1) satisfies the inequality

$$\|w(t)\|_{H^1} \le \|w_0\|_{H^1}, \quad t \in [0,T]. \tag{2.15}$$

3 Proofs of Theorems 1.2, 1.3, 1.4, and 1.5

We recall a lemma which is related to the commutator estimates.

Lemma 3.1 ([1]) *Assume $s > 0$, $1 \le p \le p_1 \le \infty$, $1 \le r \le \infty$, $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$. f and g are scalar functions on $\mathbb{R}$. Then*

$$\|[\Delta_j, f\partial_x]g\|_{B^s_{p,r}} \lesssim \|\partial_x f\|_{L^\infty}\|g\|_{B^s_{p,r}} + \|\partial_x f\|_{B^{s-1}_{p_1,r}}\|\partial_x g\|_{L^{p_2}}$$

and

$$\|[\Delta_j, f\partial_x]g\|_{B^s_{p,r}} \le C\|\partial_x f\|_{L^\infty}\|g\|_{B^s_{p,r}} \quad \textit{with } 0 < s < 1.$$

3.1 Proof of Theorem 1.2

Applying the operator Δ_q to problem (1.5) leads to

$$\big(\partial_t + (2w_x - 4w + \beta)\partial_x\big)\Delta_q n = [2w_x - 4w, \Delta_q]\partial_x n + \Delta_q G_2(t,x) - \lambda\Delta_q n, \tag{3.1}$$

where

$$G_2(t,x) = 2n^2 + (8w_x - 4w)n + 2(w + w_x)^2.$$

Utilizing $n_0 \in B^s_{p,r}(\mathbb{R})(\max(\frac{1}{2}, \frac{1}{p}) < s < 1)$ and Lemma 3.1, it yields

$$\begin{aligned}
&\big\|\big[\Delta_q, (2w_x - 4w)\partial_x\big]n\big\|_{B^s_{p,r}} \\
&\quad\lesssim \big\|\partial_x(2w_x - 4w)\big\|_{L^\infty}\|n\|_{B^s_{p,r}} \\
&\quad\lesssim \|n\|_{L^\infty}\|n\|_{B^s_{p,r}}
\end{aligned}$$

and

$$\begin{aligned}
\big\|G_2(t,x)\big\|_{B^s_{p,r}} &\lesssim \big\|2n^2 + (8w_x - 4w)n + 2(w + w_x)^2 - \lambda n\big\|_{B^s_{p,r}} \\
&\lesssim \|n\|_{L^\infty}\|n\|_{B^s_{p,r}} \\
&\quad + \|8w_x - 4w\|_{L^\infty}\|n\|_{B^s_{p,r}} + \|8w_x - 4w\|_{B^s_{p,r}}\|n\|_{L^\infty} \\
&\quad + \|w_x + w\|_{L^\infty}\|w_x + w\|_{B^s_{p,r}} \\
&\lesssim \|n\|_{L^\infty}\|n\|_{B^s_{p,r}}.
\end{aligned}$$

Multiplying (3.1) by $(\Delta_q n)^{p-1}$ and integrating on $\mathbb{R}$, we acquire

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\Delta_q n\|_{L^p}^p \lesssim{}& \big\|\partial_x(2w_x-4w+\beta)\big\|_{L^\infty}\|\Delta_q n\|_{L^p}^p\\ &+\big\|[2w_x-4w,\Delta_q]\partial_x n\big\|_{L^p}\|\Delta_q n\|_{L^p}^{p-1}\\ &+\big\|\Delta_q G_2(t,x)\big\|_{L^p}\|\Delta_q n\|_{L^p}^{p-1}-\lambda\|\Delta_q n\|_{L^p}^p.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta_q n\|_{L^p}\lesssim{}& \big\|\partial_x(2w_x-4w+\beta)\big\|_{L^\infty}\|\Delta_q n\|_{L^p}\\ &+\big\|[2w_x-4w,\Delta_q]\partial_x n\big\|_{L^p}+\big\|\Delta_q G_2(t,x)\big\|_{L^p}-\lambda\|\Delta_q n\|_{L^p}.\end{aligned}$$

Making use of Lemma 2.1 gives rise to

$$\big\|n(t)\big\|_{B^s_{p,r}}\lesssim\|n_0\|_{B^s_{p,r}}+\int_0^t\big(\big\|n(\tau)\big\|_{L^\infty}-\lambda\big)\big\|n(\tau)\big\|_{B^s_{p,r}}\,d\tau.$$

Applying the Gronwall inequality, we conclude

$$\big\|n(t)\big\|_{B^s_{p,r}}\lesssim\|n_0\|_{B^s_{p,r}}e^{\int_0^t(\|n(\tau)\|_{L^\infty}-\lambda)\,d\tau}. \tag{3.2}$$

Suppose that $T^*<\infty$ is the maximal existence time of solutions to problem (1.5). If

$$\int_0^t\big(\big\|n(\tau)\big\|_{L^\infty}-\lambda\big)\,d\tau<\infty, \tag{3.3}$$

we acquire that $\|n(T^*)\|_{B^s_{p,r}}$ is bounded in view of (3.2). The proof of Theorem 1.2 is completed.

3.2 Proof of Theorem 1.3

We illustrate the proof with density argument in the case $s=3$. Due to problem (1.5), we acquire the identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}n^2\,dx=\int_{\mathbb{R}}\big[(n+6w_x-3w)n^2+2(w+w_x)^2n-(\beta n_x+\lambda n)n\big]\,dx. \tag{3.4}$$

That is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n\|_{H^1}^2 ={}& \frac{1}{2}\frac{d}{dt}\big(\|n\|_{L^2}^2+\|n_x\|_{L^2}^2\big)\\ ={}&\int_{\mathbb{R}}\big[(n+6w_x-3w)n^2+2(w+w_x)^2n-(\beta n_x+\lambda n)n\big]\,dx\\ &+\int_{\mathbb{R}}\big[5(2w_x+n-w)n_x^2+4(w-2w_x)nn_x\big]\,dx\\ &-\int_{\mathbb{R}}8\big[(w+w_x)^2n-(w+uw_x)n^2\big]\,dx+\int_{\mathbb{R}}\big[-(\beta n_{xx}+\lambda n_x)n_x\big]\,dx\\ \lesssim{}&\|n+6w_x-3w\|_{L^\infty}\|n\|_{L^2}^2+\|w+w_x\|_{L^\infty}^2\|n\|_{L^1}\end{aligned}$$

$$
\begin{aligned}
&+\|2w_x+n-w\|_{L^\infty}\|n_x\|_{L^2}^2+\|w-2w_x\|_{L^\infty}\|n\|_{L^\infty}\|n_x\|_{L^1}\\
&+\|w+w_x\|_{L^\infty}^2\|n\|_{L^1}+\|w+w_x\|_{L^\infty}\|n\|_{L^2}^2-\lambda\|n\|_{H^1}^2\\
\lesssim{}&\big(\|n\|_{L^\infty}-\lambda\big)\|n\|_{H^1}^2.
\end{aligned}
\tag{3.5}
$$

Eventually, we deduce

$$
\big\|n(t)\big\|_{H^1}\lesssim\|n_0\|_{H^1}e^{\int_0^t(\|n(\tau)\|_{L^\infty}-\lambda)\,d\tau},
\tag{3.6}
$$

which yields a contradiction.

3.3 Proof of Theorem 1.4

Lemma 3.2 ([4]) *Let $T>0$, $w\in C^1([0,T);H^3(\mathbb{R}))$ and $n=(1-\partial_x^2)w$. Then, for all $t\in[0,T)$, there exists one point $\xi(t)\in\mathbb{R}$ such that*

$$
n_1(t)=\sup_{x\in\mathbb{R}}n(t,x)=n\big(t,\xi(t)\big)
\tag{3.7}
$$

and

$$
\frac{d}{dt}n_1(t)=n_{1,t}\big(t,\xi(t)\big),
$$

where $n_1(t)$ is absolutely continuous on $(0,T)$.

Consider the problem

$$
\begin{cases}
\frac{d}{dt}p(t,x)=(2w_x-4w)(t,p(t,x))+\beta,\\
p(0,x)=x.
\end{cases}
\tag{3.8}
$$

Lemma 3.3 ([3]) *Let $w\in C([0,T];H^s(\mathbb{R}))\cap C^1([0,T];H^{s-1}(\mathbb{R}))(s\geq 3)$, $n=w-w_{xx}$. Then problem (3.8) admits a unique solution $p(t,x)\in C^1([0,T]\times\mathbb{R},\mathbb{R})$. Moreover, the map $p(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ for all $t\in[0,T)$ and $p(t,x)$ satisfies the equality*

$$
p_x(t,x)=e^{\int_0^t(2w-2n-4w_x)(\tau,p(\tau,x))\,d\tau}.
\tag{3.9}
$$

Lemma 3.4 *Let $w_0\in H^s(\mathbb{R})(s\geq 3)$, $n_0=w_0-w_{0,xx}$, $(t,x)\in[0,T]\times\mathbb{R}$. Then*

$$
n\big(t,p(t,x)\big)p_x^2(t,x)\geq n_0(x)e^{\int_0^t(-2n(\tau,p(\tau,x))-\lambda)\,d\tau}.
\tag{3.10}
$$

Proof of Lemma 3.4 Utilizing (3.8) and Lemma 3.3 gives rise to

$$
\begin{aligned}
\frac{d}{dt}\big[n\big(t,p(t,x)\big)p_x^2(t,x)\big]&=(n_t+n_xp_t)p_x^2+2np_xp_{xt}\\
&=p_x^2\big[2(w+w_x)^2-2n^2\big]-\lambda np_x^2\\
&\geq(-2n-\lambda)np_x^2.
\end{aligned}
$$

Making use of the Gronwall inequality, we complete the proof of Lemma 3.4. □

Proof of Theorem 1.4 We present the proof by using Lemmas 3.2–3.4 with density argument in the case $s = 3$. Taking advantage of the assumption $n_0(x) > 0$ and Lemma 3.4 yields $n(t,x) > 0$. In view of $w(t,x) = g * n$ and $g(x) = \frac{1}{2}e^{-|x|}$, it satisfies

$$w(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|} n(t,\xi)\,d\xi \geq 0.$$

It follows that

$$w(t,x) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} n(t,\xi)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi} n(t,\xi)\,d\xi \tag{3.11}$$

and

$$w_x(t,x) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi} n(t,\xi)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{+\infty} e^{-\xi} n(t,\xi)\,d\xi. \tag{3.12}$$

Thus we conclude $|w_x| \leq w$ and

$$n_t + (2w_x - 4w + \beta)n_x \geq n^2 - \lambda n - 18\|w_0\|_{H^1}^2, \tag{3.13}$$

where we have used Remark 2.2 and

$$(4w_x - 2w)^2 \leq 36w^2 \leq 36\left(\frac{1}{\sqrt{2}}\|w\|_{H^1}\right)^2 \leq 18\|w_0\|_{H^1}^2.$$

Set $n_1(t) = \sup_{x\in\mathbb{R}}[n(t,x)]$. Applying Lemma 3.2, we deduce that there exists $\xi(t), t \in [0,T)$ such that

$$n_1(t) = \sup_{x\in\mathbb{R}} n(t,x) = n\bigl(t,\xi(t)\bigr).$$

Thus, we come to $n_x(t,\xi(t)) = 0$.

We recall that $p(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for all $t \in [0,T)$. There exists $x_1(t) \in \mathbb{R}$ such that $p(t,x_1(t)) = \xi(t)$. From (3.13), we acquire

$$\frac{d}{dt}n_1(t) \geq n_1^2 - \lambda n_1 - 18\|w_0\|_{H^1}^2. \tag{3.14}$$

Setting

$$n_2(t) = -\left[n_1(t) - \frac{\lambda}{2}\right] \quad \text{and} \quad K = \frac{\lambda^2}{4} + 18\|w_0\|_{H^1}^2, \tag{3.15}$$

we have

$$\frac{d}{dt}\bigl[n_2(t)\bigr] \leq -\bigl[n_2(t)\bigr]^2 + K. \tag{3.16}$$

Then $n_2(t)$ is strictly decreasing on $[0,T)$.

Recalling the condition $n_0(x_0) > \frac{\lambda}{2} + \sqrt{K}$ with x_0 defined by $n(x_0) = \sup_{x\in\mathbb{R}} n_0(x)$ in Theorem 1.4 and letting $\xi(0) = x_0$, we deduce $n_2(0) = -(n_1(0) - \frac{\lambda}{2}) = -(n_0(\xi(0)) - \frac{\lambda}{2}) = -(n_0(x_0) - \frac{\lambda}{2}) < -\sqrt{K}$. We choose $\delta \in (0,1)$ to satisfy $-\sqrt{\delta}n_2(0) = \sqrt{K}$.

Utilizing (3.16), we observe

$$\frac{d}{dt}\left(\frac{1}{n_2(t)}\right) = -\frac{1}{n_2^2(t)}\frac{dn_2(t)}{dt} \geq 1-\delta. \tag{3.17}$$

That is,

$$-\frac{1}{n_2(t)} + \frac{1}{n_2(0)} \leq -(1-\delta)t. \tag{3.18}$$

Bearing in mind $n_2(t) < 0$, $t \in [0,T]$, we come to the estimate $T \leq \frac{-1}{(1-\delta)n_2(0)} < \infty$, where $n_2(0) = -(n_0(x_0) - \frac{\lambda}{2}) < 0$. It turns out that

$$\begin{aligned} -\left[n\big(t,\xi(t)\big) - \frac{\lambda}{2}\right] &\leq \frac{n_0(x_0) - \frac{\lambda}{2}}{-1 + t(1-\delta)(n_0(x_0) - \frac{\lambda}{2})} \to -\infty \\ \text{as } t &\to \frac{1}{(1-\delta)(n_0(x_0) - \frac{\lambda}{2})}. \end{aligned} \tag{3.19}$$

The proof of Theorem 1.4 is finished. □

3.4 Proof of Theorem 1.5

Differentiating the first equation in (1.4) with x, we acquire

$$\begin{aligned} \partial_t w_x + (2w_x - 4w + \beta)w_{xx} = 2w_x^2 - 6w^2 + P_1(D)\big[w_x^2\big] \\ + P_2(D)\big[2w_x^2 + 6w^2\big] - \lambda w_x. \end{aligned} \tag{3.20}$$

Making use of Remark 2.2 leads to

$$\begin{aligned} \left|\frac{d}{dt}w\big(t,p(t,x)\big)\right| &= \big|w_t + (2w_x - 4w + \beta)w_x\big| \\ &\lesssim \|w_0\|_{H^1}^2 + \|w_0\|_{H^1} \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \left|\frac{d}{dt}w_x\big(t,p(t,x)\big)\right| &= \big|2w_x^2 - 6w^2 + P_1(D)\big[w_x^2\big] + P_2(D)\big[2w_x^2 + 6w^2\big] - \lambda w_x\big| \\ &\lesssim \|w_0\|_{H^1}^2 + \|w_0\|_{H^1}. \end{aligned} \tag{3.22}$$

Eventually, we come to the identity

$$\frac{d}{dt}n\big(t,p(t,x)\big) = 2(n + 2w_x - w)^2 + 2(w + w_x)^2 - 2(2w_x - w)^2 - \lambda n. \tag{3.23}$$

That is,

$$\begin{aligned} \frac{d}{dt}\left[n + 2w_x - w - \frac{\lambda}{4}\right]\big(t,p(t,x)\big) \geq 2\left[n + 2w_x - w - \frac{\lambda}{4}\right]^2\big(t,p(t,x)\big) \\ - \big[C_4\|w_0\|_{H^1}^2 + C_5\|w_0\|_{H^1} + C_6\big], \end{aligned}$$

where we use the inequality

$$\left|-2(w+w_x)^2+2(2w_x-w)^2-2\lambda w_x+\lambda w+\frac{1}{8}\lambda^2\right|$$
$$\leq C_4\|w_0\|_{H^1}^2+C_5\|w_0\|_{H^1}+C_6. \tag{3.24}$$

Setting

$$\begin{aligned} n_3(t,x) &= -\left[2\left(n+2w_x-w-\frac{\lambda}{4}\right)(t,p(t,x))\right], \\ K_1 &= 2\big(C_4\|w_0\|_{H^1}^2+C_5\|w_0\|_{H^1}+C_6\big) \end{aligned} \tag{3.25}$$

gives rise to

$$\frac{dn_3(t)}{dt}\leq -n_3^2(t)+K_1. \tag{3.26}$$

Let $\varepsilon\in(0,\frac{1}{2})$. Similar to the proof of Theorem 1.4, we choose certain $t_0\in(0,T)$ to satisfy $n_3(t_0)<-\sqrt{K_1+\frac{K_1}{\varepsilon}}$. Utilizing (3.26) gives rise to

$$n_3(t)<-\sqrt{K_1+\frac{K_1}{\varepsilon}}<-\sqrt{\frac{K_1}{\varepsilon}}.$$

We check

$$1-\varepsilon\leq\frac{d}{dt}\left(\frac{1}{n_3(t)}\right)\leq 1+\varepsilon. \tag{3.27}$$

Applying $\lim_{t\to T^-}n_3(t)=-\infty$, $|w_x|\leq|w|\lesssim\|v_0\|_{H^1}$ and (3.24), we conclude

$$\lim_{t\to T^-}\left[\sup_{x\in\mathbb{R}}(2w_x-w)(T-t)\right]=0.$$

Thus, we have

$$\lim_{t\to T^-}\left[\sup_{x\in\mathbb{R}}\left(n(t,x)-\frac{\lambda}{4}\right)(T-t)\right]=\frac{1}{2}, \tag{3.28}$$

which finishes the proof of Theorem 1.5.

4 Proof of Theorem 1.6

Setting $M=\sup_{t\in[0,T]}\|v(t)\|_{H^s}>0$, $s>\frac{5}{2}$, we acquire $\|v_{xx}(t)\|_{L^\infty}\leq\|v(t)\|_{H^s}\leq M$. The function

$$\varphi_N(x)=\begin{cases}1, & x\leq 0,\\ e^{\theta x}, & x\in(0,N),\\ e^{\theta N}, & x\geq N\end{cases}$$

satisfies $0 \le (\varphi_N(x))_x \le \varphi_N(x)$, where $N \in \mathbb{N}^*$, $\theta \in (0,1)$. There exists a constant $M_0 = M_0(\theta) > 0$ such that

$$\varphi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_N(y)} \, dy \le M_0.$$

The first equation in (1.1) is written as

$$w_t + (-4w + \beta) w_x = \partial_x g * \left[2w_x^2 + 6w^2 + \partial_x\left(w_x^2\right)\right] - \lambda w. \tag{4.1}$$

Then we acquire

$$\begin{aligned}
&\frac{1}{2n} \frac{d}{dt} \|w\varphi_N\|_{L^{2n}}^{2n} \\
&\quad = 4 \int_{\mathbb{R}} |w\varphi_N|^{2n} w_x \, dx - \beta \int_{\mathbb{R}} \left[\partial_x(w\varphi_N) - w(\varphi_N)_x\right] (w\varphi_N)^{2n-1} \, dx \\
&\qquad + \int_{\mathbb{R}} (w\varphi_N)^{2n-1} \varphi_N \partial_x g * \left[2w_x^2 + 6w^2 + \partial_x\left(w_x^2\right)\right] dx - \lambda \int_{\mathbb{R}} (w\varphi_N)^{2n} \, dx \\
&\quad \le 4\|w_x\|_{L^\infty} \|w\varphi_N\|_{L^{2n}}^{2n} + \beta \|w\varphi_N\|_{L^{2n}}^{2n} - \lambda \|w\varphi_N\|_{L^{2n}}^{2n} \\
&\qquad + \left\|\varphi_N \partial_x g * \left[2w_x^2 + 6w^2 + \partial_x\left(w_x^2\right)\right]\right\|_{L^{2n}} \|w\varphi_N\|_{L^{2n}}^{2n-1}.
\end{aligned} \tag{4.2}$$

Utilizing the Gronwall inequality and sending $n \to \infty$ in (4.2), we obtain

$$\begin{aligned}
\|w\varphi_N\|_{L^\infty} \le e^{(4M+\beta-\lambda)t} \Bigg[& \|w_0\varphi_N\|_{L^\infty} \\
& + \int_0^t \left\|\varphi_N \partial_x g * \left[2w_x^2 + 6w^2 + \partial_x\left(w_x^2\right)\right]\right\|_{L^\infty} d\tau \Bigg].
\end{aligned} \tag{4.3}$$

Direct computation gives rise to

$$\|w\varphi_N\|_{L^\infty} \le e^{(4M+\beta-\lambda)t} \left[\|w_0\varphi_N\|_{L^\infty} + 6M_0 M \int_0^t \left(\|w\varphi_N\|_{L^\infty} + \|w_x\varphi_N\|_{L^\infty}\right) d\tau \right]. \tag{4.4}$$

We arrive at

$$\begin{aligned}
\|w_x\varphi_N\|_{L^\infty} \le e^{(6M+\beta-\lambda)t} \Bigg[& \|w_{0,x}\varphi_N\|_{L^\infty} + (4M + 6M_0 M) \int_0^t \|w\varphi_N\|_{L^\infty} \, d\tau \\
& + \frac{5}{2} M_0 M \int_0^t \|w_x\varphi_N\|_{L^\infty} \, d\tau \Bigg]
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\|w_{xx}\varphi_N\|_{L^\infty} \le e^{(16M+\beta-\lambda)t} \Bigg[& \|w_{0,x}\varphi_N\|_{L^\infty} + 3M_0 M \int_0^t \|w\varphi_N\|_{L^\infty} \, d\tau \\
& + (12M + M_0 M) \int_0^t \|w_x\varphi_N\|_{L^\infty} \, d\tau \\
& + M_0 M \int_0^t \|w_{xx}\varphi_N\|_{L^\infty} \, d\tau \Bigg].
\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5) with (4.6), we achieve

$$\begin{aligned}
&\|w\varphi_N\|_{L^\infty} + \|w_x\varphi_N\|_{L^\infty} + \|w_{xx}\varphi_N\|_{L^\infty} \\
&\quad \le C_4\big(\|w_0\varphi_N\|_{L^\infty} + \|w_{0,x}\varphi_N\|_{L^\infty} + \|w_{0,xx}\varphi_N\|_{L^\infty}\big) \\
&\qquad + C_4\int_0^t \big(\|w\varphi_N\|_{L^\infty} + \|w_x\varphi_N\|_{L^\infty} + \|w_{xx}\varphi_N\|_{L^\infty}\big)\,d\tau,
\end{aligned} \tag{4.7}$$

which leads to the estimate

$$\begin{aligned}
&\sup_{t\in[0,T]}\big(\|e^{\theta x}w\|_{L^\infty} + \|e^{\theta x}w_x\|_{L^\infty} + \|e^{\theta x}w_{xx}\|_{L^\infty}\big) \\
&\quad \lesssim \|e^{\theta x}w_0\|_{L^\infty} + \|e^{\theta x}w_{0,x}\|_{L^\infty} + \|e^{\theta x}w_{0,xx}\|_{L^\infty}.
\end{aligned}$$

Thus, we acquire

$$|w|, |\partial_x w|, |\partial_x^2 w| \sim O(e^{-\theta x}) \quad \text{as } x\to\infty$$

uniformly on $[0,T]$.

5 Proof of Theorem 1.7

Let $s>0$. We give a scale of Banach spaces

$$E_s = \left\{ w\in C^\infty(\mathbb{R}) \mid |||w|||_s = \sup_{k\in\mathbb{N}^*} \frac{s^k\|\partial_x^k w\|_{H^2}}{k!(k+1)^{-2}} < +\infty \right\}.$$

Here, we denote $|||\cdot|||_{E_s}$ by $|||\cdot|||_s$ for simplicity. E_s is continuously embedded in $E_{s'}$ with $0<s'<s$ and $|||w|||_{s'} \le |||w|||_s$. A function w in E_s is a real analytic function on $\mathbb{R}$.

We present several related lemmas.

Lemma 5.1 ([12]) *Assume $s>0$. Then, for all $u,v\in E_s$, it holds that*

$$|||uv|||_s \le C|||u|||_s|||v|||_s,$$

where $C>0$ is independent of s.

Lemma 5.2 ([12]) *There exists a positive constant C, for all $0<s'<s\le 1$, such that*

$$\begin{aligned}
&|||\partial_x u|||_{s'} \le \frac{C}{s-s'}|||u|||_s, \\
&|||P_1(D)u|||_{s'} \le |||u|||_s, \qquad |||P_2(D)u|||_{s'} \le |||u|||_s.
\end{aligned}$$

Lemma 5.3 ([12]) *Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces. $X_s \hookrightarrow X_{s'}$ for all $s'<s$. T, R, and C are positive constants. Consider the Cauchy problem*

$$\frac{du}{dt} = F(t,u(t)), \quad u(0)=0. \tag{5.1}$$

$F(t,u)$ satisfies the following conditions:

(1) *Let* $0 < s' < s < 1$. $u(t)$ *is holomorphic for* $|t| < T$ *and continuous on* $|t| < T$ *with values in* X_s. $u(t)$ *satisfies* $\sup_{|t|<T} |||u(t)|||_s < R$. *Then* $t \to F(t, u(t))$ *is holomorphic on* $|t| < T$ *with values in* $X_{s'}$.

(2) *For* $0 < s' < s \le 1$ *and* $u, v \in X_s$ *with* $|||u|||_s < R$ *and* $|||v|||_s < R$, *it holds that*

$$\sup_{|t|\le T} \big|\big|\big|F(t,u) - F(t,v)\big|\big|\big|_{s'} \le \frac{C}{s-s'} |||u - v|||_s.$$

(3) *Let* $T_0 \in (0, T)$. *There exists* $M > 0$, *for all* $0 < s < 1$, *such that*

$$\sup_{|t|<T} \big|\big|\big|F(t,0)\big|\big|\big|_s < \frac{M}{1-s}.$$

Then problem (5.1) *admits a unique solution* $u(t)$ *which is holomorphic for* $|t| < (1-s)T_0$ *with values in* X_s *for all* $s \in (0,1)$.

Let $u_1 = w$, $u_2 = w_x$. The pair (u_1, u_2) satisfies the problem

$$\begin{cases} u_{1,t} = 4u_1u_2 - u_2^2 + F_1(u_1,u_2), \\ u_{2,t} = 4\partial_x(u_1u_2) - \partial_x(u_2^2) + F_2(u_1,u_2), \\ u_1(0,x) = u_{10}(x) = w_0(x), \\ u_2(0,x) = u_{20}(x) = w_{0,x}(x), \end{cases} \tag{5.2}$$

where

$$F_1(u_1,u_2) = P_1(D)\big[2u_2^2 + 6u_1^2\big] + P_2(D)\big[u_2^2\big] - \lambda u_1 - \beta u_2,$$
$$F_2(u_1,u_2) = \partial_x P_1(D)\big[2u_2^2 + 6u_1^2\big] + \partial_x P_2(D)\big[u_2^2\big] - \lambda\partial_x(u_1) - \beta\partial_x(u_2).$$

Proof of Theorem 1.7 We acquire that $F_1(u_1,u_2)$ and $F_2(u_1,u_2)$ do not depend on t explicitly. We only need to verify conditions (1) and (2) in Lemma 5.3 for $F_1(u_1,u_2)$ and $F_2(u_1,u_2)$. Making use of Lemmas 5.1 and 5.2 gives rise to

$$\begin{aligned} \big|\big|\big|F_1(u_1,u_2)\big|\big|\big|_{s'} &\le C|||u_1|||_s|||u_2|||_s + |||u_2|||_s^2 + \frac{C}{s-s'}\big(2|||u_2|||_s^2 + 6|||u_1|||_s^2\big) \\ &\quad + \lambda|||u_1|||_s + \beta|||u_2|||_s, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \big|\big|\big|F_2(u_1,u_2)\big|\big|\big|_{s'} &\le \frac{C}{s-s'}|||u_1|||_s|||u_2|||_s + \frac{C}{s-s'}|||u_2|||_s^2 \\ &\quad + \frac{C}{s-s'}\big(2|||u_2|||_s^2 + 6|||u_1|||_s^2\big) + \frac{C}{s-s'}\lambda|||u_1|||_s + \frac{C}{s-s'}\beta|||u_2|||_s, \end{aligned} \tag{5.4}$$

where C is a positive constant. Then condition (1) in Lemma 5.3 holds.

In order to verify condition (2) in Lemma 5.3, we obtain

$$\begin{aligned} &\big|\big|\big|F_1(u_1,u_2) - F_1(\bar{u}_1,\bar{u}_2)\big|\big|\big|_{s'} \\ &\quad \le \big|\big|\big|F_1(u_1,u_2) - F_1(\bar{u}_1,u_2)\big|\big|\big|_{s'} + \big|\big|\big|F_1(\bar{u}_1,u_2) - F_1(\bar{u}_1,\bar{u}_2)\big|\big|\big|_{s'}, \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&\left|\left|\left|F_2(u_1,u_2)-F_2(\bar{u}_1,\bar{u}_2)\right|\right|\right|_{s'} \\
&\quad\leq \left|\left|\left|F_2(u_1,u_2)-F_2(\bar{u}_1,u_2)\right|\right|\right|_{s'}+\left|\left|\left|F_2(\bar{u}_1,u_2)-F_2(\bar{u}_1,\bar{u}_2)\right|\right|\right|_{s'}.
\end{aligned}
\tag{5.6}
$$

Taking advantage of Lemmas 5.1, 5.2 and the assumptions $|||u_1|||_s \leq |||u_{10}|||_s + R$ and $|||u_2|||_s \leq |||u_{20}|||_s + R$ yields

$$
\begin{aligned}
&\left|\left|\left|F_1(u_1,u_2)-F_1(\bar{u}_1,u_2)\right|\right|\right|_{s'} \\
&\quad\leq C|||u_1-\bar{u}_1|||_s|||u_2|||_s+\frac{C}{s-s'}\left|\left|\left|u_1^2-\bar{u}_1^2\right|\right|\right|_s+\lambda|||u_1-\bar{u}_1|||_s \\
&\quad\leq C(|||u_{20}|||_s+R)|||u_1-\bar{u}_1|||_s+C(|||u_{10}|||_s+R+\lambda)|||u_1-\bar{u}_1|||_s,
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
&\left|\left|\left|F_1(\bar{u}_1,u_2)-F_1(\bar{u}_1,\bar{u}_2)\right|\right|\right|_{s'} \\
&\quad\leq |||\bar{u}_1|||_s|||u_2-\bar{u}_2|||_s+\left|\left|\left|u_2^2-\bar{u}_2^2\right|\right|\right|_s+\frac{C}{s-s'}\left|\left|\left|u_2^2-\bar{u}_2^2\right|\right|\right|_s+\beta|||u_2-\bar{u}_2|||_s \\
&\quad\leq C(|||u_{10}|||_s+R)|||u_1-\bar{u}_1|||_s+C(|||u_{20}|||_s+R+\beta)|||u_1-\bar{u}_1|||_s,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
&\left|\left|\left|F_2(u_1,u_2)-F_2(\bar{u}_1,u_2)\right|\right|\right|_{s'} \\
&\quad\leq \frac{C}{s-s'}|||u_1-\bar{u}_1|||_s|||u_2|||_s+\frac{C}{s-s'}\left|\left|\left|u_1^2-\bar{u}_1^2\right|\right|\right|_s+\frac{C}{s-s'}\lambda|||u_1-\bar{u}_1|||_s \\
&\quad\leq C(|||u_{20}|||_s+R)|||u_1-\bar{u}_1|||_s+C(|||u_{10}|||_s+R+\lambda)|||u_1-\bar{u}_1|||_s,
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
&\left|\left|\left|F_2(\bar{u}_1,u_2)-F_2(\bar{u}_1,\bar{u}_2)\right|\right|\right|_{s'} \\
&\quad\leq \frac{C}{s-s'}|||\bar{u}_1|||_s|||u_2-\bar{u}_2|||_s+\frac{C}{s-s'}\left|\left|\left|u_2^2-\bar{u}_2^2\right|\right|\right|_s \\
&\qquad+\frac{C}{s-s'}\left|\left|\left|u_1^2-\bar{u}_1^2\right|\right|\right|_s+\frac{C}{s-s'}\beta|||u_2-\bar{u}_2|||_s \\
&\quad\leq C(|||u_{10}|||_s+R)|||u_2-\bar{u}_2|||_s+(|||u_{20}|||_s+R+\beta)|||u_2-\bar{u}_2|||_s.
\end{aligned}
\tag{5.10}
$$

From (5.5)–(5.10), we check that condition (2) in Lemma 5.3 holds. Replacing s' with s and s with 1 and applying condition (2) in Lemma 5.3 give rise to that condition (3) in Lemma 5.3 holds. This finishes the proof of Theorem 1.7. □

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