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Construction of invariant solutions and conservation laws to the $(2 + 1)$ -dimensional integrable coupling of the KdV equation

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Abstract

Under investigation in this paper is the $(2 + 1)$ -dimensional integrable coupling of the KdV equation which has applications in wave propagation on the surface of shallow water. Firstly, based on the Lie symmetry method, infinitesimal generators and an optimal system of the obtained symmetries are presented. At the same time, new analytical exact solutions are computed through the tanh method. In addition, based on Ibragimov's approach, conservation laws are established. In the end, the objective figures of the solutions of the coupling of the KdV equation are performed.

Keywords: The $(2 + 1)$ -dimensional integrable coupling of the KdV equation; Lie group of symmetry; Optimal system; Tanh method; Conservation laws

1 Introduction

It is well known that the Korteweg–de Vries (KdV) equation describes the propagation of long waves on the surface of water with a small amplitude and is widely used to explain many complex science phenomena [1, 2]. Various forms of the expansion for the KdV equation have been proposed because of its importance, such as the KdV-Burgers equation [3], the KdV-BBM equation [4], the Rosenau–KdV equation [5], the modified KdV equation [6], KdV-hierarchy [7], and the $(2 + 1)$ -dimensional KdV equation [8]. In this research article, we consider the following $(2 + 1)$ -dimensional integrable coupling of the KdV equation which has the bi-Hamiltonian structure for the $(2 + 1)$ -dimensional perturbation equations of the KdV hierarchy [9]:

$$\begin{cases} u_t = u_{xxx} + 6uu_x, \\ v_t = v_{xxx} + 3u_{xy} + 6(uv)_x + 6uv_y, \end{cases} \quad (1)$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ are the unknown real functions, the subscripts denote the partial derivatives, and the variable y is called a slow variable. Equation (1) plays an important role in many analyses of physical phenomena such as stratified internal waves and lattice dynamics [10, 11], and it has aroused worldwide interest. The $(2 + 1)$ -dimensional hereditary recursion operators were examined in [12], its integrability was verified by

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using Painlevé in [13], some traveling wave solutions were established in [14], the auto-Bäcklund transformation, doubly periodic solutions and new non-traveling wave solutions were analyzed in [15].

As is well known, some methods have been used to explore exact solutions for models of nonlinear partial differential equations (PDEs) [16–18], the Lie group method is considered to be one of the most important methods to study the properties of solutions of PDEs [19, 20]. The main idea of the symmetry method is to construct an invariance condition and obtain reductions to differential equations [21–23]. Once reduction equations have been given, one can get a large number of corresponding exact solutions. In order to obtain the classification of all reduction equations, we require an optimal system of the one-dimensional subalgebra of the Lie algebra constructed by the Lie group method [19]. Using a symmetry analysis, we will get an optimal system of (1), from which the fascinating special solutions are inferred. Another important area is the conservation laws of PDEs which have an important impact on constructing solutions of PDEs [24–26]. We will obtain conservation laws of Eq. (1) by using Ibragimov's approach [27].

The rest of this paper is organized as follows. In Sect. 2, symmetries of the $(2 + 1)$ -dimensional integrable coupling of the KdV equation are discussed; Sect. 3 considers the reduced equations by means of similar variables; in Sect. 4, some new explicit solutions are presented with the help of the tanh method, and some objective features of the solutions are presented; in Sect. 5, the nonlinearly self-adjointness of Eq. (1) is proved and its conservation laws are established by using Ibragimov's method. Finally, concluding remarks are given at the end of the paper.

2 Lie point symmetry

In this section, we apply Lie's theory of symmetries for Eq. (1), and get its infinitesimal generators, commutator of Lie algebra.

First, let us consider a Lie algebra of infinitesimal symmetries of Eq. (1) of the form

$$\begin{aligned} X = & \xi^1(x, y, t, u, v) \partial_x + \xi^2(x, y, t, u, v) \partial_y + \xi^3(x, y, t, u, v) \partial_t \\ & + \phi(x, y, t, u, v) \partial_u + \varphi(x, y, t, u, v) \partial_v. \end{aligned} \quad (2)$$

According to the invariance conditions for Eq. (1) with respect to the transformation (2), we have [19, 28]

$$\text{pr}^{(3)} X(\Delta_1)|_{\Delta_1=0} = 0, \quad \text{pr}^{(3)} X(\Delta_2)|_{\Delta_2=0} = 0,$$

where $\text{pr}^{(3)} X$ is the third-order prolongation of X [19, 28] and $\Delta_1 = u_t - u_{xxx} - 6uu_x$, $\Delta_2 = v_t - v_{xxx} - 3u_{xy} - 6(uv)_x - 6uv_y$, on this condition,

$$\begin{aligned} \text{pr}^{(3)} X = & X + \phi_x^{(1)} \frac{\partial}{\partial u_x} + \phi_y^{(1)} \frac{\partial}{\partial u_y} + \phi_t^{(1)} \frac{\partial}{\partial u_t} + \phi_{xxx}^{(3)} \frac{\partial}{\partial u_{xxx}} + \phi_{xy}^{(3)} \frac{\partial}{\partial u_{xy}} \\ & + \varphi_x^{(1)} \frac{\partial}{\partial v_x} + \varphi_t^{(1)} \frac{\partial}{\partial v_t} + \varphi_{xxx}^{(3)} \frac{\partial}{\partial v_{xxx}}, \end{aligned}$$

where

$$\phi_x^{(1)} = D_x \phi - u_x D_x \xi^1 - u_y D_x \xi^2 - u_t D_x \xi^3,$$

$$\begin{aligned}
\phi_y^{(1)} &= D_y \phi - u_x D_y \xi^1 - u_y D_y \xi^2 - u_t D_y \xi^3, \\
\phi_t^{(1)} &= D_t \phi - u_x D_t \xi^1 - u_y D_t \xi^2 - u_t D_t \xi^3, \\
\varphi_x^{(1)} &= D_x \varphi - v_x D_x \xi^1 - v_y D_x \xi^2 - v_t D_x \xi^3, \\
\varphi_t^{(1)} &= D_t \varphi - v_x D_t \xi^1 - v_y D_t \xi^2 - v_t D_t \xi^3, \\
\phi_{xxx}^{(3)} &= D_x^3 (\phi - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t) + \xi^1 u_{xxxx} + \xi^2 u_{xxxxy} + \xi^3 u_{xxxxt}, \\
\phi_{xxy}^{(3)} &= D_x^2 D_y (\phi - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t) + \xi^1 u_{xxxy} + \xi^2 u_{xxyy} + \xi^3 u_{xyxt}, \\
\varphi_{xxx}^{(3)} &= D_x^3 (\varphi - \xi^1 v_x - \xi^2 v_y - \xi^3 v_t) + \xi^1 v_{xxxx} + \xi^2 v_{xxxxy} + \xi^3 v_{xxxxt},
\end{aligned}$$

and D_x, D_y, D_t stand for the operators of the total differentiation, for instance,

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + u_{tt} \frac{\partial}{\partial u_t} + v_{ty} \frac{\partial}{\partial v_y} + \cdots.$$

Next, we get a system of over-determined linear equations of $\xi^1, \xi^2, \xi^3, \phi$ and φ ,

$$\begin{aligned}
\xi_x^1 &= \frac{1}{3} \xi_t^3, & \xi_y^1 &= \xi_t^1 = \xi_u^1 = \xi_v^1 = 0, \\
\xi_x^2 &= \xi_t^2 = \xi_u^2 = \xi_v^2 = 0, \\
\xi_x^3 &= \xi_y^3 = \xi_u^3 = \xi_v^3 = 0, & \xi_{tt}^3 &= 0, \\
\phi &= -\frac{2}{3} u \xi_t^3, & \varphi &= -\frac{1}{3} v (\xi_t^3 + 3 \xi_y^2).
\end{aligned}$$

Solving these equations, one can get

$$\begin{aligned}
\xi^1 &= \frac{1}{3} c_1 x + c_3, & \xi^2 &= F(y), & \xi^3 &= c_1 t + c_2, \\
\phi &= -\frac{2}{3} c_1 u, & \varphi &= -\frac{1}{3} v (c_1 + 3 F_y),
\end{aligned}$$

where c_1, c_2, c_3 are real constants, $F(y)$ is an arbitrary function. To obtain physically crucial solutions, we take $F_1(z) = c_4 y + c_5$, then on substituting the above obtaining

$$\begin{aligned}
\xi^1 &= \frac{1}{3} c_1 x + c_3, & \xi^2 &= c_4 y + c_5, & \xi^3 &= c_1 t + c_2, \\
\phi &= -\frac{2}{3} c_1 u, & \varphi &= -\frac{1}{3} v (c_1 + 3 c_4).
\end{aligned}$$

Therefore, the Lie algebra L_5 of the transformations of Eq. (1) is spanned by the following generators:

$$\begin{aligned}
X_1 &= \frac{1}{3} x \partial_x + t \partial_t - \frac{2}{3} u \partial_u - \frac{1}{3} v \partial_v, & X_2 &= \partial_t, \\
X_3 &= \partial_x, & X_4 &= y \partial_y - v \partial_v, & X_5 &= \partial_y.
\end{aligned}$$

In order to classify all the group-invariant solutions, we need an optimal system of one-dimensional subalgebras. In this section, the optimal system of subgroups for Eq. (1)

Table 1 Table of Lie brackets

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	$-\frac{1}{3}X_3$	0	0
X_2	X_2	0	0	0	0
X_3	$\frac{1}{3}X_3$	0	0	0	0
X_4	0	0	0	0	$-X_5$
X_5	0	0	0	X_5	0

is constructed by only using the commutator table [29]. First, using the commutator $[X_m, X_n] = X_m X_n - X_n X_m$, we attained the commutation relations of X_1, X_2, X_3, X_4, X_5 listed in Table 1.

An arbitrary operator $X \in L_5$ is given as

$$X = l_1 X_1 + l_2 X_2 + l_3 X_3 + l_4 X_4 + l_5 X_5.$$

To establish the linear transformations of the vector $l = (l_1, l_2, l_3, l_4, l_5)$, we denote

$$E_i = c_{ij}^k l_j \partial_{l_k}, \quad i = 1, 2, 3, 4, 5, \quad (3)$$

where c_{ij}^k is constructed by the formula $[X_i, X_j] = c_{ij}^k X_k$. Based on Eq. (3) and Table 1, E_1, E_2, E_3, E_4, E_5 can be written as

$$E_1 = -l_2 \partial_{l_2} - \frac{1}{3} l_3 \partial_{l_3},$$

$$E_2 = l_1 \partial_{l_2},$$

$$E_3 = \frac{1}{3} l_1 \partial_{l_3},$$

$$E_4 = -l_5 \partial_{l_5},$$

$$E_5 = l_4 \partial_{l_5}.$$

For E_1, E_2, E_3, E_4, E_5 , the Lie equations with parameters a_1, a_2, a_3, a_4, a_5 and the initial condition $\tilde{l}|_{a_i=0} = l, i = 1, 2, 3, 4, 5$ are given as

$$\begin{aligned} \frac{d\tilde{l}_1}{da_1} &= 0, & \frac{d\tilde{l}_2}{da_1} &= -\tilde{l}_2, & \frac{d\tilde{l}_3}{da_1} &= -\frac{1}{3}\tilde{l}_3, & \frac{d\tilde{l}_4}{da_1} &= 0, & \frac{d\tilde{l}_5}{da_1} &= 0, \\ \frac{d\tilde{l}_1}{da_2} &= 0, & \frac{d\tilde{l}_2}{da_2} &= \tilde{l}_1, & \frac{d\tilde{l}_3}{da_2} &= 0, & \frac{d\tilde{l}_4}{da_2} &= 0, & \frac{d\tilde{l}_5}{da_2} &= 0, \\ \frac{d\tilde{l}_1}{da_3} &= 0, & \frac{d\tilde{l}_2}{da_3} &= 0, & \frac{d\tilde{l}_3}{da_3} &= \frac{1}{3}\tilde{l}_1, & \frac{d\tilde{l}_4}{da_3} &= 0, & \frac{d\tilde{l}_5}{da_3} &= 0, \\ \frac{d\tilde{l}_1}{da_4} &= 0, & \frac{d\tilde{l}_2}{da_4} &= 0, & \frac{d\tilde{l}_3}{da_4} &= 0, & \frac{d\tilde{l}_4}{da_4} &= 0, & \frac{d\tilde{l}_5}{da_4} &= -\tilde{l}_5, \\ \frac{d\tilde{l}_1}{da_5} &= 0, & \frac{d\tilde{l}_2}{da_5} &= 0, & \frac{d\tilde{l}_3}{da_5} &= 0, & \frac{d\tilde{l}_4}{da_5} &= 0, & \frac{d\tilde{l}_5}{da_5} &= \tilde{l}_4. \end{aligned}$$

The solutions of the above equations are associated with the transformations

$$T_1: \quad \tilde{l}_1 = l_1, \quad \tilde{l}_2 = e^{-a_1} l_2, \quad \tilde{l}_3 = e^{-\frac{1}{3}a_1} l_3, \quad \tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,$$

$$\begin{aligned}
T_2: \quad \tilde{l}_1 &= l_1, & \tilde{l}_2 &= a_2 l_1 + l_2, & \tilde{l}_3 &= l_3, & \tilde{l}_4 &= l_4, & \tilde{l}_5 &= l_5, \\
T_3: \quad \tilde{l}_1 &= l_1, & \tilde{l}_2 &= l_2, & \tilde{l}_3 &= \frac{1}{3} a_3 l_1 + l_3, & \tilde{l}_4 &= l_4, & \tilde{l}_5 &= l_5, \\
T_4: \quad \tilde{l}_1 &= l_1, & \tilde{l}_2 &= l_2, & \tilde{l}_3 &= l_3, & \tilde{l}_4 &= l_4, & \tilde{l}_5 &= e^{-a_4} l_5, \\
T_5: \quad \tilde{l}_1 &= l_1, & \tilde{l}_2 &= l_2, & \tilde{l}_3 &= l_3, & \tilde{l}_4 &= l_4, & \tilde{l}_5 &= a_5 l_4 + l_5.
\end{aligned}$$

The establishment of the optimal system requires a simplification of the vector

$$l = (l_1, l_2, l_3, l_4, l_5), \quad (4)$$

by applying the transformations $T_1 - T_5$. Our task is to construct a simplest representative of each class of similar vectors (4). Two cases will be considered separately.

Case 2.1. $l_1 \neq 0$

By making $a_2 = -\frac{l_2}{l_1}$ and $a_3 = -\frac{3l_3}{l_1}$ in T_2 and T_3 , we enable $\tilde{l}_2, \tilde{l}_3 = 0$. The vector (4) becomes

$$(l_1, 0, 0, l_4, l_5). \quad (5)$$

2.1.1. $l_4 \neq 0$

By making $a_5 = -\frac{l_5}{l_4}$ in T_5 , we can enable $\tilde{l}_5 = 0$. The vector (5) is equivalent to

$$(l_1, 0, 0, l_4, 0).$$

We get the following representatives:

$$X_1 \pm X_4. \quad (6)$$

2.1.2. $l_4 = 0$

The vector (5) is equivalent to

$$(l_1, 0, 0, 0, l_5). \quad (7)$$

We get the following representatives:

$$X_1, \quad X_1 \pm X_5. \quad (8)$$

Case 2.2. $l_1 = 0$

The vector (4) becomes

$$(0, l_2, l_3, l_4, l_5). \quad (9)$$

2.2.1. $l_4 \neq 0$

By making $a_5 = -\frac{l_5}{l_4}$ in T_5 , we can enable $\tilde{l}_5 = 0$. The vector (9) is equivalent to

$$(0, l_2, l_3, l_4, 0).$$

Making all the possible combinations, we get the following representatives:

$$X_4, \quad X_2 \pm X_4, \quad X_3 \pm X_4, \quad X_2 \pm X_3 \pm X_4. \quad (10)$$

2.2.2. $l_4 = 0$

The vector (9) becomes

$$(0, l_2, l_3, 0, l_5).$$

We get the following representatives:

$$X_2, \quad X_3, \quad X_5, \quad X_2 \pm X_3, \quad X_2 \pm X_5, \quad X_3 \pm X_5, \quad X_2 \pm X_3 \pm X_5. \quad (11)$$

Finally, by gathering the operators (6, 8, 10 and 11), we obtain the following theorem.

Theorem 2.1 *An optimal system of $\{X_1, X_2, X_3, X_4, X_5\}$ is generated by*

$$\begin{aligned} &X_1 \pm X_4, \quad X_1, \quad X_1 \pm X_5, \quad X_4, \quad X_2 \pm X_4, \\ &X_3 \pm X_4, \quad X_2 \pm X_3 \pm X_4, \quad X_2, \quad X_3, \quad X_5, \\ &X_2 \pm X_3, \quad X_2 \pm X_5, \quad X_3 \pm X_5, \quad X_2 \pm X_3 \pm X_5. \end{aligned}$$

3 Similarity reductions of the (2 + 1)-dimensional integrable coupling of the KdV equation

In this section, based on Theorem 2.1, we will find some reduced equations of Eq. (1) by using similarity variables.

Case 3.1. Reduction by $X_2 + X_4$.

Integrating the characteristic equation for $X_2 + X_4$, we get the invariance

$$\tilde{x} = x, \quad \tilde{y} = t - \ln y, \quad \tilde{u} = u, \quad \tilde{v} = \nu y,$$

and the invariant solution takes the form $\tilde{u} = f(\tilde{x}, \tilde{t})$, $\tilde{v} = g(\tilde{x}, \tilde{t})$, that is, $u = f(\tilde{x}, \tilde{y})$, $\nu = \frac{g(\tilde{x}, \tilde{t})}{y}$, Eq. (1) can be reduced to

$$\begin{cases} f_{\tilde{y}} = f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}}, \\ g_{\tilde{y}} = g_{\tilde{x}\tilde{x}\tilde{x}} - 3f_{\tilde{x}\tilde{x}\tilde{y}} + 6f_{\tilde{x}}g + 6fg_{\tilde{x}} - 6ff_{\tilde{y}}. \end{cases} \quad (12)$$

Case 3.2. Reduction by X_2 .

Similarly, we have $\tilde{x} = x$, $\tilde{y} = y$, $u = f(\tilde{x}, \tilde{y})$, $\nu = g(\tilde{x}, \tilde{y})$. Equation (1) is reduced to

$$\begin{cases} f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}} = 0, \\ g_{\tilde{x}\tilde{x}\tilde{x}} + 3f_{\tilde{x}\tilde{x}\tilde{y}} + 6f_{\tilde{x}}g + 6fg_{\tilde{x}} + 6ff_{\tilde{y}} = 0. \end{cases} \quad (13)$$

Case 3.3. For the generator $X_2 + X_3$, we have $\tilde{y} = y$, $\tilde{t} = x - t$, $u = f(\tilde{y}, \tilde{t})$, $\nu = g(\tilde{y}, \tilde{t})$. Equation (1) is reduced to

$$\begin{cases} -f_{\tilde{t}} = f_{\tilde{t}\tilde{t}\tilde{t}} + 6ff_{\tilde{t}}, \\ -g_{\tilde{t}} = g_{\tilde{t}\tilde{t}\tilde{t}} + 3f_{\tilde{y}\tilde{t}\tilde{t}} + 6f_{\tilde{t}}g + 6fg_{\tilde{t}} + 6ff_{\tilde{y}}. \end{cases} \quad (14)$$

Case 3.4. For the generator $X_2 + X_5$, we have $\tilde{x} = x$, $\tilde{y} = y - t$, $u = f(\tilde{x}, \tilde{y})$, $v = g(\tilde{x}, \tilde{y})$. Equation (1) can be reduced to

$$\begin{cases} -f_{\tilde{y}} = f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}}, \\ -g_{\tilde{y}} = g_{\tilde{x}\tilde{x}\tilde{x}} + 3f_{\tilde{x}\tilde{x}\tilde{y}} + 6f_{\tilde{x}}g + 6fg_{\tilde{x}} + 6ff_{\tilde{y}}. \end{cases} \quad (15)$$

Case 3.5. For the generator $X_2 + X_3 + X_5$, we have $\tilde{x} = x - y$, $\tilde{t} = x - t$, $u = f(\tilde{x}, \tilde{t})$, $v = g(\tilde{x}, \tilde{t})$. Equation (1) becomes

$$\begin{cases} -f_{\tilde{t}} = f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}}, \\ -g_{\tilde{t}} = g_{\tilde{x}\tilde{x}\tilde{x}} - 3f_{\tilde{x}\tilde{x}\tilde{x}} + 6f_{\tilde{x}}g + 6fg_{\tilde{x}} - 6ff_{\tilde{x}}. \end{cases} \quad (16)$$

Case 3.6. For the generator X_4 , we have $\tilde{x} = x$, $\tilde{t} = t$, $u = f(\tilde{x}, \tilde{t})$, $v = \frac{g(\tilde{x}, \tilde{t})}{y}$. Equation (1) can be reduced to

$$\begin{cases} f_{\tilde{t}} = f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}}, \\ g_{\tilde{t}} = g_{\tilde{x}\tilde{x}\tilde{x}} + 6fg_{\tilde{x}} + 6f_{\tilde{x}}g. \end{cases} \quad (17)$$

Case 3.7. For the generator $X_3 + X_5$, we have $\tilde{x} = x - y$, $\tilde{t} = t$, $u = f(\tilde{x}, \tilde{t})$, $v = g(\tilde{x}, \tilde{t})$. Equation (1) becomes

$$\begin{cases} f_{\tilde{t}} = f_{\tilde{x}\tilde{x}\tilde{x}} + 6ff_{\tilde{x}}, \\ g_{\tilde{t}} = g_{\tilde{x}\tilde{x}\tilde{x}} + 6fg_{\tilde{x}} + 6f_{\tilde{x}}g. \end{cases} \quad (18)$$

Case 3.8. For the generator X_1 , we have $\tilde{y} = y$, $\tilde{t} = \frac{t}{x^3}$, $u = \frac{f(\tilde{y}, \tilde{t})}{x^2}$, $v = \frac{g(\tilde{y}, \tilde{t})}{x}$. Equation (1) can be reduced to

$$\begin{cases} f_{\tilde{t}} = -24f - 186\tilde{t}f_{\tilde{t}} - 162\tilde{t}^2f_{\tilde{t}\tilde{t}} - 27\tilde{t}^3f_{\tilde{t}\tilde{t}\tilde{t}} - 12f^2 - 18\tilde{t}ff_{\tilde{t}}, \\ g_{\tilde{t}} = -6g - 114\tilde{t}g_{\tilde{t}} - 135\tilde{t}^2g_{\tilde{t}\tilde{t}} - 27\tilde{t}^3g_{\tilde{t}\tilde{t}\tilde{t}} + 18f_{\tilde{y}} + 84\tilde{t}f_{\tilde{t}\tilde{y}}. \end{cases} \quad (19)$$

4 The exact solutions of reduced equations

In the previous section, we have dealt with the similarity reductions and derived the corresponding reduced equations. In this section, we use the tanh method on reduced equations, obtaining some exact solutions of Eq. (1). With the help of exact solutions, we can understand some motion rules of waves of the (2 + 1)-dimensional integrable coupling of KdV equation.

The main steps of the tanh method [24, 25] are expressed as follows:

1. Consider the following nonlinear differential equations:

$$\begin{cases} F_1(u, v, u_x, v_x, u_{xx}, v_{xx}, \dots, u_y, v_y, \dots, u_t, v_t, \dots) = 0, \\ F_2(u, v, u_x, v_x, u_{xx}, v_{xx}, \dots, u_y, v_y, \dots, u_t, v_t, \dots) = 0, \end{cases} \quad (20)$$

where F_1, F_2 are polynomials of the u, v and their derivatives.

2. By using the wave transformations

$$\begin{cases} u(x, y, t) = \Phi(\xi), \\ v(x, y, t) = \Psi(\xi), \end{cases} \quad (21)$$

where $\xi = lx + ky + ct$, and l, k, c are unknown constants, and substituting (21) into Eq. (20), we obtain the following nonlinear ordinary differential equations:

$$\begin{cases} F_1(\Phi, \Psi, l\Phi', l\Psi', l^2\Phi'', l^2\Psi'', \dots, k\Phi', k\Psi', \dots, c\Phi', c\Psi', \dots) = 0, \\ F_2(\Phi, \Psi, l\Phi', l\Psi', l^2\Phi'', l^2\Psi'', \dots, k\Phi', k\Psi', \dots, c\Phi', c\Psi', \dots) = 0. \end{cases} \quad (22)$$

3. Next, we introduce the independent variable

$$Y = \tanh(\xi), \quad (23)$$

which leads to the following changes:

$$\begin{aligned} \frac{d}{d\xi} &= (1 - Y^2) \frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= (1 - Y^2) \left[-2Y \frac{d}{dY} + (1 - Y^2) \frac{d^2}{dY^2} \right], \\ \frac{d^3}{d\xi^3} &= (1 - Y^2) \left[(6Y^2 - 2) \frac{d}{dY} - 6Y(1 - Y^2) \frac{d^2}{dY^2} + (1 - Y^2)^2 \frac{d^3}{dY^3} \right]. \end{aligned}$$

4. We assume that the solution of Eq. (22) is written as the following form:

$$\Phi(Y) = \sum_{i=0}^n a_i Y^i, \quad \Psi(Y) = \sum_{i=0}^m b_i Y^i, \quad (24)$$

where n, m are positive integers, which are decided by balancing the highest order nonlinear terms with the derivative terms in the resulting equations. After deciding n, m , taking (23) and (24) into (22), we obtain a polynomial concerning Y^i ($i = 0, 1, 2, \dots$). Then we gather all terms of Y^i ($i = 0, 1, 2, \dots$) and make all them equal to zero. Solving these algebraic equations, we get the values of the unknown numbers a_i, b_i ($i = 0, 1, \dots$), l, k and c . Then, putting these values into the equations, we get exact solutions of equations.

Case 4.1. For Eq. (12), substituting Eq. (21) into (12), we get the following equations:

$$\begin{cases} k\Phi' = l^3\Phi^{(3)} + 6l\Phi\Phi', \\ k\Psi' = l^3\Psi^{(3)} - 3l^2k\Phi^{(3)} + 6l\Phi'\Psi + 6l\Phi\Psi' - 6k\Phi\Phi'. \end{cases} \quad (25)$$

Concerning (25), balancing $\Phi^{(3)}$ with $\Phi\Phi'$, we have

$$2 \times 3 + n - 3 = n + 2 \times 1 + n - 1 \implies n = 2,$$

balancing $\Phi^{(3)}$ with $\Phi'\Psi$, we have

$$2 \times 3 + n - 3 = 2 \times 1 + n - 1 + m \implies m = 2.$$

Hence, according to Eq. (24), the solution of Eq. (12) is assumed to be

$$\begin{cases} \Phi(Y) = a_0 + a_1 Y + a_2 Y^2, \\ \Psi(Y) = b_0 + b_1 Y + b_2 Y^2. \end{cases} \quad (26)$$

Then, substituting Eq. (23) and Eq. (26) into Eq. (25), we collect all terms of Y^i and obtain the algebraic equations including unknown numbers a_i , b_i ($i = 0, 1, 2$), l and k . By solving these equations, we have the following solutions:

$$\begin{aligned} l &= l, & k &= k, & a_0 &= \frac{8l^3 + k}{6l}, & a_1 &= 0, & a_2 &= -2l^2, \\ b_0 &= \frac{k(-16l^3 + k)}{6l^2}, & b_1 &= 0, & b_2 &= 4lk. \end{aligned} \quad (27)$$

Putting (27) into Eq. (12), we obtain the exact solution as follows:

$$\begin{cases} u(x, y, t) = \frac{8l^3 + k}{6l} - 2l^2 \tanh^2(lx + k(t - \ln y)), \\ v(x, y, t) = \frac{\frac{k(-16l^3 + k)}{6l^2} + 4lk \tanh^2(lx + k(t - \ln y))}{y}, \end{cases} \quad (28)$$

where $l \neq 0$, k are arbitrary constants.

Figures 1 and 2 depict the exact solution of Eq. (12), which is obtained by taking $l = 1$, $k = 1$ at $t = 1$.

Case 4.2. For Eq. (13), similarly, substituting Eq. (21) into (13), we have the following ordinary differential equations:

$$\begin{cases} l^3 \Phi^{(3)} + 6l\Phi\Phi' = 0, \\ l^3 \Psi^{(3)} + 3l^2 k \Phi^{(3)} + 6l\Phi'\Psi + 6l\Phi\Psi' + 6k\Phi\Phi' = 0. \end{cases} \quad (29)$$

Then, balancing $\Phi^{(3)}$ and $\Phi\Phi'$, $\Phi^{(3)}$ and $\Phi'\Psi'$ for (29), we have $n = m = 2$.

Figure 1 $u(x, y, t)$ for $l = 1$, $k = 1$ at $t = 1$

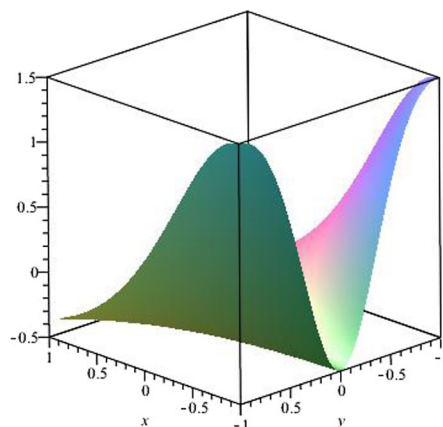
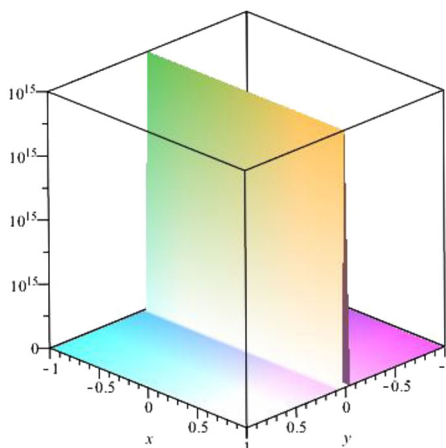


Figure 2 $v(x, y, t)$ for $l = 1, k = 1$ at $t = 1$ 

Therefore, on the basis of Eq. (24), the solution of Eq. (13) can be assumed to be

$$\begin{cases} \Phi(Y) = a_0 + a_1 Y + a_2 Y^2, \\ \Psi(Y) = b_0 + b_1 Y + b_2 Y^2. \end{cases} \quad (30)$$

Next, substituting Eq. (23) and Eq. (30) into Eq. (29), we make all coefficients of Y^i vanish and obtain the algebraic equations including the unknown numbers a_i , b_i ($i = 0, 1, 2$), l and k . Solving these equations, we have the following solutions:

$$\begin{aligned} l &= l, & k &= k, & a_0 &= \frac{4l^2}{3}, & a_1 &= 0, & a_2 &= -2l^2, \\ b_0 &= \frac{8lk}{3}, & b_1 &= 0, & b_2 &= -4lk. \end{aligned} \quad (31)$$

So, the exact solution of Eq. (13) is

$$\begin{cases} u(x, y, t) = \frac{4l^2}{3} - 2l^2 \tanh^2(lx + ky), \\ v(x, y, t) = \frac{8lk}{3} - 4lk \tanh^2(lx + ky), \end{cases} \quad (32)$$

where l, k are arbitrary constants. This solution is a static solution of Eq. (1).

When we take $l = 1, k = 1$, the values of u, v are illustrated in Figs. 3 and 4.

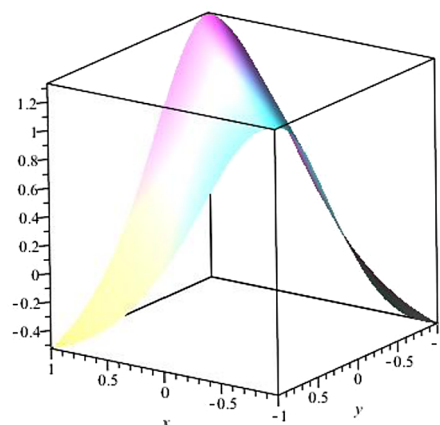
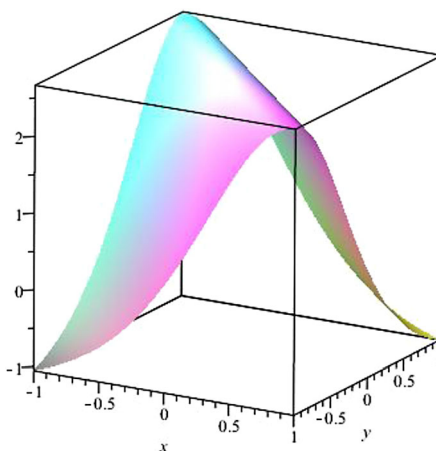
Case 4.3. For Eq. (14), equally, substituting Eq. (21) into (14), we get the following ordinary differential equations:

$$\begin{cases} -\Phi' = c^2 \Phi^{(3)} + 6\Phi\Phi', \\ -c\Psi' = c^3 \Psi^{(3)} + 3kc^2 \Phi^{(3)} + 6c\Phi'\Psi + 6c\Phi\Psi' + 6k\Phi\Phi'. \end{cases} \quad (33)$$

Furthermore, balancing $\Phi^{(3)}$ and $\Phi\Phi'$, $\Phi^{(3)}$ and $\Phi'\Psi'$ for (33), we have $n = m = 2$.

Therefore, based on Eq. (24), the solution of Eq. (14) can be assumed to be

$$\begin{cases} \Phi(Y) = a_0 + a_1 Y + a_2 Y^2, \\ \Psi(Y) = b_0 + b_1 Y + b_2 Y^2. \end{cases} \quad (34)$$

Figure 3 $u(x, y, t)$ for $l = 1, k = 1$ at $t = 1$ **Figure 4** $v(x, y, t)$ for $l = 1, k = 1$ at $t = 1$ 

Next, substituting Eq. (23) and Eq. (34) into Eq. (33), we make all coefficients of Y^i vanish and obtain the algebraic equations including unknown numbers a_i , b_i ($i = 0, 1, 2$), k and c . Solving these equations, we have the following solutions:

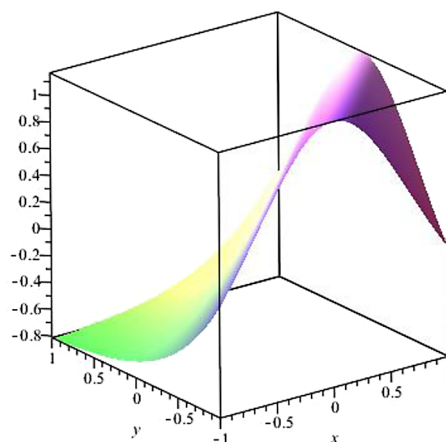
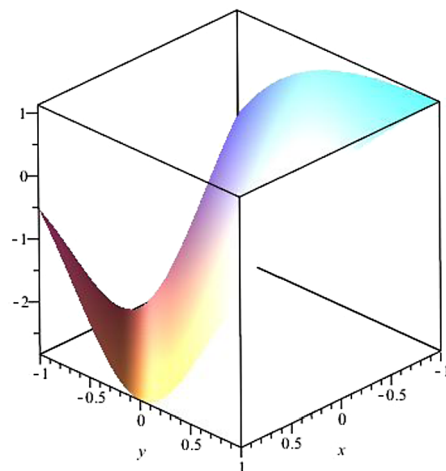
$$\begin{aligned} k &= k, & c &= c, & a_0 &= \frac{8c^2 - 1}{6}, & a_1 &= 0, & a_2 &= -2c^2, \\ b_0 &= \frac{k(16c^2 + 1)}{6c}, & b_1 &= 0, & b_2 &= -4kc. \end{aligned} \quad (35)$$

So, the exact solution of Eq. (14) is

$$\begin{cases} u(x, y, t) = \frac{8c^2 - 1}{6} - 2c^2 \tanh^2(ky + c(x - t)), \\ v(x, y, t) = \frac{k(16c^2 + 1)}{6c} - 4kc \tanh^2(ky + c(x - t)), \end{cases} \quad (36)$$

where $c \neq 0$ and k are arbitrary constants.

Figures 5 and 6 portray the solution of Eq. (14), which is obtained by taking $k = -1$, $c = 1$ at $t = 1$.

Figure 5 $u(x, y, t)$ for $k = -1, c = 1$ at $t = 1$ **Figure 6** $v(x, y, t)$ for $k = -1, c = 1$ at $t = 1$ 

Case 4.4. For Eq. (15), in the same way, substituting Eq. (21) into (15), we have the following ordinary differential equations:

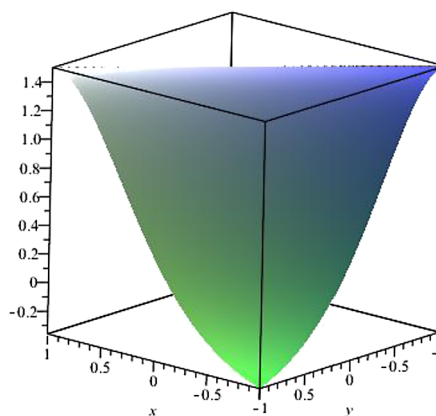
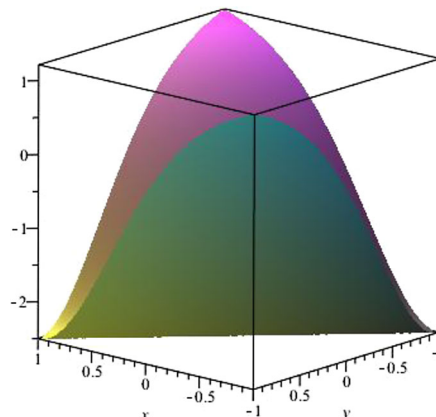
$$\begin{cases} -k\Phi' = l^3\Phi^{(3)} + 6l\Phi\Phi', \\ -k\Psi' = l^3\Psi^{(3)} + 3l^2k\Phi^{(3)} + 6l\Phi'\Psi + 6l\Phi\Psi' + 6k\Phi\Phi'. \end{cases} \quad (37)$$

Then, balancing $\Phi^{(3)}$ and $\Phi\Phi'$, $\Phi^{(3)}$ and $\Phi'\Psi'$ for (33), we have $n = m = 2$.

Therefore, based on Eq. (24), the solution of Eq. (15) can be assumed to be

$$\begin{cases} \Phi(Y) = a_0 + a_1Y + a_2Y^2, \\ \Psi(Y) = b_0 + b_1Y + b_2Y^2. \end{cases} \quad (38)$$

Next, substituting Eq. (23) and Eq. (38) into Eq. (37), we make all coefficients of Y^i vanish and obtain the algebraic equations including unknown numbers a_i, b_i ($i = 0, 1, 2$), l and k .

Figure 7 $u(x, y, t)$ for $l = 1, k = -1$ at $t = 0$ **Figure 8** $v(x, y, t)$ for $l = 1, k = -1$ at $t = 0$ 

Solving these equations, we have the following solutions:

$$\begin{aligned} l &= l, & k &= k, & a_0 &= \frac{8l^3 - k}{6l}, & a_1 &= 0, & a_2 &= -2l^2, \\ b_0 &= \frac{k(16l^3 + k)}{6l^2}, & b_1 &= 0, & b_2 &= -4lk. \end{aligned} \quad (39)$$

So, the exact solution of Eq. (15) is

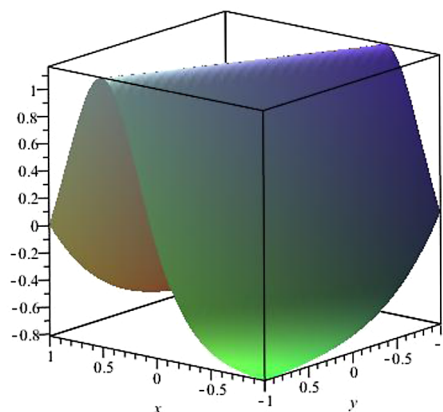
$$\begin{cases} u(x, y, t) = \frac{8l^3 - k}{6l} - 2l^2 \tanh^2(lx + k(y - t)), \\ v(x, y, t) = \frac{k(16l^3 + k)}{6l^2} - 4lk \tanh^2(lx + k(y - t)), \end{cases} \quad (40)$$

where $l \neq 0, k$ are arbitrary constants.

When we take $l = 1, k = -1$ at $t = 0$, the values of u, v are illustrated in Figs. 7 and 8.

Case 4.5. For Eq. (16), likewise, substituting Eq. (21) into (16), we get the following ordinary differential equations:

$$\begin{cases} -c\Phi' = l^3\Phi^{(3)} + 6l\Phi\Phi', \\ -c\Psi' = l^3\Psi^{(3)} - 3l^3\Phi^{(3)} + 6l\Phi'\Psi + 6l\Phi\Psi' - 6l\Phi\Phi'. \end{cases} \quad (41)$$

Figure 9 $u(x, y, t)$ for $l = 1, c = 1$ at $t = 0$ 

Then, balancing $\Phi^{(3)}$ and $\Phi\Phi'$, $\Phi^{(3)}$ and $\Phi'\Psi'$ for (41), we have $n = m = 2$.

Therefore, based on Eq. (24), the solution of Eq. (16) can be assumed to be

$$\begin{cases} \Phi(Y) = a_0 + a_1 Y + a_2 Y^2, \\ \Psi(Y) = b_0 + b_1 Y + b_2 Y^2. \end{cases} \quad (42)$$

Next, substituting Eq. (23) and Eq. (42) into Eq. (41), we make all coefficients of Y^i vanish and obtain the algebraic equations including unknown numbers a_i , b_i ($i = 0, 1, 2$), l and c . Solving these equations, we have the following solutions:

$$\begin{aligned} l &= l, & c &= c, & a_0 &= \frac{8l^3 - c}{6l}, & a_1 &= 0, & a_2 &= -2l^2, \\ b_0 &= -\frac{16l^3 + c}{6l}, & b_1 &= 0, & b_2 &= 4l^2. \end{aligned} \quad (43)$$

So, the exact solution of Eq. (16) is

$$\begin{cases} u(x, y, t) = \frac{8l^3 - c}{6l} - 2l^2 \tanh^2(l(x - y) + c(x - t)), \\ v(x, y, t) = -\frac{16l^3 + c}{6l} + 4l^2 \tanh^2(l(x - y) + c(x - t)), \end{cases} \quad (44)$$

where $l \neq 0$, c are arbitrary constants.

Figures 9 and 10 depict the exact solution of Eq. (16), which is obtained by taking $l = 1$, $c = 1$ at $t = 0$.

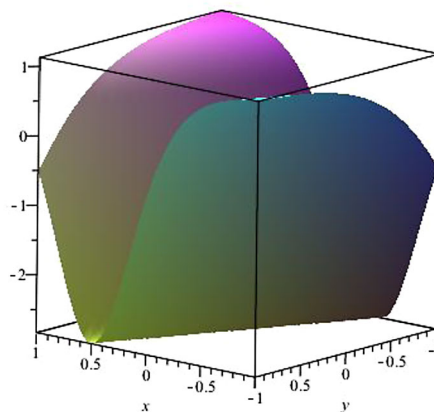
5 Construction of conservation laws

In this section, we chiefly construct conservation laws of Eq. (1) using Ibragimov's method [27, 30]. First, we prove that Eq. (1) is nonlinear self-adjoint.

5.1 Proof of nonlinear self-adjointness

With regard to Eq. (1), conservation laws multipliers have the following form:

$$\Lambda_1 = \Lambda_1(x, y, t, u, v), \quad \Lambda_2 = \Lambda_2(x, y, t, u, v).$$

Figure 10 $v(x, y, t)$ for $l = 1, c = 1$ at $t = 0$ 

Moreover,

$$\begin{cases} E_u[\Lambda_1(u_t - u_{xxx} - 6uu_x) + \Lambda_2(v_t - v_{xxx} - 3u_{xy} - 6(uv)_x - 6uu_y)] = 0, \\ E_v[\Lambda_1(u_t - u_{xxx} - 6uu_x) + \Lambda_2(v_t - v_{xxx} - 3u_{xy} - 6(uv)_x - 6uu_y)] = 0, \end{cases} \quad (45)$$

where the Euler operators E_u, E_v are expressed as

$$\begin{cases} E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} - D_t \frac{\partial}{\partial u_t} + D_x^2 \frac{\partial}{\partial u_{xx}} \cdots, \\ E_v = \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_y \frac{\partial}{\partial v_y} - D_t \frac{\partial}{\partial v_t} + D_x^2 \frac{\partial}{\partial v_{xx}} \cdots. \end{cases} \quad (46)$$

Substituting (46) into (45), we obtain the following system which only has the unknown variables Λ_1, Λ_2 :

$$\begin{aligned} \Lambda_{1v} &= 0, & \Lambda_{1xu} &= 0, & \Lambda_{1uu} &= 0, \\ \Lambda_{2u} &= 0, & \Lambda_{2v} &= 0, \\ -\Lambda_{1t} + 6u\Lambda_{1x} + 6v\Lambda_{2x} + 6u\Lambda_{2y} + \Lambda_{1xxx} + 3\Lambda_{2xy} &= 0, \\ -\Lambda_{2t} + 6u\Lambda_{2x} + \Lambda_{2xxx} &= 0. \end{aligned}$$

Solving this system, we have $\Lambda_1 = 6tuF_{1y} + (6tu + x)F_2(y) + uF_3(y) + F_4(y)$, $\Lambda_2 = F_1(y)$, where $F_1(y), F_2(y), F_3(y)$ and $F_4(y)$ are arbitrary functions.

Consider a PDE system of order m ,

$$\mathcal{R}^\alpha(x, u, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (47)$$

where $x = (x^1, x^2, \dots, x^n)$, $u = (u^1, u^2, \dots, u^m)$ and $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ represent the set of all first, second, ..., k th-order derivatives of u in regard to x .

The adjoint equations of Eq. (47) are written as

$$(\mathcal{R}^\alpha)^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad v = v(x).$$

Besides,

$$(\mathcal{R}^\alpha)^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha},$$

where \mathcal{L} is a formal Lagrangian of the following form:

$$\mathcal{L} = v^\beta \mathcal{R}^\beta(x, u, \dots, u_{(k)}), \quad \beta = 1, 2, \dots, m,$$

and the Euler–Lagrange operator is expressed as

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^{\infty} (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial u_{i_1 \dots i_j}^\alpha}, \quad \alpha = 1, 2, \dots, m.$$

Definition 5.1 ([31]) The system (47) is said to be nonlinearly self-adjoint if the adjoint system is satisfied for all the solutions u of system (47) upon a substitution $v = \varphi(x, u)$ such that $\varphi(x, u) \neq 0$. Particularly, the system

$$(\mathcal{R}^\alpha)^*(x, u, \varphi, \dots, u_{(k)}, \varphi_{(k)}) = 0, \quad \alpha = 1, \dots, m,$$

is identical to the system

$$\lambda_\alpha^\beta \mathcal{R}^\beta(x, u, u, \dots, u_{(k)}, u_{(k)}) = 0, \quad \beta = 1, \dots, m,$$

that is,

$$(\mathcal{R}^\alpha)^*|_{v=\varphi(x,u)} = \lambda_\alpha^\beta \mathcal{R}^\beta, \quad \beta = 1, \dots, m,$$

where λ_α^β is a certain function.

Theorem 5.1 ([32]) *The determining system of the multiplier $\Lambda(x, u)$ of system (47) is identical to the system of nonlinearly self-adjoint substitution.*

If the formal Lagrangian of Eq. (1) is given as

$$\begin{aligned} \mathcal{L} = & \varphi_1(x, y, t, u, v)(u_t - u_{xxx} - 6uu_x) \\ & + \varphi_2(x, y, t, u, v)(v_t - v_{xxx} - 3u_{xy} - 6(uv)_x - 6uu_y), \end{aligned}$$

based on Theorem 5.1, we can get

$$\begin{cases} \varphi_1(x, y, t, u, v) = \Lambda_1(x, y, t, u, v) = 6tuF_{1y} + (6tu + x)F_2(y) + uF_3(y) + F_4(y), \\ \varphi_2(x, y, t, u, v) = \Lambda_2(x, y, t, u, v) = F_1(y). \end{cases} \quad (48)$$

Therefore, Eq. (1) is nonlinearly self-adjoint with substitution (48).

5.2 Construction of conservation laws

Theorem 5.2 ([31]) *The system of differential Eq. (47) is nonlinearly self-adjoint. Then every Lie point, the Lie–Bäcklund, nonlocal symmetry*

$$X = \xi^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^\alpha},$$

admitted by the system of Eq. (47), gives rise to a conservation law, where the components C^i of the conserved vector $C = (C^1, \dots, C^n)$ are determined by

$$\begin{aligned} C^i = & W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\ & + D_j (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right], \end{aligned}$$

and $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$. The formal Lagrangian \mathcal{L} should be written in the symmetric form concerning all mixed derivatives $u_{ij}^\alpha, u_{ijk}^\alpha, \dots$

The Lagrangian \mathcal{L} of Eq. (1) is given as follows:

$$\mathcal{L} = \Lambda_1(u_t - u_{xxx} - 6uu_x) + \Lambda_2(v_t - v_{xxx} - 3u_{xxy} - 6(uv)_x - 6uu_y).$$

For the generator $X = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_t + \phi \partial_u + \varphi \partial_v$, in line with the Theorem 5.2, we obtain $W^1 = \phi - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t$, $W^2 = \varphi - \xi^1 v_x - \xi^2 v_y - \xi^3 v_t$, so the components of the conservation vector have the following form:

$$\begin{aligned} C^x = & W^1 \left[\frac{\partial \mathcal{L}}{\partial u_x} + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xxx}} \right) + D_x D_y \left(\frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) \right] \\ & + D_x (W^1) \left[-D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xxx}} \right) - D_y \left(\frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) \right] \\ & + D_x^2 (W^1) \frac{\partial \mathcal{L}}{\partial u_{xxx}} + D_x D_y (W^1) \frac{\partial \mathcal{L}}{\partial u_{xxy}} + W^2 \left[\frac{\partial \mathcal{L}}{\partial v_x} + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial v_{xxx}} \right) \right] \\ & + D_x (W^2) \left[-D_x \left(\frac{\partial \mathcal{L}}{\partial v_{xxx}} \right) \right] + D_x^2 (W^2) \frac{\partial \mathcal{L}}{\partial v_{xxx}}, \\ C^y = & W^1 \left[\frac{\partial \mathcal{L}}{\partial u_y} + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) \right] + D_x (W^1) \left[-D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xxy}} \right) \right] + D_x^2 (W^1) \frac{\partial \mathcal{L}}{\partial u_{xxy}}, \\ C^t = & W^1 \frac{\partial \mathcal{L}}{\partial u_t} + W^2 \frac{\partial \mathcal{L}}{\partial v_t}. \end{aligned}$$

By substituting the Lagrangian \mathcal{L} into above components of the conservation vector, C^x, C^y, C^t are simplified as

$$\begin{aligned} C^x = & -W^1 [6(u\Lambda_1 + v\Lambda_2) + D_x^2(\Lambda_1) + 3D_x D_y(\Lambda_2)] \\ & + D_x(W^1) [D_x(\Lambda_1) + 3D_y(\Lambda_2)] - D_x^2(W^1) \Lambda_1 - 3D_x D_y(W^1) \Lambda_2 \\ & - W^2 [6u\Lambda_2 + D_x^2(\Lambda_2)] + D_x(W^2) D_x(\Lambda_2) - D_x^2(W^2) \Lambda_2, \end{aligned} \quad (49)$$

$$C^y = -W^1 [6u\Lambda_2 + 3D_x^2(\Lambda_2)] + 3D_x(W^1) D_x(\Lambda_2) - 3D_x^2(W^1) \Lambda_2, \quad (50)$$

$$C^t = W^1 \Lambda_1 + W^2 \Lambda_2. \quad (51)$$

For the generator $X_1 = \frac{1}{3}x\partial_x + t\partial_t - \frac{2}{3}u\partial_u - \frac{1}{3}v\partial_v$, we have $W^1 = -\frac{2}{3}u - \frac{1}{3}xu_x - tu_t$, $W^2 = -\frac{1}{3}v - \frac{1}{3}xv_x - tv_t$. According to Eqs. (49)–(51), the components of the conserved vector of generator X_1 have the following form:

$$\begin{aligned} C_1^x &= \left(\frac{2}{3}u + \frac{1}{3}xu_x + tu_t \right) \{ 6[u(6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)) + vF_1(y)] \\ &\quad + u_{xx}[6t(F_1)_y + 6tF_2(y) + F_3(y)] \} \\ &\quad - \left(u_x + \frac{1}{3}xu_{xx} + tu_{xt} \right) \{ F_2(y) + u_x[6t(F_1)_y + 6tF_2(y) + F_3(y)] \\ &\quad + 3(F_1)_y \} + \left(\frac{4}{3}u_{xx} + \frac{1}{3}xu_{xxx} + tu_{xxt} \right) [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] \\ &\quad + (3u_{xy} + xu_{xy} + 3tu_{xyt})F_1(y) + (2v + 2xv_x + 6tv_t)uF_1(y) \\ &\quad + \left(v_{xx} + \frac{1}{3}xv_{xxx} + tv_{xxt} \right) F_1(y), \\ C_1^y &= (4u + 2xu_x + 6tu_t)uF_1(y) + (4u_{xx} + xu_{xxx} + 3tu_{xxt})F_1(y), \\ C_1^t &= \left(-\frac{2}{3}u - \frac{1}{3}xu_x - tu_t \right) [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] \\ &\quad - \left(\frac{1}{3}v + \frac{1}{3}xv_x - tv_t \right) F_1(y). \end{aligned}$$

For the generator $X_2 = \partial_t$, we have $W^1 = -u_t$, $W^2 = -v_t$. According to Eqs. (49)–(51), the components of the conserved vector of generator X_2 can be expressed as follows:

$$\begin{aligned} C_2^x &= u_t \{ 6[u(6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)) + vF_1(y)] \\ &\quad + u_{xx}[6t(F_1)_y + 6tF_2(y) + F_3(y)] \} \\ &\quad - u_{xt} \{ F_2(y) + u_x[6t(F_1)_y + 6tF_2(y) + F_3(y)] + 3(F_1)_y \} \\ &\quad + u_{xxt} [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] \\ &\quad + 3u_{xyt}F_1(y) + 6uv_tF_1(y) + v_{xxt}F_1(y), \\ C_2^y &= 6uu_tF_1(y) + 3u_{xxt}F_1(y), \\ C_2^t &= -u_t [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] - v_tF_1(y). \end{aligned}$$

For the generator $X_3 = \partial_x$, we have $W^1 = -u_x$, $W^2 = -v_x$. According to Eqs. (49)–(51), the components of the conserved vector of generator X_3 can be written the following form:

$$\begin{aligned} C_3^x &= u_x \{ 6[u(6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)) + vF_1(y)] \\ &\quad + u_{xx}[6t(F_1)_y + 6tF_2(y) + F_3(y)] \} \\ &\quad - u_{xx} \{ F_2(y) + u_x[6t(F_1)_y + 6tF_2(y) + F_3(y)] + 3(F_1)_y \} \\ &\quad + u_{xxx} [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] \\ &\quad + 3u_{xy}F_1(y) + 6uv_xF_1(y) + v_{xxx}F_1(y), \end{aligned}$$

$$\begin{aligned}C_3^y &= 6uu_x F_1(y) + 3u_{xxx} F_1(y), \\C_3^t &= -u_x [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] - v_x F_1(y).\end{aligned}$$

For the generator $X_4 = y\partial_y - v\partial_v$, we have $W^1 = -yu_y$, $W^2 = -v - yv_y$. According to Eqs. (49)–(51), the components of the conserved vector of generator X_4 are given as

$$\begin{aligned}C_4^x &= yu_y \{6[u(6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)) + vF_1(y)] \\&\quad + u_{xx} [6t(F_1)_y + 6tF_2(y) + F_3(y)]\} \\&\quad - yu_{xy} \{F_2(y) + u_x [6t(F_1)_y + 6tF_2(y) + F_3(y)] + 3(F_1)_y\} \\&\quad + yu_{xxy} [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] + 3(u_{xy} + yu_{xyy})F_1(y) \\&\quad + 6u(v + yv_y)F_1(y) + (v_{xx} + yv_{xxy})F_1(y), \\C_4^y &= 6yuu_y F_1(y) + 3yu_{xxy} F_1(y), \\C_4^t &= -yu_y [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] - (v + yv_y)F_1(y).\end{aligned}$$

For the generator $X_5 = \partial_y$, we have $W^1 = -u_y$, $W^2 = -v_y$. According to Eqs. (49)–(51), the components of the conserved vector of the generator X_5 can be expressed as

$$\begin{aligned}C_5^x &= u_y \{6[u(6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)) + vF_1(y)] \\&\quad + u_{xx} [6t(F_1)_y + 6tF_2(y) + F_3(y)]\} \\&\quad - u_{xy} \{F_2(y) + u_x [6t(F_1)_y + 6tF_2(y) + F_3(y)] + 3(F_1)_y\} \\&\quad + u_{xxy} [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] + 3u_{xyy} F_1(y) \\&\quad + 6uv_y F_1(y) + v_{xxy} F_1(y), \\C_5^y &= 6uu_y F_1(y) + 3u_{xxy} F_1(y), \\C_5^t &= -u_y [6tu(F_1)_y + (6tu + x)F_2(y) + uF_3(y) + F_4(y)] - v_y F_1(y).\end{aligned}$$

6 Conclusions

In this paper, Lie group analysis is applied to the $(2 + 1)$ -dimensional integrable coupling of the KdV equation. The optimal system of the obtained symmetries and reduced equations are obtained based on symmetry method. Moreover, explicit solutions of the reduced equations are constructed by using the tanh method. Through the figures related to solutions, we can show the rules of the wave propagation corresponding to Eq. (1). Finally, nonlinearly self-adjointness of Eq. (1) is manifested and its conservation laws are derived by using Ibragimov's method.

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Authors' contributions

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