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Global existence and exponential stability of coupled Lamé system with distributed delay and source term without memory term

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Abstract

In this paper, we prove the global existence and exponential energy decay results of a coupled Lamé system with distributed time delay, nonlinear source term, and without memory term by using the Faedo–Galerkin method. In addition, an appropriate Lyapunov functional, more general relaxation functions, and some properties of convex functions are considered.

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1 Introduction

In this work, we study the following Lamé system in $\Omega \times \mathbb{R}^+$:

$$\begin{cases} u_{tt} - \Delta_e u + k_1 u_t + \int_{\tau_1}^{\tau_2} \mu_1(\varrho) u_t(x, t - \varrho) d\varrho = f_1(u, v), \\ v_{tt} - \Delta_e v + k_2 v_t + \int_{\tau_1}^{\tau_2} \mu_2(\varrho) v_t(x, t - \varrho) d\varrho = f_2(u, v), \\ u(x, t) = v(x, t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad u_t(x, 0) = u_1(x), \\ v_t(x, 0) = v_1(x), \quad x \in \Omega, \\ (u_t(x, -t), v_t(x, -t)) = (f_0(x, t), g_0(x, t)) \quad \text{in } \Omega \times (0, \tau_2), \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n = 1, 2, 3$) with smooth boundary $\partial\Omega$. The elasticity differential operator Δ_e is given by

$$\Delta_e u = \mu \Delta u + (\mu + \lambda) \nabla(\operatorname{div} u),$$

and the Lamé constants μ and λ satisfy the following conditions:

$$\mu > 0, \quad \mu + \lambda > 0.$$

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The parameters k_1, k_2, τ_1 , and τ_2 are positive constants, with $\tau_1 < \tau_2$. The functions $\mu_1, \mu_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ are bounded. The functions $f_1(u, v)$ and $f_2(u, v)$, which represent the source terms, will be specified later.

After several authors have studied the problems of coupled systems and hyperbolic systems, their stability is associated with velocities and is proven under some given conditions (see, for example, [1–11]). In recent years, several authors have been interested in studying the existence and stability for Lamé systems, we refer to [12–14]. The Lamé system with localized nonlinear damping and a general decay result of energy have been considered by some recent works (see, for example, [12, 14], and [15]). Bchatnia et al. in [16] investigated Lamé systems with past history. Then, Taouaf et al. in [17] established the well-posedness and asymptotic stability for the Lamé system with internal distributed delay.

Beniani et al. [13] proved the well-posedness and exponential stability of the following coupled Lamé system:

$$\begin{cases} u_{tt} + \alpha v - \Delta_e u + \int_0^t g_1(t-s) \Delta u(s) ds - \mu_1 \Delta u_t = 0, \\ v_{tt} + \alpha u - \Delta_e v + \int_0^t g_2(t-s) \Delta v(s) ds - \mu_2 \Delta v_t = 0. \end{cases} \quad (1.2)$$

After that, Baowei et al. in [18] considered the same problem with more general assumption on the relaxation functions. They established an explicit and general decay result, which are optimal, to the system.

Boulaaras et al. in [14] considered the previous problem with a source term, where under some suitable conditions on the initial data and the relaxation functions, they proved an asymptotic stability result of global solution taking into account that the kernel is not necessarily decreasing.

In the present work, we prove the existence and general decay results of problem (1.1) with respect to the presence of distributed term delay in order to ensure fast stability under some given conditions. We establish the exponential energy decay results to the system by using an appropriate Lyapunov functional.

This paper is organized as follows. In the second section, we give some preliminaries related to problem (1.1). In Sect. 3, we prove the global existence by using Faedo–Galerkin method. In the fourth section, we prove our main result of exponential energy decay.

2 Preliminaries

In this section, we provide some materials and necessary assumptions which we need in the proof of our results. We use the standard Lebesgue and Sobolev spaces with their scalar products and norms. For simplicity, we would write $\|\cdot\|$ instead of $\|\cdot\|_2$.

(A1) For the source terms f_1 and f_2 , we take

$$f_1(u, v) = \alpha |u + v|^{p-1} (u + v) + \beta |u|^{\frac{p-3}{2}} |u|v|^{\frac{p+1}{2}}, \quad \forall (u, v) \in [\mathbb{R}^n]^2$$

and

$$f_2(u, v) = \alpha |u + v|^{p-1} (u + v) + \beta |v|^{\frac{p-3}{2}} |v|u|^{\frac{p+1}{2}}, \quad \forall (u, v) \in [\mathbb{R}^n]^2,$$

with $\alpha, \beta > 0$. Clearly,

$$uf_1(u, v) + vf_2(u, v) = (p+1)F(u, v), \quad \forall (u, v) \in [\mathbb{R}^n]^2, \quad (2.1)$$

where

$$F(u, v) = \frac{1}{(p+1)} [\alpha |u+v|^{p+1} + 2\beta |uv|^{\frac{p+1}{2}}], \quad \forall (u, v) \in [\mathbb{R}^n]^2, \quad (2.2)$$

and

$$f_1(u, v) = \frac{\partial F}{\partial u}, \quad f_2(u, v) = \frac{\partial F}{\partial v}. \quad (2.3)$$

(A2)

$$\text{If } n = 1, 2; \quad p \geq 3, \quad \text{if } n = 3; \quad p = 3. \quad (2.4)$$

So, we have the embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for } 2 \leq q \leq \frac{2n}{n-2} \text{ if } n \geq 3 \quad \text{or} \quad q \geq 2 \quad \text{if } n = 1, 2$$

and

$$L^r \hookrightarrow L^q \quad \text{for } q < r.$$

Let c_s be the same embedding constant, so we have

$$\|v\|_q \leq c_s \|\nabla v\|_2, \quad \|v\|_q \leq c_s \|v\|_r \quad \text{for } v \in H_0^1(\Omega). \quad (2.5)$$

As in many papers, we introduce the following new variables:

$$\begin{cases} z(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho), \\ y(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{cases}$$

then we obtain

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = u_t(x, t), \end{cases} \quad (2.6)$$

and

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = v_t(x, t). \end{cases} \quad (2.7)$$

Consequently, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - \Delta_e u + k_1 u_t + \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho = f_1(u, v), \\ v_{tt} - \Delta_e v + k_2 v_t + \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y(x, 1, \varrho, t) d\varrho = f_2(u, v), \\ \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \end{cases} \quad (2.8)$$

with the initial data and boundary conditions

$$\begin{cases} (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)) & \text{in } \Omega, \\ (u_t(x, -t), v_t(x, -t)) = (f_0(x, t), g_0(x, t)) & \text{in } \Omega \times (0, \tau_2), \\ u(x, t) = v(x, t) = 0 & \text{in } \partial\Omega \times (0, \infty), \\ (z(x, \rho, \varrho, 0), y(x, \rho, \varrho, 0)) = (f_0(x, \rho, \varrho), g_0(x, \rho, \varrho)) & \text{in } \Omega \times (0, 1) \times (0, \tau_2), \end{cases} \quad (2.9)$$

where

$$(x, \rho, \varrho, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

The energy associated with problem (2.8) is defined by

$$\begin{aligned} E(t) = & \frac{1}{2} [\|u_t\|^2 + \|v_t\|^2 + \mu(\|\nabla u\|^2 + \|\nabla v\|^2) + (\lambda + \mu)(\|\operatorname{div} u\|^2 + \|\operatorname{div} v\|^2)] \\ & + \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho [|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2] d\varrho d\rho dx \\ & - \int_{\Omega} F(u, v) dx. \end{aligned} \quad (2.10)$$

First, we prove in the following theorem, the result of energy identity.

Lemma 2.1 *Assume that*

$$\int_{\tau_1}^{\tau_2} |\mu_i(\varrho)| d\varrho < k_i, \quad i = 1, 2. \quad (2.11)$$

Then the energy modified defined by (2.10) satisfies, along the solution (u, v, z, y) of (2.8), the estimate

$$\begin{aligned} \frac{d}{dt} E(t) \leq & - \left[k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right] \|u_t\|^2 - \left[k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right] \|v_t\|^2 \\ & \leq 0. \end{aligned} \quad (2.12)$$

Proof First multiplying equation (2.8)₁ by u_t and integrating by parts over Ω , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|u_t\|^2 + \mu \|\nabla u\|^2 + (\lambda + \mu) \|\operatorname{div} u\|^2] + k_1 \|u_t\|^2 \\ + \int_{\Omega} u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z(x, 1, \varrho, t) d\varrho dx \\ = \int_{\Omega} u_t f_1(u, v) dx. \end{aligned} \quad (2.13)$$

Similarly, multiplying equation (2.8)₂ by v_t and integrating over Ω , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|v_t\|^2 + \mu \|\nabla v\|^2 + (\lambda + \mu) \|\operatorname{div} v\|^2] + k_1 \|v_t\|^2 \\ & + \int_{\Omega} v_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho dx \\ & = \int_{\Omega} v_t f_2(u, v) dx. \end{aligned} \quad (2.14)$$

Multiplying equation (2.8)₃ by $|\mu_1(\varrho)|z(x, \rho, \varrho, t)$ and integrating by parts over $\Omega \times (0, 1) \times (\tau_1, \tau_2)$, we obtain

$$\begin{aligned} & \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| z(x, \rho, \varrho, t) z_t(x, \rho, \varrho, t) d\varrho d\rho dx \\ & = - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z(x, \rho, \varrho, t) z_{\rho}(x, \rho, \varrho, t) d\varrho d\rho dx, \end{aligned}$$

therefore

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & = - \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \frac{d}{d\rho} |z(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & = - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |u_t(x, t)|^2 d\varrho dx. \end{aligned} \quad (2.15)$$

Similarly, multiplying the fourth equation of (2.8) by $|\mu_2(\varrho)|y(x, \rho, \varrho, t)$ and integrating over $\Omega \times (0, 1) \times (\tau_1, \tau_2)$, we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \right) \\ & = - \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \frac{d}{d\rho} |y(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & = - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |v_t(x, t)|^2 d\varrho dx. \end{aligned} \quad (2.16)$$

For the source term, we have

$$\begin{aligned} & \int_{\Omega} u_t f_1(u, v) dx + \int_{\Omega} v_t f_2(u, v) dx \\ & = \int_{\Omega} u_t (\alpha |u + v|^{p-1} (u + v) + \beta |u|^{\frac{p-3}{2}} |u|^{p+1}) \\ & \quad + \int_{\Omega} v_t (\alpha |u + v|^{p-1} (u + v) + \beta |v|^{\frac{p-3}{2}} |v|^{p+1}) \\ & = \int_{\Omega} (\alpha |u + v|^{p-1} (u + v) (u_t + v_t) + \beta (|u|^{\frac{p-3}{2}} u \cdot u_t) |v|^{\frac{p+1}{2}} \\ & \quad + \beta (|v|^{\frac{p-3}{2}} v \cdot v_t) |u|^{\frac{p+1}{2}}) dx \end{aligned} \quad (2.17)$$

$$= \frac{d}{dt} \int_{\Omega} \left(\frac{\alpha}{p+1} |u+v|^{p+1} + \frac{2\beta}{p+1} |uv|^{\frac{p+1}{2}} \right) dx = \frac{d}{dt} \int_{\Omega} F(u, v) dx.$$

By collecting the previous equations (2.13)–(2.17), we get

$$\begin{aligned} \frac{d}{dt} E(t) &= -k_1 \|u_t\|^2 - \int_{\Omega} u_t \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho dx \\ &\quad - k_2 \|v_t\|^2 - \int_{\Omega} v_t \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y(x, 1, \varrho, t) d\varrho dx \\ &\quad - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_t\|^2 \\ &\quad - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_t\|^2. \end{aligned} \quad (2.18)$$

Using Young's inequality, we obtain

$$\begin{aligned} &\int_{\Omega} u_t \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho dx \\ &\leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_t\| + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 d\varrho dx, \end{aligned} \quad (2.19)$$

similarly

$$\begin{aligned} &\int_{\Omega} v_t \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y(x, 1, \varrho, t) d\varrho dx \\ &\leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_t\| + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 d\varrho dx, \end{aligned} \quad (2.20)$$

This completes the proof. $\square$

3 Global existence

Theorem 3.1 (Global existence) *Let $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$ and $(f_0, g_0) \in (H^1(\Omega \times (0, 1) \times (\tau_1, \tau_2)))^2$ satisfying the compatibility condition*

$$(f_0(\cdot, 0), g_0(\cdot, 0)) = (u_1, v_1).$$

Assume that (A1)–(A2) hold. Then, problem (2.8)–(2.9) admits a weak solution such that $u, v \in L^\infty(0, \infty; H^2(\Omega) \cap H_0^1(\Omega))$, $u_t, v_t \in L^\infty(0, \infty; H_0^1(\Omega))$, and $u_{tt}, v_{tt} \in L^2(0, \infty, H_0^1(\Omega))$.

Throughout this section we assume $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$ and $f_0, g_0 \in H^2(\Omega; H^1(0, 1)) \cap H_0^1(\Omega; H^1(0, 1))$. We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \dots, w_k\}$, where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$. Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x, \rho)$ as follows:

$$\phi_j(x, 0) = w_j. \quad (3.1)$$

Then, we may extend $\phi_j(x, 0)$ by $\phi_j(x, \rho)$ over $L^2(\Omega \times [0, 1])$ and denote by Z_k the space generated by $\{\phi_1, \phi_2, \dots, \phi_k\}$. We construct approximate solutions (u^k, v^k, z^k, y^k) $k = 1, 2, \dots$ in the form

$$u^k(t) = \sum_{j=1}^k g_{jk}(t) w_j(x), \quad z^k(t) = \sum_{j=1}^k c_{jk}(t) \phi_j(x, \rho, \varrho) \quad (3.2)$$

$$v^k(t) = \sum_{j=1}^k h_{jk}(t) w_j(x), \quad y^k(t) = \sum_{j=1}^k d_{jk}(t) \phi_j(x, \rho, \varrho), \quad (3.3)$$

where g_{jk}, h_{jk}, c_{jk} , and d_{jk} , $j = 1, 2, \dots$, are determined by the following ordinary differential equations:

$$\begin{cases} \langle u_t^k, w_j \rangle + \mu \langle \nabla u^k, \nabla w_j \rangle + (\lambda + \mu) \langle \operatorname{div} u^k, \operatorname{div} w_j \rangle + k_1 \langle u_t^k, w_j \rangle \\ \quad + \langle \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z^k(x, 1, \varrho, t) d\varrho, w_j \rangle = \langle f_1(u^k, v^k), w_j \rangle, \quad 1 \leq j \leq k, \\ \langle v_t^k, w_j \rangle + \mu \langle \nabla v^k, \nabla w_j \rangle + (\lambda + \mu) \langle \operatorname{div} v^k, \operatorname{div} w_j \rangle + k_2 \langle v_t^k, w_j \rangle \\ \quad + \langle \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^k(x, 1, \varrho, t) d\varrho, w_j \rangle = \langle f_2(u^k, v^k), w_j \rangle, \quad 1 \leq j \leq k, \\ \langle \varrho z_t^k + \frac{\partial}{\partial \rho} z^k, \phi_j \rangle = 0, \quad 1 \leq j \leq k, \\ \langle \varrho y_t^k + \frac{\partial}{\partial \rho} y^k, \phi_j \rangle = 0, \quad 1 \leq j \leq k, \end{cases} \quad (3.4)$$

and

$$\begin{cases} (u^k(0), u_{t,k}(0)) = (u_{0k}, u_{1k}), \\ (v^k(0), v_{t,k}(0)) = (u_{0k}, u_{1k}), \\ z^k(x, 0, \varrho, t) = u_t^k(x, t), \\ y^k(x, 0, \varrho, t) = v_t^k(x, t). \end{cases} \quad (3.5)$$

Suppose that

$$w_j \in H^2(\Omega). \quad (3.6)$$

We choose u_0^k, v_0^k, u_1^k and $v_1^k \in [w_1, \dots, w_k]$, where

$$u^k(0) = u_0^k = \sum_{j=1}^k (u_0, w_j) w_j \rightarrow u_0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \rightarrow +\infty, \quad (3.7)$$

$$u_t^k(0) = u_1^k = \sum_{j=1}^k (u_1, w_j) w_j \rightarrow u_1 \quad \text{in } H_0^1(\Omega) \text{ as } k \rightarrow +\infty, \quad (3.8)$$

$$v^k(0) = v_0^k = \sum_{j=1}^k (v_0, w_j) w_j \rightarrow v_0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \rightarrow +\infty, \quad (3.9)$$

$$v_t^k(0) = v_1^k = \sum_{j=1}^k (v_1, w_j) w_j \rightarrow v_1 \quad \text{in } H_0^1(\Omega) \text{ as } k \rightarrow +\infty, \quad (3.10)$$

$$z^k(\rho, 0) = z_0^k = \sum_{j=1}^k (f_0, \phi_j) \phi_j \rightarrow f_0 \quad \text{in } L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)) \text{ as } k \rightarrow +\infty, \quad (3.11)$$

$$y^k(\rho, 0) = y_0^k = \sum_{j=1}^k (g_0, \phi_j) \phi_j \rightarrow g_0 \quad \text{in } L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2)) \text{ as } k \rightarrow +\infty. \quad (3.12)$$

By virtue of the theory of ordinary differential equations, system (3.4)–(3.12) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \leq +\infty$). We can utilize a standard compactness argument for the limiting procedure.

The first estimate.

Lemma 3.2 *There exists a constant $T > 0$ such that the approximate solution satisfies, for all $k \geq 1$:*

$$u^k, v^k \quad \text{are bounded in } L^\infty(0, T; H_0^1(\Omega)), \quad (3.13)$$

$$u_t^k, v_t^k \quad \text{are bounded in } L^\infty(0, T; L^2(\Omega)), \quad (3.14)$$

$$z^k(x, \rho, \varrho, t), y^k(x, \rho, \varrho, t) \quad \text{are bounded in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))). \quad (3.15)$$

Proof Multiplying the first and second equations of (3.4) by (g'_{jk}) and (h'_{jk}) respectively and summing with respect to j , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|u_t^k\|^2 + \mu \|\nabla u^k\|^2 + (\lambda + \mu) \|\operatorname{div} u^k\|^2 + \|v_t^k\|^2 \\ & \quad + \mu \|\nabla v^k\|^2 + (\lambda + \mu) \|\operatorname{div} v^k\|^2] \\ & \quad + \int_{\Omega} u_t^k \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z^k(x, 1, \varrho, t) d\varrho dx + \int_{\Omega} v_t^k \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y^k(x, 1, \varrho, t) d\varrho dx \\ & \quad + k_1 \|u_t^k\|^2 + k_2 \|v_t^k\|^2 = \langle f_1(u^k, v^k), u_t^k \rangle + \langle f_2(u^k, v^k), v_t^k \rangle, \quad 1 \leq j \leq k. \end{aligned} \quad (3.16)$$

Multiplying (3.4) by $|\mu_1(\varrho)|(c_{jk})$ and $|\mu_2(\varrho)|(d_{jk})$ respectively, iterating over $\Omega \times (0, 1) \times (\tau_1, \tau_2)$, and summing with respect to j , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z^k|^2 d\varrho d\rho dx \\ & \quad + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| |y^k|^2 d\varrho d\rho dx \\ & = -\frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z^k(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_t^k\|^2 \\ & \quad - \frac{1}{2} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| |y^k(x, 1, \varrho, t)|^2 d\varrho dx + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_t^k\|^2. \end{aligned} \quad (3.17)$$

By summing (3.16)–(3.17), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|u_t^k\|^2 + \mu \|\nabla u^k\|^2 + (\lambda + \mu) \|\operatorname{div} u^k\|^2 + \|v_t^k\|^2 + \mu \|\nabla v^k\|^2 + (\lambda + \mu) \|\operatorname{div} v^k\|^2] \\ & \quad + \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \right. \\ & \quad \left. + \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| |y^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \right] \end{aligned}$$

$$\begin{aligned}
& + k_1 \|u_t^k\|^2 + k_2 \|v_t^k\|^2 \\
& + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z^k(x, 1, \varrho, t)|^2 d\varrho dx - \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_t^k\|^2 \\
& + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y^k(x, 1, \varrho, t)|^2 d\varrho dx - \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_t^k\|^2 \\
& = \int_{\Omega} (u_t^k f_1(u^k, v^k) + v_t^k f_2(u^k, v^k)) dx \\
& \quad - \int_{\Omega} u_t^k \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z_k(x, 1, \varrho, t) d\varrho dx - \int_{\Omega} v_t^k \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y_k(x, 1, \varrho, t) d\varrho dx.
\end{aligned} \tag{3.18}$$

By using Holder and Young's inequalities, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} [\|u_t^k\|^2 + \mu \|\nabla u^k\|^2 + (\lambda + \mu) \|\operatorname{div} u^k\|^2 \\
& \quad + \|v_t^k\|^2 + \mu \|\nabla v^k\|^2 + (\lambda + \mu) \|\operatorname{div} v^k\|^2] \\
& + \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho [|\mu_1(\varrho)| \|z^k(x, \rho, \varrho, t)\|^2 \\
& \quad + \varrho |\mu_2(\varrho)| \|y^k(x, \rho, \varrho, t)\|^2] d\varrho d\rho \\
& + \left[k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right] \|u_t^k\|^2 + \left[k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right] \|v_t^k\|^2 \\
& \leq \int_{\Omega} (u_t^k f_1(u^k, v^k) + v_t^k f_2(u^k, v^k)) dx.
\end{aligned} \tag{3.19}$$

Integrating over $(0, t)$, $0 < t < T_k$, we obtain

$$\begin{aligned}
& \|u_t^k(t)\|^2 + \mu \|\nabla u^k(t)\|^2 + (\lambda + \mu) \|\operatorname{div} u^k(t)\|^2 + \|v_t^k(t)\|^2 \\
& + \mu \|\nabla v^k(t)\|^2 + (\lambda + \mu) \|\operatorname{div} v^k(t)\|^2 \\
& + \int_0^1 \int_{\tau_1}^{\tau_2} \varrho [|\mu_1(\varrho)| \|z^k(x, \rho, \varrho, t)\|^2 + \varrho |\mu_2(\varrho)| \|y^k(x, \rho, \varrho, t)\|^2] d\varrho d\rho \\
& + 2 \left[k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right] \int_0^t \|u_t^k(s)\|^2 ds \\
& + 2 \left[k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right] \int_0^t \|v_t^k(s)\|^2 ds \\
& \leq C_0 + c \int_0^t \int_{\Omega} [u_t^k f_1(u^k, v^k) + v_t^k f_2(u^k, v^k)] dx ds,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
C_0 = C(& \|u_0^k\|_{H^1(\Omega)}, \|v_0^k\|_{H^1(\Omega)}, \|u_1^k\|_{L^2(\Omega)}, \|v_1^k\|_{L^2(\Omega)}, \\
& \|f_0\|_{L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))}, \|g_0\|_{L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))})
\end{aligned}$$

is a positive constant. We just need to estimate the right-hand terms of (3.20). Applying Hölder's inequality, Sobolev's embedding theorem, and Young's inequality, we obtain

$$\begin{aligned}
 & \left| \int_0^t \int_{\Omega} u_t^k(s) f_1(u^k(s), v^k(s)) \, ds \right| \\
 & \leq C \int_0^t \int_{\Omega} (|u^k(s)|^p + |v^k(s)|^p + |u^k(s)|^{\frac{p-1}{2}} |v^k(s)|^{\frac{p+1}{2}}) |u_t^k(s)| \, ds \\
 & \leq C \int_0^t \int_{\Omega} (\|u^k(s)\|_{2p}^p + \|v^k(s)\|_{2p}^p \\
 & \quad + \|u^k(s)\|_{\frac{3(p-1)}{2}}^{\frac{p-1}{2}} \|v^k(s)\|_{\frac{3(p+1)}{2}}^{\frac{p+1}{2}}) \|u_t^k(s)\|_2 \, ds \\
 & \leq C \int_0^t \int_{\Omega} (\|\nabla u^k(s)\|_2^p + \|\nabla v^k(s)\|_2^p \\
 & \quad + \|\nabla u^k(s)\|_2^{\frac{p-1}{2}} \|\nabla v^k(s)\|_2^{\frac{p+1}{2}}) \|u_t^k(s)\|_2 \, ds \\
 & \leq C \int_0^t \int_{\Omega} (\|u_t^k(s)\|_2^2 + \|\nabla u^k(s)\|_2^{2p} + \|\nabla v^k(s)\|_2^{2p} \\
 & \quad + \|\nabla u^k(s)\|_2^{(p-1)} \|\nabla v^k(s)\|_2^{(p+1)}) \, ds,
 \end{aligned} \tag{3.21}$$

when we have used in (3.21) the Sobolev imbedding in (2.5) and the fact when $n = 3$ then $2p = 3(p-1) = \frac{3(p+1)}{2} = 6$.

Likewise, we obtain

$$\begin{aligned}
 & \left| \int_0^t \int_{\Omega} v_t^k(s) f_2(u^k(s), v^k(s)) \, ds \right| \\
 & \leq C \int_0^t \int_{\Omega} (\|v_t^k(s)\|_2^2 + \|\nabla u^k(s)\|_2^{2p} + \|\nabla v^k(s)\|_2^{2p} \\
 & \quad + \|\nabla u^k(s)\|_2^{p+1} \|\nabla v^k(s)\|_2^{p-1}) \, ds.
 \end{aligned} \tag{3.22}$$

Let

$$X_k(t) = \|u_t^k(t)\|_2^2 + \|v_t^k(t)\|_2^2 + \|\nabla u^k(t)\|_2^2 + \|\nabla v^k(t)\|_2^2. \tag{3.23}$$

From assumptions of Lemma 2.1, we can find positive constants such that

$$\begin{aligned}
 & X_k(t) + c_1 \|\operatorname{div} u^k(t)\|^2 + c_1 \|\operatorname{div} v^k(t)\|^2 \\
 & \quad + c_2 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho [|\mu_1(\varrho)| \|z^k(x, \rho, \varrho, t)\|^2 + |\mu_2(\varrho)| \|y^k(x, \rho, \varrho, t)\|^2] \, d\varrho \, d\rho \\
 & \quad + c_3 \int_0^t \|u_t^k(s)\|_2^2 \, ds + c_3 \int_0^t \|v_t^k(s)\|_2^2 \, ds \\
 & \leq C_0 + c \int_0^t \int_{\Omega} (X_k(t))^p \, dx \, ds.
 \end{aligned} \tag{3.24}$$

Particularly, we have

$$X_k(t) \leq C_0 + c \int_0^t \int_{\Omega} (X_k(t))^p \, dx \, ds. \tag{3.25}$$

Using Gronwall-type inequality, we can get

$$X_k(t) \leq [C_0 - (p-1)Ct]^{-1/(p-1)}. \quad (3.26)$$

Thus, we deduce from (3.26) that there exists a time $T > 0$ such that

$$X_k(t) \leq C_1, \quad \forall t \in [0, T], \quad (3.27)$$

where C_1 is a positive constant independent of k . Then inequality (3.27) established the first two parts of lemma. The last part of lemma immediately follows from (3.24). $\square$

The second estimate: First, we are going to estimate $u_{tt}^k(0)$ and $v_{tt}^k(0)$. Testing the first and second equations in (3.4) by $g_{j,k}''(t)$ and $h_{j,k}''(t)$ respectively and taking $t = 0$, we obtain

$$\begin{aligned} & \|u_{tt}^k(0)\|_2^2 + \|v_{tt}^k(0)\|_2^2 \\ & \leq \|\Delta u_0^k\|_2^2 + \|\Delta v_0^k\|_2^2 + c\|\operatorname{div} u_0^k\|_2 + c\|\operatorname{div} v_0^k\|_2^2 + \|\Delta u_t^k(0)\|_2^2 \\ & \quad + \|\Delta v_t^k(0)\|_2^2 + c \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\Delta z^k(x, 1, \varrho, 0)|^2 d\varrho dx \\ & \quad + c \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\Delta y^k(x, 1, \varrho, 0)|^2 d\varrho dx. \end{aligned} \quad (3.28)$$

From (3.7)–(3.12), we have

$$\|u_{tt}^k(0)\|_2^2 + \|v_{tt}^k(0)\|_2^2 \leq C. \quad (3.29)$$

In order to calculate the second estimate, we take the derivatives of the first and second equations of system (3.4) with respect to t , we get

$$\begin{aligned} & \langle u_{ttt}^k, w_j \rangle + \mu \langle \nabla u_t^k, \nabla w_j \rangle + (\lambda + \mu) \langle \operatorname{div} u_t^k, \operatorname{div} w_j \rangle \\ & + k_1 \langle u_{tt}^k, w_j \rangle + \left\langle \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z_t^k(x, 1, \varrho, t) d\varrho, w_j \right\rangle = \langle Df_1(u_k, v_k), w_j \rangle, \quad 1 \leq j \leq k \end{aligned}$$

and

$$\begin{aligned} & \langle v_{ttt}^k, w_j \rangle + \mu \langle \nabla v_t^k, \nabla w_j \rangle + (\lambda + \mu) \langle \operatorname{div} v_t^k, \operatorname{div} w_j \rangle \\ & + k_2 \langle v_{tt}^k, w_j \rangle + \left\langle \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_t^k(x, 1, \varrho, t) d\varrho, w_j \right\rangle = \langle Df_2(u_k, v_k), w_j \rangle, \quad 1 \leq j \leq k. \end{aligned}$$

Multiplying by $(g_{j,k}''(t))$ and $(h_{j,k}''(t))$ respectively and summing with respect to j from 1 to k , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|u_{tt}^k(t)\|_2^2 + \mu \|\nabla u_t^k(t)\|_2^2 + (\lambda + \mu) \|\operatorname{div} u_t^k(t)\|_2^2] + k_1 \|u_{tt}^k(t)\|_2^2 \\ & + \int_{\Omega} u_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z_t^k(x, 1, \varrho, t) d\varrho dx \\ & = \int_{\Omega} Df_1(u_k, v_k) u_{tt}^k(t) dx, \quad 1 \leq j \leq k, \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|v_{tt}^k\|^2 + \mu \|\nabla v_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k\|^2] + k_2 \|v_{tt}^k\|^2 \\ & + \int_{\Omega} v_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_t^k(x, 1, \varrho, t) d\varrho dx \\ & = \int_{\Omega} Df_2(u_k, v_k) v_{tt}^k(t) dx, \quad 1 \leq j \leq k. \end{aligned} \quad (3.31)$$

Differentiating the third and fourth equations in (3.4) with respect to t , we get

$$\begin{aligned} \left\langle \varrho z_{tt}^k + \frac{\partial}{\partial \rho} z_t^k, \phi_j \right\rangle &= 0, \quad 1 \leq j \leq k, \\ \left\langle \varrho y_{tt}^k + \frac{\partial}{\partial \rho} y_t^k, \phi_j \right\rangle &= 0, \quad 1 \leq j \leq k. \end{aligned}$$

Multiplying by $|\mu_1(\varrho)| c'_{jk}$ and $|\mu_2(\varrho)| d'_{jk}$ respectively, integrating over $(0, 1) \times (\tau_1, \tau_2)$, and summing over j from 1 to k , it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |z_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & + \frac{1}{2} \int_0^1 \int_{\Omega} |\mu_1(\varrho)| \left| \frac{d}{d\rho} z_t^k(x, \rho, \varrho, t) \right|^2 d\varrho d\rho dx = 0, \\ & \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |y_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & + \frac{1}{2} \int_0^1 \int_{\Omega} |\mu_2(\varrho)| \left| \frac{d}{d\rho} y_t^k(x, \rho, \varrho, t) \right|^2 d\varrho d\rho dx = 0, \end{aligned}$$

then we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |z_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z_t^k(x, 1, \varrho, t)|^2 d\varrho dx \\ & - \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_{tt}^k(t)\|^2 dx = 0 \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_2(\varrho)| |y_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \\ & + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y_t^k(x, 1, \varrho, t)|^2 d\varrho dx \\ & - \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_{tt}^k(t)\|^2 dx = 0. \end{aligned} \quad (3.33)$$

Taking the sum of (3.30), (3.31), (3.32), and (3.33), we get

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left[\|u_{tt}^k\|^2 + \mu \|\nabla u_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} u_t^k\|^2 + \|v_{tt}^k\|^2 + \mu \|\nabla v_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k\|^2 \right] \\
 & + \frac{1}{2} \frac{d}{dt} \left[\int_0^1 \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \right. \\
 & \left. + \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_2(\varrho)| |y_t^k(x, \rho, \varrho, t)|^2 d\varrho d\rho dx \right] \\
 & + k_1 \|u_{tt}^k\|^2 + k_2 \|v_{tt}^k\|^2 + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z_t^k(x, 1, \varrho, t)|^2 d\varrho dx \\
 & + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y_t^k(x, 1, \varrho, t)|^2 d\varrho dx \\
 & = \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_{tt}^k(t)\|^2 + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_{tt}^k(t)\|^2 \\
 & - \int_{\Omega} u_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z_t^k(x, 1, \varrho, t) d\varrho dx \\
 & - \int_{\Omega} v_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_t^k(x, 1, \varrho, t) d\varrho dx \\
 & + \int_{\Omega} Df_1(u_k, v_k) u_{tt}^k(t) dx + \int_{\Omega} Df_2(u_k, v_k) v_{tt}^k(t) dx.
 \end{aligned} \tag{3.34}$$

Using Cauchy–Schwarz and Young’s inequalities, we conclude

$$\begin{aligned}
 & \left| \int_{\Omega} u_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z_t^k(x, 1, \varrho, t) d\varrho dx \right| \\
 & \leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_{tt}^k(t)\|^2 \\
 & + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z_t^k(x, 1, \varrho, t)\|^2 d\varrho.
 \end{aligned} \tag{3.35}$$

Similarly,

$$\begin{aligned}
 & \left| \int_{\Omega} v_{tt}^k(t) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_t^k(x, 1, \varrho, t) d\varrho dx \right| \\
 & \leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_{tt}^k(t)\|^2 \\
 & + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y_t^k(x, 1, \varrho, t)\|^2 d\varrho.
 \end{aligned} \tag{3.36}$$

For the source term

$$\begin{aligned}
 & \int_{\Omega} Df_1(u_k, v_k) u_{tt}^k(t) dx \\
 & \leq C[(\|u^k\|^{p-1} + \|v^k\|^{p-1}) \|u_t^k\| + (\|u^k\|^{p-1} + \|v^k\|^{p-1}) \|v_t^k\|] \|u_{tt}^k\| \\
 & \leq C[\|u^k\|_2^{2(p-1)} + \|v^k\|_2^{2(p-1)} + \|u_t^k\|_2^2 + \|v_t^k\|_2^2] \|u_{tt}^k\|_2
 \end{aligned} \tag{3.37}$$

$$\leq \|u_{tt}^k(t)\|_2^2 + c$$

and

$$\begin{aligned} & \int_{\Omega} Df_2(u_k, v_k) v_{tt}^k(t) dx \\ & \leq C[(\|u^k\|^{p-1} + \|v^k\|^{p-1}) \|u_t^k\| + (\|u^k\|^{p-1} + \|v^k\|^{p-1}) \|v_t^k\|] \|v_{tt}^k\| \\ & \leq C[\|u^k\|_2^{2(p-1)} + \|v^k\|_2^{2(p-1)} + \|u_t^k\|_2^2 + \|v_t^k\|_2^2] \|v_{tt}^k\|_2 \\ & \leq \|v_{tt}^k(t)\|_2^2 + c. \end{aligned} \quad (3.38)$$

Taking into account (3.35)–(3.38) into (3.34), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|u_{tt}^k\|^2 + \mu \|\nabla u_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} u_t^k\|^2 + \|v_{tt}^k\|^2 + \mu \|\nabla v_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k\|^2] \\ & + \frac{1}{2} \frac{d}{dt} \left[\int_0^1 \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right. \\ & \left. + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |y_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right] \\ & + k_1 \|u_{tt}^k\|^2 + k_2 \|v_{tt}^k\|^2 + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z_t^k(x, 1, \varrho, t)\|^2 d\varrho \\ & + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y_t^k(x, 1, \varrho, t)\|^2 d\varrho \\ & \leq \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_{tt}^k(t)\|^2 + \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_{tt}^k(t)\|^2 \\ & + \left(\frac{\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho}{2} \right) \|u_{tt}^k\|^2 + \left(\frac{\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho}{2} \right) \|v_{tt}^k\|^2 \\ & + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|z_t^k(x, 1, \varrho, t)\|^2 d\varrho + \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y_t^k(x, 1, \varrho, t)\|^2 d\varrho \\ & + \|u_{tt}^k(t)\|^2 + \|v_{tt}^k(t)\|^2 + c. \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} [\|u_{tt}^k\|^2 + \mu \|\nabla u_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} u_t^k\|^2 + \|v_{tt}^k\|^2 + \mu \|\nabla v_t^k\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k\|^2] \\ & + \frac{1}{2} \frac{d}{dt} \left[\int_0^1 \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right. \\ & \left. + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |y_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right] \\ & + k_1 \|u_{tt}^k\|^2 + k_2 \|v_{tt}^k\|^2 \\ & \leq \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_{tt}^k(t)\|^2 + \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_{tt}^k(t)\|^2 \\ & + \|u_{tt}^k(t)\|^2 + \|v_{tt}^k(t)\|^2 + c. \end{aligned} \quad (3.39)$$

Integrating (3.39) over $(0, t)$, we get

$$\begin{aligned}
 & \left[\|u_{tt}^k(t)\|^2 + \mu \|\nabla u_t^k(t)\|^2 + (\lambda + \mu) \|\operatorname{div} u_t^k(t)\|^2 + \|v_{tt}^k(t)\|^2 \right. \\
 & \quad \left. + \mu \|\nabla v_t^k(t)\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k(t)\|^2 \right] \\
 & \quad \times \left[\int_0^1 \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right. \\
 & \quad \left. + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |y_t^k(x, \varrho, \rho, t)|^2 d\varrho d\rho dx \right] \\
 & \quad + \left(k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right) \int_0^t \|u_{tt}^k(s)\|^2 ds \\
 & \quad + \left(k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right) \int_0^t \|v_{tt}^k(s)\|^2 ds \\
 & \leq \left[\|u_{tt}^k(0)\|^2 + \mu \|\nabla u_t^k(0)\|^2 + (\lambda + \mu) \|\operatorname{div} u_t^k(0)\|^2 + \|v_{tt}^k(0)\|^2 \right. \\
 & \quad \left. + \mu \|\nabla v_t^k(0)\|^2 + (\lambda + \mu) \|\operatorname{div} v_t^k(0)\|^2 \right] \\
 & \quad \times \left[\int_0^1 \int_{\Omega} \int_{\tau_1}^{\tau_2} \varrho |\mu_1(\varrho)| |z_t^k(x, \varrho, \rho, 0)|^2 d\varrho d\rho dx \right. \\
 & \quad \left. + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho |\mu_1(\varrho)| |y_t^k(x, \varrho, \rho, 0)|^2 d\varrho d\rho dx \right] \\
 & \quad + \int_0^t \|u_{tt}^k(s)\|^2 ds + \int_0^t \|v_{tt}^k(s)\|^2 ds + c.
 \end{aligned}$$

Taking

$$y^k(t) = \|u_{tt}^k(t)\|^2 + \|v_{tt}^k(t)\|^2$$

and by using Gronwall's inequality, we conclude that

$$u_t^k \text{ and } v_t^k \text{ are bounded in } L^\infty(0, T; H_0^1(\Omega)), \quad (3.40)$$

$$u_{tt}^k \text{ and } v_{tt}^k \text{ are bounded in } L^\infty(0, T; L^2(\Omega)), \quad (3.41)$$

$$z_t^k \text{ and } y_t^k \text{ are bounded in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))). \quad (3.42)$$

Applying Dunford–Pettis' theorem, we deduce from (3.13), (3.14), (3.15), (3.40), (3.41), and (3.42), replacing the sequence u_k with a subsequence if necessary, that

$$u^k \rightarrow u, v^k \rightarrow v \text{ weak-star in } L^\infty(0, T; H_0^1(\Omega)), \quad (3.43)$$

$$z^k \rightarrow z, y^k \rightarrow y \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))), \quad (3.44)$$

$$u_t^k \rightarrow u_t, v_t^k \rightarrow v_t \text{ weak-star in } L^\infty(0, T; H_0^1(\Omega)), \quad (3.45)$$

$$u_{tt}^k \rightarrow u_{tt}, v_{tt}^k \rightarrow v_{tt} \text{ weak-star in } L^\infty(0, T; L^2(\Omega)), \quad (3.46)$$

$$z_t^k \rightarrow z_t, y_t^k \rightarrow y_t \text{ weak-star in } L^\infty(0, T; L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))). \quad (3.47)$$

Corollary 3.3 *The sequences of approximate solutions $\{u_k, v_k\}$ satisfy, as $k \rightarrow \infty$,*

$$\begin{cases} f_1(u_k, v_k) \rightarrow f_1(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)), \\ f_2(u_k, v_k) \rightarrow f_2(u, v) \text{ strongly in } L^\infty(0, T; L^2(\Omega)). \end{cases} \quad (3.48)$$

Proof The proof is similar to that of [11]. $\square$

We can complete the proof of theorem as in [2].

4 General decay

In this section we prove that the solution of problem (2.8)–(2.9) decays generally to a trivial solution using the energy method and a suitable Lyapunov functional.

In the following, we present our main stability result.

Theorem 4.1 *The solution of (2.8) satisfies, for two positive constants α, β , the estimate*

$$E(t) \leq \beta e^{-\alpha t}, \quad \forall t \geq 0. \quad (4.1)$$

To prove the desired result, we create a Lyapunov functional equivalent to E . For this, we define some functions that allow us to construct this Lyapunov function.

Lemma 4.2 *Let (u, v, z, y) be a solution of problem (2.8). Then the functional*

$$\varphi(t) = \int_{\Omega} u(t)u_t(t) dx + \int_{\Omega} v(t)v_t(t) dx \quad (4.2)$$

satisfies the estimate

$$\begin{aligned} \varphi'(t) \leq & \left(1 + \frac{Ck_1^2}{\mu}\right) \|u_t(t)\|^2 + \left(1 + \frac{Ck_2}{\mu}\right) \|v_t(t)\|^2 - \frac{\mu}{2} \|\nabla u(t)\|^2 - \frac{\mu}{2} \|\nabla v(t)\|^2 \\ & - (\lambda + \mu) \|\operatorname{div} u(t)\|^2 - (\lambda + \mu) \|\operatorname{div} v(t)\|^2 \\ & + \frac{Ck_1}{\mu} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x, 1, \varrho, t)|^2 d\varrho dx \\ & + \frac{Ck_2}{\mu} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x, 1, \varrho, t)|^2 d\varrho dx \\ & + (p+1) \int_{\Omega} F(u(t), v(t)) dx. \end{aligned} \quad (4.3)$$

Proof Taking the derivative of (4.2), we obtain

$$\varphi'(t) = \int_{\Omega} |u_t(t)|^2 dx + \int_{\Omega} u(t)u_{tt}(t) dx + \int_{\Omega} |v_t(t)|^2 dx + \int_{\Omega} v(t)v_{tt}(t) dx.$$

From problem (2.8) and using integration by parts, we get

$$\begin{aligned}
 \varphi'(t) &= \|u_t(t)\|^2 + \|v_t(t)\|^2 \\
 &\quad + \int_{\Omega} u(t) \left(\Delta_e u - k_1 u_t - \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho + f_1(u, v) \right) dx \\
 &\quad + \int_{\Omega} v(t) \left(\Delta_e v - k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\varrho) \Delta y(x, 1, \varrho, t) d\varrho + f_2(u, v) \right) dx \\
 &= \|u_t(t)\|^2 + \|v_t(t)\|^2 - k_1 \int_{\Omega} u u_t dx - k_2 \int_{\Omega} v v_t dx \\
 &\quad - \mu \|\nabla u(t)\|^2 - (\lambda + \mu) \|\operatorname{div} u(t)\|^2 - \mu \|\nabla v(t)\|^2 - (\lambda + \mu) \|\operatorname{div} v(t)\|^2 \\
 &\quad - \int_{\Omega} u(t) \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho - \int_{\Omega} v(t) \int_{\tau_1}^{\tau_2} \mu_2(\varrho) y(x, 1, \varrho, t) d\varrho \\
 &\quad + \int_{\Omega} v(t) f_2(u, v) dx + \int_{\Omega} u(t) f_1(u, v) dx.
 \end{aligned} \tag{4.4}$$

By using Hölder's and Young's inequalities, we have

$$\begin{aligned}
 &\int_{\Omega} u(t) \int_{\tau_1}^{\tau_2} \mu_1(\varrho) z(x, 1, \varrho, t) d\varrho dx \\
 &\leq \frac{\varepsilon}{2} \|u(t)\|^2 + \frac{1}{2\varepsilon} \int_{\Omega} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| z(x, 1, \varrho, t) d\varrho \right)^2 dx \\
 &\leq \frac{C\varepsilon}{2} \|\nabla u(t)\|^2 + \frac{1}{2\varepsilon} \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x, 1, \varrho, t)|^2 d\varrho \\
 &\leq \frac{C\varepsilon}{2} \|\nabla u(t)\|^2 + \frac{k_1}{2\varepsilon} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 d\varrho.
 \end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned}
 &\int_{\Omega} v(t) \int_{\tau_1}^{\tau_2} \mu_1(\varrho) y(x, 1, \varrho, t) d\varrho dx \\
 &\leq \frac{C\varepsilon}{2} \|\nabla v(t)\|^2 + \frac{k_2}{2\varepsilon} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |y(x, 1, \varrho, t)|^2 d\varrho.
 \end{aligned} \tag{4.6}$$

Young's and Poincaré's inequalities give

$$k_1 \int_{\Omega} u(t) u_t(t) dx \leq \frac{C\varepsilon}{2} \|\nabla u(t)\|^2 + \frac{k_1^2}{2\varepsilon} \|u_t(t)\|^2 \tag{4.7}$$

and

$$k_2 \int_{\Omega} v(t) \nabla v_t(t) dx \leq \frac{C\varepsilon}{2} \|\nabla v(t)\|^2 + \frac{k_2^2}{2\varepsilon} \|v_t(t)\|^2. \tag{4.8}$$

For the source term, we have

$$\int_{\Omega} u(t) f_1(u, v) dx + \int_{\Omega} v(t) f_2(u, v) dx = (p+1) \int_{\Omega} F(u, v) dx. \tag{4.9}$$

Combining equations (4.4)–(4.9) and taking $\varepsilon = \frac{\mu}{2C}$, thus, our proof is completed. $\square$

Lemma 4.3 Let (u, v, z, y) be a solution of problem (2.8). Then the functional

$$I(t) = \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} \left[|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] dx d\varrho d\rho \quad (4.10)$$

satisfies the estimate

$$\begin{aligned} I'(t) &\leq -e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 dx d\varrho \\ &\quad - e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 dx d\varrho \\ &\quad + k_1 \|u_t(t)\|^2 + k_2 \|v_t(t)\|^2 - I(t). \end{aligned} \quad (4.11)$$

Proof Differentiating (4.10) with respect to t , we get

$$\begin{aligned} \frac{d}{dt} I(t) &= 2 \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} |\mu_1(\varrho)| z(x, \rho, \varrho, t) z_t(x, \rho, \varrho, t) dx d\varrho d\rho \\ &\quad + 2 \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} |\mu_2(\varrho)| y(x, \rho, \varrho, t) y_t(x, \rho, \varrho, t) dx d\varrho d\rho. \end{aligned}$$

By using (2.6)–(2.7), we have

$$\begin{aligned} \frac{d}{dt} I(t) &= - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho \rho} \left[|\mu_1(\varrho)| \frac{\partial}{\partial \rho} |z(x, \rho, \varrho, t)|^2 \right. \\ &\quad \left. + |\mu_2(\varrho)| \frac{\partial}{\partial \rho} |y(x, \rho, \varrho, t)|^2 \right] dx d\varrho d\rho \\ &= - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \left[|\mu_1(\varrho)| \frac{\partial}{\partial \rho} (e^{-\varrho \rho} |z(x, \rho, \varrho, t)|^2) \right. \\ &\quad \left. + |\mu_2(\varrho)| \frac{\partial}{\partial \rho} (e^{-\varrho \rho} |y(x, \rho, \varrho, t)|^2) \right] dx d\varrho d\rho \\ &\quad - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} \left[|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] dx d\varrho d\rho. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt} I(t) &= - \int_{\Omega} \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 dx d\varrho + \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \|u_t(x, \varrho, t)\|^2 \\ &\quad - \int_{\Omega} \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 dx d\varrho + \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \|v_t(x, \varrho, t)\|^2 \\ &\quad - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} \left[|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2 \right] dx d\varrho d\rho. \end{aligned}$$

Since $e^{-\varrho}$ is a decreasing function over (τ_1, τ_2) , the desired estimate (4.11) follows immediately from (2.11). $\square$

Proof of Theorem 4.1 Now, we define the following functional:

$$\mathcal{F}(t) = NE(t) + \varphi(t) + N_2 I(t), \quad (4.12)$$

where N and N_2 are positive constants. It is easy to prove $\mathcal{F}(t)$ and $E(t)$ are equivalent, namely there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 E(t) \leq \mathcal{F}(t) \leq \kappa_2 E(t). \quad (4.13)$$

From Lemmas 4.2 and 4.3, we have, for any $t \geq 0$,

$$\begin{aligned} \mathcal{F}'(t) \leq & -N \left[k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right] \|u_t\|^2 - N \left[k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right] \|v_t\|^2 \\ & + \left(1 + \frac{Ck_1^2}{\mu} \right) \|u_t(t)\|^2 + \left(1 + \frac{Ck_2}{\mu} \right) \|v_t(t)\|^2 - \frac{\mu}{2} \|\nabla u(t)\|^2 - \frac{\mu}{2} \|\nabla v(t)\|^2 \\ & - (\lambda + \mu) \|\operatorname{div} u(t)\|^2 - (\lambda + \mu) \|\operatorname{div} v(t)\|^2 \\ & + \frac{Ck_1}{\mu} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |\nabla z(x, 1, \varrho, t)|^2 d\varrho dx \\ & + \frac{Ck_2}{\mu} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |\nabla y(x, 1, \varrho, t)|^2 d\varrho dx \\ & + (p+1) \int_{\Omega} F(u(t), v(t)) dx \\ & - N_2 e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 dx d\varrho \\ & - N_2 e^{-\tau_2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 dx d\varrho \\ & - N_2 \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} [|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 \\ & + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2] dx d\varrho d\rho \\ & + N_2 k_1 \|u_t(t)\|^2 + N_2 k_2 \|v_t(t)\|^2. \end{aligned} \quad (4.14)$$

Thus

$$\begin{aligned} \mathcal{F}'(t) \leq & - \left(N\sigma_1 - \left(1 + \frac{Ck_1^2}{\mu} \right) - N_2 k_1 \right) \|u_t\|^2 - \left(N\sigma_2 - \left(1 + \frac{Ck_2^2}{\mu} \right) - N_2 k_2 \right) \|v_t\|^2 \\ & - \frac{\mu}{2} \|\nabla u(t)\|^2 - \frac{\mu}{2} \|\nabla v(t)\|^2 - (\lambda + \mu) \|\operatorname{div} u(t)\|^2 - (\lambda + \mu) \|\operatorname{div} v(t)\|^2 \\ & - \left(N_2 e^{-\tau_2} - \frac{Ck_1}{\mu} \right) \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| |z(x, 1, \varrho, t)|^2 d\varrho dx \\ & - \left(N_2 e^{-\tau_2} - \frac{Ck_2}{\mu} \right) \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| |y(x, 1, \varrho, t)|^2 d\varrho dx \\ & + (p+1) \int_{\Omega} F(u(t), v(t)) dx \\ & - N_2 \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} [|\mu_1(\varrho)| |z(x, \rho, \varrho, t)|^2 \\ & + |\mu_2(\varrho)| |y(x, \rho, \varrho, t)|^2] dx d\varrho d\rho, \end{aligned} \quad (4.15)$$

where

$$\sigma_1 = \left[k_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho \right) \right], \quad \sigma_2 = \left[k_2 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \right) \right].$$

First, we take N_2 large such that

$$N_2 e^{-\tau_2} - \frac{Ck_1}{\mu} > 0 \quad \text{and} \quad N_2 e^{-\tau_2} - \frac{Ck_2}{\mu} > 0.$$

We choose $N > 0$ large enough so that

$$N\sigma_1 - \left(1 + \frac{Ck_1^2}{\mu} \right) - N_2 k_1 > 0$$

and

$$N\sigma_2 - \left(1 + \frac{Ck_2^2}{\mu} \right) - N_2 k_2 > 0$$

and $\mathcal{F} \sim E$. Thus we arrive at

$$\mathcal{F}'(t) \leq -\alpha \mathcal{F}(t),$$

which yields, by integration,

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\alpha t}, \quad \forall t \geq 0.$$

The use of $\mathcal{F} \sim E$ gives

$$E(t) \leq \beta e^{-\alpha t}, \quad \forall t \geq 0.$$

□

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