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The mixed problem in the theory of strain gradient thermoelasticity approached with the Lagrange identity

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Abstract

In our paper we address the thermoelasticity theory of the strain gradient. First, we define the mixed problem with initial and boundary data in this context. Then, with the help of an identity of Lagrange type, we prove some uniqueness theorems with regards to the solution of this problem and two theorems with regards to the continuous dependence of solutions on loads and on initial data. We want to highlight that the use of the approach proposed in this work enables obtaining results without recourse to any boundedness assumptions on the coefficients or to any laws of conservation of energy. Also, we do not impose restrictions on thermoelastic coefficients regarding their positive definition.

Keywords: Strain gradient thermoelasticity; Lagrange's identity; Uniqueness; Continuous dependence

1 Introduction

It is known that in the classical theory of elasticity, the internal structure of the bodies is not taken into account. However, the reaction of different kinds of bodies to different actions depends on a significant extent on inner structure of respective material. In order to take into account these kinds of effects, new mathematical models of continuum bodies have been designed. This is where the materials from strain gradient thermoelasticity theory fall.

As we can see in the published papers on this subject, for these media it is characteristic consideration of superior gradients of the displacement vector in main relations. A basic motivation for the emergence of this theory is that it is useful to modeling the micro-scale structures. On the other hand, these configurations are widely used in various concrete situations caused by their important advantages, like small dimensions, high durability, low power consumption, or low manufacturing cost. The initial form of the theory of the strain gradient was proposed by Eshel and Mindlin in [1]. Some other remarkable papers intending to capture the micro-scale structure are those by Mindlin [2], Lam et al. [3], Yang et al. [4]. For example, in [3] a generalized theory which intends to describe the deviatoric stretch, the symmetric rotation gradient tensors, and the dilatation gradient vector

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appears. To this aim in this theory there are included some third order material length scale parameters. The motivation for introduction of these high order derivatives is given by the possibility that some media configurations can be better characterized with the help of these higher gradients.

Let us underline that in the constitutive equations in the strain gradient thermoelasticity theory the second order gradient is contained, of course, along with the first gradient, because both have contributions to dissipation. The strain gradient theory of thermoelasticity is suitable to approach main problems regarding the size effects and to characterize the evolution of chiral elastic materials which include auxenic media, carbon nanotube, bones, porous composite, and honeycomb structures. Some papers in this domain are [5–14]. In [5] Pata and Quintanilla linearized this theory and have presented a uniqueness result for the solution of mixed problem in this context. A study where we find an approach to the problem of increment in the theory of thermoelasticity of this kind of elastic bodies is presented in the paper [6] by Martinez and Quintanilla. For the extended thermodynamics, from the paper [7] of Ciarletta, we can find a solution of Galerkin type for the differential equations and, also, some fundamental solutions for the vibrations of steady type. There are many papers in the theory of elasticity or in the theory of thermoelasticity dedicated to the uniqueness of solutions or/and to continuous dependence results, but we need to say that these results are based almost exclusively on the hypothesis that the tensors of the thermoelastic coefficients are positive definite. In other studies, the uniqueness or continuous dependence of solutions are obtained by using a specific law for the conservation of energy. Green and Lindsay supplemented in [15] the conditions arising from thermoelasticity with some assumptions on positive or negative definiteness in order to prove a uniqueness theorem. We want to consider that our study is the continuation of many studies which are based on the different improvements of the Lagrange identity, of which it is worth mentioning the papers [16–18]. From the studies dedicated to Cesaro means, to uniqueness and to continuous dependence results, we remember [19–21]. Other results for different kinds of micro-structures can be found in the papers [22–32].

In our study we address the mixed problem in the context of strain gradient thermoelasticity in a new manner, namely our approach is based on the identity of Lagrange. So, we can prove that the problem in this context admits at most one solution, and we demonstrate that the solutions to the problem depend continuously on loads, that is, the heat supply and mass forces. Another continuous dependence of solutions result is obtained with respect to the initial data. All the results are deduced in the case of bounded domains from the Euclidean three-dimensional space, but they can be extended without much difficulty in the case of boundless domains, with some restrictions on behavior to infinity. Again, we outline that the results are obtained without recourse to any hypotheses regarding the boundedness of the coefficients or to a law for the conservation of energy. In addition, we avoid using definiteness assumptions on the thermoelastic coefficients.

2 Basic equations

Let us consider a bounded domain D of three-dimensional Euclidean space R^3 which is occupied by the reference configuration of an anisotropic homogeneous linear elastic body from strain gradient thermoelasticity theory. The closure of the regular domain D is denoted by \bar{D} and the boundary of D is ∂D , that is, we have $\bar{D} = D \cup \partial D$. As usual, a fixed system of Cartesian axes is used. The surface ∂D is piecewise smooth, and the components

of the outward unit normal vector to it are denoted by n_i . Letters in boldface are used to designate fields of tensors and vectors. The vector of the displacement \mathbf{v} has the components designated by v_i . By convention, for the material time derivative, a superposed dot is used. The usual summation and differentiation conventions are employed. Summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

When there is no likelihood of confusion, we omit the time argument or/and the spatial argument of a function.

To describe the evolution of such type of media, we consider a vector of displacement having the components v_i and the difference of temperature, denoted by ϑ , which is measured from absolute temperature in the reference state T_0 , which is assumed be a constant. We use two strain tensors of components e_{mn} and μ_{mnr} , respectively, which are also called the kinematic characteristics of the body. These are introduced with the help of the kinematic equations

$$e_{mn} = \frac{1}{2}(v_{m,n} + v_{n,m}), \quad \mu_{mnr} = v_{r,mn}. \tag{1}$$

We also use two tensors of stress, namely the classic stress tensor of components t_{mn} and the hyperstress tensor of components σ_{mnr} , both tensors defined over D . Having the stress tensors and the strain tensors, we can highlight the connections between them through the constitutive equations which, for an anisotropic and homogeneous strain gradient thermoelastic body, have the following form:

$$\begin{aligned} t_{mn} &= a_{mnkl}e_{kl} + b_{mnrkl}\mu_{rkl} - \alpha_{mn}\vartheta, \\ \sigma_{mnr} &= b_{klmnr}e_{kl} + c_{klsmnr}\mu_{kls} - \beta_{mnr}\vartheta, \\ \eta &= \alpha_{mn}e_{mn} + \beta_{mnr}\mu_{mnr} + \frac{a}{T_0}\vartheta - d_m\vartheta_{,m}, \\ q_m &= T_0(d_m\dot{\vartheta} - \kappa_{mn}\vartheta_{,n}). \end{aligned} \tag{2}$$

In our following considerations we use some basic notations and theoretical results in a manner similar to that used by Iesan in his known book [33].

First, the equations of motion in strain gradient thermoelasticity have the general form (see also [33])

$$t_{mn,n} + \sigma_{mnr,nr} + F_m = \rho\ddot{v}_m. \tag{3}$$

The equation of energy is given by

$$\vartheta_0\dot{\eta} = \frac{1}{\rho}q_{m,m} + r. \tag{4}$$

The functions used in the previous equations have the following meanings:

- F_m are the components of the body force per unit volume;
- ρ is the density of mass;
- η is the notation for the entropy;

- r is the supply of heat;
- q_m are the components of the heat flux.

The thermoelastic coefficients a_{mnkl} , b_{mnrkl} , c_{mnrkls} , α_{mn} , β_{mnk} , d_m , a , and κ_{mn} are constants for the characterization of materials from a constitutive point of view, and these obey the following relations of symmetry:

$$\begin{aligned}
 a_{mnkl} &= a_{klmn} = a_{nmkl}, & b_{mnrkl} &= b_{nmrkl} = b_{mnrk}, \\
 c_{mnrkls} &= c_{klsnmr} = c_{nmrks}, & \alpha_{mn} &= \alpha_{nm}, & \beta_{mnr} &= \beta_{nmr}, & \kappa_{mn} &= \kappa_{nm}.
 \end{aligned}
 \tag{5}$$

From Clausius–Duhem inequality, called also the inequality of production of entropy, we can write

$$\kappa_{mn} \xi_m \xi_n \geq 0, \quad \forall \xi_m.$$

In all well-defined points of the set ∂D we consider a surface traction of components t_k and a scalar heat flux denoted by q with the help of notations

$$t_k = t_{kl} n_l, \quad q = q_l n_l,$$

where n_l are the components of the normal vector of ∂D .

Together with differential relations (1)–(4) we introduce a system of initial data of the form

$$v_m(0, x) = a_m(x), \quad \dot{v}_m(0, x) = b_m(x), \quad \vartheta(0, x) = \sigma(x), \quad x \in \bar{D}.
 \tag{6}$$

Also, there are prescribed the following boundary conditions:

$$\begin{aligned}
 v_m &= \bar{v}_m \quad \text{on } [0, t_0) \times \partial D_1, & t_k &= t_{kl} n_l = \bar{t}_k \quad \text{on } [0, t_0) \times \partial D_1^c, \\
 \vartheta &= \bar{\vartheta} \quad \text{on } [0, t_0) \times \partial D_2, & q &= q_l n_l = \bar{q} \quad \text{on } [0, t_0) \times \partial D_2^c,
 \end{aligned}
 \tag{7}$$

where the instant of time t_0 can take the infinite value.

Also, the subsets ∂D_1 and ∂D_2 , respectively ∂D_1^c and ∂D_2^c , are subsurfaces of the set ∂D satisfying the following conditions:

$$\begin{aligned}
 \partial D_1 \cap \partial D_1^c &= \partial D_2 \cap \partial D_2^c = \emptyset, \\
 \partial D_1 \cup \partial D_1^c &= \partial D_2 \cup \partial D_2^c = \partial D.
 \end{aligned}$$

Assume that a_m , b_m , σ , \bar{v}_m , \bar{t}_k , $\bar{\vartheta}$, and \bar{q} are given regular functions on the domain on which they are defined.

In the following considerations we use some hypotheses of regularity as follows:

- (i) the functions which define the thermoelastic coefficients of class C^1 on \bar{D} ;
- (ii) ϱ is a function of class C^0 on \bar{D} ;
- (iii) F_m and r are functions of class C^0 on $[0, t_0) \times \bar{D}$;
- (iv) a_m , b_m , and σ are functions of class C^0 on \bar{D} ;

- (v) \bar{v}_m and $\bar{\vartheta}$ are functions of class C^0 on $[0, t_0] \times \partial D_1$ and $[0, t_0] \times \partial D_2$, respectively;
- (vi) \bar{t}_k and \bar{q} are piecewise regular functions on $[0, t_0] \times \partial D_1^c$ and $[0, t_0] \times \partial D_2^c$, respectively, and are continuous functions in time.

By using the constitutive relations (2), the motion equation (3) and the equation of energy (4) become

$$\begin{aligned}
 & a_{mnkl}v_{k,ln} + b_{mnrkl}v_{k,lnr} + \alpha_{mn}\vartheta_{,n} + b_{mnrkl}v_{k,lnr} \\
 & + c_{mnrks}v_{k,lnrs} + \beta_{mnr}\vartheta_{,nr} + F_m = \varrho\ddot{v}_m, \tag{8} \\
 & (\kappa_{mn}\vartheta_{,n})_{,m} + \varrho r = c\dot{\vartheta} + T_0(\alpha_{mn}\dot{v}_{m,n} + \beta_{mnr}\dot{v}_{r,mm}).
 \end{aligned}$$

Let us denote by \mathcal{P} the problem having initial and boundary data, from the strain gradient theory of thermoelasticity, in the domain $D_0 = [0, t_0] \times D$, which consists of the system of partial differential relations (8) for all $(t, x) \in D_0$, the boundary data (7), and the initial relations (6). Any solution of this problem is an ordered array (v_m, ϑ) .

3 Verification of theorems

Let $v(t, x)$ and $w(t, x)$ be two functions of class C^1 regarding the variable t . By a simple check, we can see that the following equality takes place:

$$\frac{d}{dt} [v(t)\dot{w}(t) - \dot{v}(t)w(t)] = v(t)\ddot{w}(t) - \ddot{v}(t)w(t),$$

where, for simplicity, we have omitted the time variable and the spatial variables of the functions $v(t, x)$ and $w(t, x)$.

In the previous identity, we replace the functions $v(t, x)$ and $w(t, x)$ with the functions $V_m(t, x)$ and $W_m(t, x)$, respectively, considering that the two new functions are also of class C^1 regarding the variable t . If we integrate the resulting equality, we deduce the following equality known as the identity of Lagrange:

$$\begin{aligned}
 & \int_B \varrho(x) [V_m(t, x)\dot{W}_m(t, x) - \dot{V}_m(t, x)W_m(t, x)] dV = \\
 & = \int_D \varrho(x) [V_m(0, x)\dot{W}_m(0, x) - \dot{V}_m(0, x)W_m(0, x)] dV + \tag{9} \\
 & + \int_0^t \int_B \varrho(x) [V_m(\tau, x)\ddot{W}_m(\tau, x) - \ddot{V}_m(\tau, x)W_m(\tau, x)] dV d\tau.
 \end{aligned}$$

We introduce the following notations:

$$\begin{aligned}
 w_m &= v_m^{(2)} - v_m^{(1)}, & \mu &= \vartheta^{(2)} - \vartheta^{(1)} \\
 t_{mn} &= t_{mn}^{(2)} - t_{mn}^{(1)}, & \sigma_{mnk} &= \sigma_{mnk}^{(2)} - \sigma_{mnk}^{(1)}, & S &= \eta^{(2)} - \eta^{(1)}, \tag{10} \\
 p_m &= q_m^{(2)} - q_m^{(1)}, & f_m &= F_m^{(2)} - F_m^{(1)}, & R &= r^{(2)} - r^{(1)},
 \end{aligned}$$

where we denoted by $(v_m^{(\nu)}, \vartheta^{(\nu)})$, $\nu = 1, 2$, two solutions that verify the above problem \mathcal{P} corresponding to the same boundary relations and same initial relations, but to heat supplies and to different body forces, namely $(F_m^{(\nu)}, r^{(\nu)})$, $\nu = 1, 2$, respectively.

The constitutive equations become:

$$\begin{aligned}
 t_{mn} &= a_{mnkl}w_{k,l} + b_{mnrkl}w_{r,kl} + \alpha_{mn}\mu, \\
 \sigma_{mnr} &= b_{klmnr}w_{k,l} + c_{mnrkls}w_{k,ls} + \beta_{mnr}\mu, \\
 \eta &= -\alpha_{mn}v_{m,n} - \beta_{mnr}v_{r,mn} + \frac{a}{T_0}\mu - b_m\mu_{,m}, \\
 p_m &= T_0(b_m\dot{\mu} - \kappa_{mn}\mu_{,n}).
 \end{aligned}
 \tag{11}$$

In this way, we see that the differences (w_m, μ) verify the equations and conditions that follow:

– the equation of motion:

$$\begin{aligned}
 \varrho \ddot{w}_m &= a_{mnkl}w_{k,ln} + b_{mnrkl}w_{r,kln} + \alpha_{mn}\mu_{,n} \\
 &\quad + b_{klmnr}w_{r,klm} + c_{klsmnr}w_{r,nskl} + \beta_{mnr}\mu_{,nr} + f_m;
 \end{aligned}
 \tag{12}$$

– the equation of energy:

$$a\dot{\chi} + \vartheta_0(\alpha_{mn}\dot{w}_{m,n} + \beta_{mnr}\dot{w}_{r,mn}) = (\kappa_{mn}\mu_{,n})_{,m} + \varrho R;
 \tag{13}$$

– the initial conditions:

$$w_m(0, x) = 0, \quad \dot{w}_m(0, x) = 0, \quad \mu(0, x) = 0, \quad x \in \bar{D};
 \tag{14}$$

– the boundary conditions:

$$\begin{aligned}
 w_m(t, x) &= 0 \quad \text{on } [0, t_0) \times \partial D_1, & t_{mk}(t, x)n_k &= 0 \quad \text{on } [0, t_0) \times \partial D_1^c, \\
 \mu(t, x) &= 0 \quad \text{on } [0, t_0) \times \partial D_2, & p_k(t, x)n_k &= 0 \quad \text{on } [0, t_0) \times \partial D_2^c.
 \end{aligned}
 \tag{15}$$

Now, we can find a Lagrange identity corresponding to the difference of two solutions of the mixed initial boundary value problem \mathcal{P} .

Theorem 1 *Let us consider the difference (w_m, χ) of two solutions of the mixed problem \mathcal{P} . Corresponding to this difference, the Lagrange identity receives the form that follows:*

$$\begin{aligned}
 &2 \int_D \varrho w_m(t)\dot{w}_m(t) dV + \int_D \frac{1}{T_0} \kappa_{mn} \left(\int_0^t \mu_{,m}(s) ds \right) \left(\int_0^t \mu_{,n}(s) ds \right) dV \\
 &= \int_0^t d\tau \int_D \varrho [w_m(2t - \tau)f_m(\tau) - w_m(\tau)f_m(2t - \tau)] dV + \\
 &\quad + \int_0^t \int_D \frac{\varrho}{T_0} \left[\mu(\tau) \int_0^{2t-\tau} R(s) ds - \mu(2t - \tau) \int_0^\tau R(s) ds \right] dV d\tau, \quad t \in \left[0, \frac{t_0}{2} \right).
 \end{aligned}
 \tag{16}$$

Proof Clearly, the mixed problem \mathcal{P} is linear such that the difference (w_m, μ) represents also a solution of a mixed initial boundary value problem, which is analogous to the problem \mathcal{P} , but consists of a system of relations (12), (13) with the charges f_m and R , the homogeneous boundary conditions (15), and null initial conditions (14). Identity (9) receives

the following simpler form:

$$2 \int_B \varrho v_m(t) v_m(t) dV = \int_0^t \int_B \varrho [v_m(2t - \tau) \ddot{v}_m(\tau) - \ddot{v}_m(2t - \tau) v_m(\tau)] dV d\tau \tag{17}$$

after using the substitutions

$$V_m(\tau) = w_m(\tau), \quad W_m(\tau) = w_m(2t - \tau), \quad \tau \in [0, 2t], t \in \left[0, \frac{t_0}{2}\right).$$

In order to obtain equality (17) we considered that both the initial data and the boundary conditions are zero.

If we use the motion equations for the respective differences (w_m, μ) , we can substitute the inertial terms which appear in identity (17), in its right-hand side. To this end we consider equation (12) so that we deduce

$$\begin{aligned} & \varrho [w_m(2t - \tau) \ddot{w}_m(\tau) - \ddot{w}_m(2t - \tau) w_m(\tau)] \\ &= \left\{ w_m(2t - \tau) [a_{mnkl} w_{k,l}(\tau) + b_{mnrkl} w_{r,kl}(\tau) + \alpha_{mn} \mu(\tau) \right. \\ & \quad \left. + b_{klmnr} w_{r,kl}(\tau) + c_{mnrkls} w_{r,kls}(\tau) + \beta_{mnr} \mu_{,r}(\tau)] \right\}_{,n} \\ & \quad - \left\{ w_m(\tau) [a_{mnkl} v_{k,l}(2t - \tau) + b_{mnrkl} w_{r,kl}(2t - \tau) + \alpha_{mn} \mu(2t - \tau) \right. \\ & \quad \left. + b_{rklmn} w_{r,kl}(2t - \tau) + c_{mnrkls} w_{r,kls}(2t - \tau) + \beta_{mnr} \mu_{,r}(2t - \tau)] \right\}_{,n} \\ & \quad - a_{mnkl} w_{k,l}(\tau) w_{m,n}(2t - \tau) - b_{mnrkl} w_{r,kl}(\tau) w_{m,n}(2t - \tau) - \alpha_{mn} \mu(\tau) w_{m,n}(2t - \tau) \\ & \quad - b_{klmnr} w_{r,kl}(\tau) w_{m,n}(2t - \tau) - c_{mnrkls} w_{r,kls}(\tau) w_{m,n}(2t - \tau) \\ & \quad - \beta_{mnr} \mu_{,r}(\tau) w_{m,n}(2t - \tau) \\ & \quad + a_{mnkl} w_{k,l}(2t - \tau) w_{m,n}(\tau) + b_{mnrkl} w_{r,kl}(2t - \tau) w_{m,n}(\tau) + \alpha_{mn} \mu(2t - \tau) w_{m,n}(\tau) \\ & \quad + b_{klmnr} w_{r,kl}(2t - \tau) w_{m,n}(\tau) + c_{mnrkls} w_{r,kls}(2t - \tau) w_{m,n}(\tau) \\ & \quad + \beta_{mnr} \mu_{,r}(2t - \tau) w_{m,n}(\tau) \\ & \quad + \varrho [f_m(\tau) w_m(2t - \tau) - f_m(2t - \tau) w_m(\tau)]. \end{aligned}$$

We wish to get a simpler form of the previous equality. In this regard, we use the symmetry relations (5):

$$\begin{aligned} & \varrho [w_m(2t - \tau) \ddot{w}_m(\tau) - \ddot{w}_m(2t - \tau) w_m(\tau)] \\ &= \left\{ w_m(2t - \tau) [a_{mnkl} v_{k,l}(\tau) + b_{mnrkl} w_{r,kl}(\tau) + \alpha_{mn} \mu(\tau) \right. \\ & \quad \left. + b_{klmnr} w_{r,kl}(\tau) + c_{mnrkls} w_{r,kls}(\tau) + \beta_{mnr} \mu_{,r}(\tau)] \right\}_{,n} \\ & \quad - \left\{ w_m(\tau) [a_{mnkl} w_{k,l}(2t - \tau) + b_{mnrkl} w_{r,kl}(2t - \tau) + \alpha_{mn} \mu(2t - \tau) \right. \\ & \quad \left. + b_{klmnr} w_{r,kl}(2t - \tau) + c_{mnrkls} w_{r,kls}(2t - \tau) + \beta_{mnr} \mu_{,r}(2t - \tau)] \right\}_{,n} \\ & \quad + \alpha_{mn} [\mu(2t - \tau) w_{m,n}(\tau) - \mu(\tau) w_{m,r}(2t - \tau)] \\ & \quad + \beta_{mnr} \mu_{,r}(2t - \tau) w_{m,n}(\tau) - \beta_{mnr} \mu_{,r}(\tau) w_{m,n}(2t - \tau) \\ & \quad + \varrho [f_m(\tau) w_m(2t - \tau) - f_m(2t - \tau) w_m(\tau)]. \end{aligned} \tag{18}$$

In this equality we integrate by parts equality (18) on $[0, t] \times D$, so that after we use the theorem of divergence and the null boundary relations (15), we obtain the identity

$$\begin{aligned} & \int_D \varrho [w_m(2t - \tau)\ddot{w}_m(\tau) - \dot{w}_m(2t - \tau)w_m(\tau)] dV \\ &= \int_0^t \int_D [\alpha_{mn}w_{m,n}(\tau) + \beta_{mnr}w_{m,nr}(\tau)]\mu(2t - \tau) dV d\tau \\ & \quad - \int_0^t \int_D [\alpha_{mn}w_{m,n}(2t - \tau) + \beta_{mnr}w_{m,nr}(2t - \tau)]\mu(\tau) dV d\tau \\ & \quad + \int_0^t \int_D \varrho [f_m(\tau)w_m(2t - \tau) - f_m(2t - \tau)w_m(\tau)] dV d\tau. \end{aligned} \tag{19}$$

Let us now integrate the equation of energy (13) on $[0, \tau]$. Taking into account that in (14) the initial data are null, we deduce the equality

$$\begin{aligned} & \alpha_{mn}w_{m,n}(\tau) + \beta_{mnr}w_{m,nr}(\tau) \\ &= \frac{a}{T_0} \mu(\tau) - \frac{1}{T_0} \left(\int_0^\tau \mu_{,n}(\xi) d\xi \right)_{,m} - \frac{\varrho}{T_0} \int_0^\tau \mathcal{P}(z) d\xi, \quad \tau \in [0, t_0]. \end{aligned} \tag{20}$$

In a similar manner, we can get also the equality

$$\begin{aligned} & \alpha_{mn}w_{m,n}(2t - \tau) + \beta_{mnr}w_{m,nr}(2t - \tau) \\ &= \frac{a}{T_0} \mu(2t - \tau) \\ & \quad - \frac{1}{T_0} \left(\int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right)_{,m} - \frac{\varrho}{T_0} \int_0^{2t-\tau} \mathcal{P}(\xi) d\xi, \quad \tau \in [0, t_0]. \end{aligned} \tag{21}$$

Now, we multiply identity (20) by $\mu(2t - \tau)$ and identity (21) by $\mu(\tau)$, then we add term by term the equalities that are obtained. We get a new equality that we integrate on $[0, t] \times D$ so that we obtain the equality

$$\begin{aligned} & \int_0^t \int_D [\alpha_{mn}w_{m,n}(\tau) + \beta_{mnr}w_{m,nr}(\tau)]\mu(2t - \tau) dV d\tau \\ & \quad - \int_0^t \int_D [\alpha_{mn}w_{m,n}(2t - \tau) + \beta_{mnr}w_{m,nr}(2t - \tau)]\mu(\tau) dV d\tau \\ &= \int_D \frac{1}{T_0} \left[\kappa_{mn}\mu_{,m}(2t - \tau) \int_0^\tau \mu_{,n}(\xi) d\xi - \kappa_{mn}\mu_{,m}(\tau) \int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right] dV \\ & \quad + \int_D \frac{\varrho}{T_0} \left[\mu(\tau) \int_0^{2t-\tau} R(\xi) d\xi - \mu(2t - \tau) \int_0^\tau R(\xi) d\xi \right] dV. \end{aligned} \tag{22}$$

Considering (22), from (19) we deduce

$$\begin{aligned} & \int_D \varrho [w_m(2t - \tau)\ddot{w}_m(\tau) - \dot{w}_m(2t - \tau)w_m(\tau)] dV \\ &= \int_D \frac{1}{T_0} \left[\kappa_{mn}\mu_{,m}(2t - \tau) \int_0^\tau \mu_{,n}(\xi) d\xi - \kappa_{mn}\mu_{,m}(\tau) \int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right] dV \end{aligned} \tag{23}$$

$$\begin{aligned}
 &+ \int_D \frac{\varrho}{T_0} \left[\mu(\tau) \int_0^{2t-\tau} R(\xi) d\xi - \mu(2t-\tau) \int_0^\tau R(\xi) d\xi \right] dV \\
 &+ \int_0^t \int_D \varrho [f_m(\tau) w_m(2t-\tau) - f_i(2t-\tau) w_m(\tau)] dV d\tau.
 \end{aligned}$$

Since the tensor κ_{mn} is symmetrical, we obtain

$$\begin{aligned}
 &\int_0^t \int_D \frac{1}{T_0} \kappa_{mn} \frac{d}{d\tau} \left[\left(\int_0^\tau \mu_{,m}(\xi) d\xi \right) \left(\int_0^{2t-\tau} \mu_{,j}(\xi) d\xi \right) \right] dV d\tau \\
 &= \int_D \frac{1}{T_0} \kappa_{mn} \int_0^t \frac{d}{d\tau} \left[\left(\int_0^\tau \mu_{,m}(\xi) d\xi \right) \left(\int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right) \right] d\tau dV \tag{24} \\
 &= \int_D \frac{1}{T_0} \kappa_{mn} \left(\int_0^t \mu_{,m}(\xi) d\xi \right) \left(\int_0^t \mu_{,n}(\xi) d\xi \right) dV,
 \end{aligned}$$

and this identity can be rewritten in the following form:

$$\begin{aligned}
 &\int_0^t d\tau \int_D \frac{1}{T_0} \kappa_{mn} \left[\mu_{,m}(\tau) \int_0^{2t-\tau} \mu_{,n}(\xi) d\xi - \mu_{,n}(2t-\tau) \int_0^\tau \mu_{,m}(\xi) d\xi \right] dV \\
 &= \int_0^t d\tau \int_D \frac{1}{T_0} \kappa_{mn} \frac{d}{d\tau} \left[\left(\int_0^\tau \mu_{,m}(\xi) d\xi \right) \left(\int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right) \right] dV \tag{25}
 \end{aligned}$$

after we performed integration by parts.

After using the derivation of the integral with the parameter in the integral from equality (25), we obtain

$$\begin{aligned}
 &\int_0^t \int_D \frac{1}{T_0} \kappa_{mn} \left[\mu_{,m}(2t-\tau) \int_0^\tau \mu_{,m}(\xi) d\xi - \mu_{,i}(2t-\tau) \int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right] dV d\tau \\
 &= - \int_D \frac{1}{T_0} \kappa_{mn} \left[\left(\int_0^\tau \mu_{,m}(\xi) d\xi \right) \left(\int_0^{2t-\tau} \mu_{,n}(\xi) d\xi \right) \right] dV. \tag{26}
 \end{aligned}$$

Considering (26), from (23) we obtain equality (16), and so the proof of Theorem 1 is finished. □

Although it is an auxiliary result, identity (16) of Theorem 1 is very important because based on it we prove all the results of our study, both the one ensuring that the solution to the problem is unique and the three results regarding the continuous dependence of the solution. First application of identity (16) is the result of uniqueness, from the next theorem, with regards to the solution of the mixed initial boundary value problem \mathcal{P} . For this we need to suppose that the tensor of conductivity κ_{mn} is positive definite, that is, we have

$$\kappa_{mn} x_m x_n \geq k_0 x_m x_m, \quad \forall x_m,$$

where $k_0 > 0$ is a constant.

Theorem 2 *Let us suppose that the relations of symmetry (5) are fulfilled and the set ∂D_2 is not empty or the specific heat $a(x)$ is nonzero on B . Thus, the mixed problem \mathcal{P} in strain gradient thermoelasticity has at most one solution.*

Proof Assume that the above problem \mathcal{P} is satisfied by two supposed solutions $(v_m^{(v)}, \vartheta^{(v)})$, $v = 1, 2$, corresponding to the same data to the limit, the identical initial conditions, the identical heat supply, and the identical body force.

We use the notations

$$w_m = v_m^{(2)} - v_m^{(1)}, \quad \mu = \vartheta^{(2)} - \vartheta^{(1)}, \tag{27}$$

and the proof will end if we show that

$$w_m(t, x) = 0, \quad \mu(t, x) = 0, \quad \forall (t, x) \in [0, t_0) \times B. \tag{28}$$

Taking into account the linearity of the problem \mathcal{P} , the above differences (w_m, μ) also satisfy this problem in the special case in which the heat supply is null and the body force is zero. In this particular case, equality (16) can be rewritten in the following simpler form:

$$2 \int_D \varrho w_m(t) \dot{w}_m(t) dV + \int_D \frac{1}{T_0} \kappa_{mm} \left(\int_0^t \mu_{,i}(\xi) d\xi \right) \left(\int_0^t \mu_{,j}(\xi) d\xi \right) dV = 0,$$

which, after integration on $[0, t]$, $t \in [0, t_0/2)$, becomes

$$\int_D \varrho w_m(t) w_m(t) dV + \int_0^t \int_D \frac{1}{T_0} \kappa_{mm} \left(\int_0^s \mu_{,m}(\xi) d\xi \right) \left(\int_0^s \mu_{,m}(\xi) d\xi \right) dV ds = 0.$$

Considering that the tensor κ_{mm} is positive defined and $\varrho > 0$, from the above identity we deduce

$$w_m(t, x) = 0, \quad \mu_{,m}(t, x) = 0, \quad \forall (t, x) \in [0, t_0/2) \times D. \tag{29}$$

If the surface ∂B_2 is not empty, based on the boundary data (7), from (29) we obtain that statements (28) are true. In the case $a(x) \neq 0$, the energy equation (for the above differences) leads to the conclusion that $\dot{\mu} = 0$. But the initial value of μ is zero, such that we have again that (28) holds true.

If the time t_0 were infinite, the proof of Theorem 2 would be ready. In the case t_0 is finite, the above considerations can be repeated, but on the interval $[t_0/2, t_0/2 + t_0/4]$, by setting

$$w_m \left(\frac{t_0}{2}, x \right) = \dot{w}_m \left(\frac{t_0}{2}, x \right) = 0, \quad \mu \left(\frac{t_0}{2}, x \right) = 0.$$

In this way we obtain again conclusion (28), but on the interval $B \times [0, 3t_0/4)$. The procedure can be repeated as many times as possible, each time by halving the interval.

At long last, we can conclude that conclusions (28) are true on $[0, t_0) \times D$ and the proof of Theorem 2 ends. \square

In our next theorem, we prove the first result regarding the continuous dependence of solutions of the problem \mathcal{P} in relation to heat supply and body force on the compact sub-intervals of the interval $[0, t_0)$.

To this aim, we consider two solutions $(v_m^{(\nu)}, \vartheta^{(\nu)})$, $\nu = 1, 2$, of the mixed problem \mathcal{P} , corresponding to the same boundary data and the same initial conditions, but to different heat supply and different body force, namely $(F_m^{(\nu)}, r^{(\nu)})$, $\nu = 1, 2$. We use the notations

$$f_m = F_m^{(2)} - F_m^{(1)}, \quad R = r^{(2)} - r^{(1)}.$$

Theorem 3 *Suppose that the relations of symmetry (5) take place. Assuming there exist the constants Q_1, Q_2, M_1 , and M_2 and there exists $t^* \in (0, t_0)$ such that*

$$\begin{aligned} \int_0^{t^*} \int_D \varrho w_m(t) w_m(t) dV dt &\leq Q_1^2, & \int_0^{t^*} \int_D \frac{\varrho}{T_0} \mu^2(t) dV dt &\leq Q_2^2, \\ \int_0^{t^*} \int_D \varrho f_m(t) f_m(t) dV dt &\leq M_1^2, & \int_0^{t^*} \int_D \frac{\varrho}{T_0} \left(\int_0^t R(\xi) d\xi \right)^2 dV dt &\leq M_2^2, \end{aligned} \tag{30}$$

we deduce the following estimation:

$$\begin{aligned} &\int_D \varrho w_m(\tau) w_m(\tau) dV + \int_0^\tau \int_D \frac{1}{\vartheta_0} \kappa_{mn} \left(\int_0^s \mu_{,m}(\xi) d\xi \right) \left(\int_0^s \mu_{,n}(\xi) d\xi \right) dV ds \\ &\leq t^* Q_1 \left[\int_0^{t^*} \int_D \varrho f_m(s) f_m(s) dV ds \right]^{1/2} \\ &\quad + t^* Q_2 \left[\int_0^{t^*} \int_D \frac{\varrho}{T_0} \left(\int_0^s R(\xi) d\xi \right)^2 dV ds \right]^{1/2}, \end{aligned} \tag{31}$$

where $\tau \in [0, t^*/2]$ and $w_m(\tau)$ and $\mu(\tau)$ are the differences defined in (27).

Proof As we anticipated, the proof is based on identity (16). With the help of the Schwarz inequality, we deduce some estimations for each integral which appears in the right-hand side of this identity.

For example, the following estimates are easy to follow:

$$\begin{aligned} &\int_0^t \int_D \varrho w_m(2t - \tau) f_m(\tau) dV d\tau \\ &\leq \left[\int_0^t \int_D \varrho f_m(\tau) f_m(\tau) dV d\tau \right]^{1/2} \left[\int_0^t \int_D \varrho w_m(2t - \tau) w_m(2t - \tau) dV d\tau \right]^{1/2} \\ &= \left[\int_0^t \int_D \varrho f_m(\tau) f_m(\tau) dV d\tau \right]^{1/2} \left[\int_t^{2t} \int_B \varrho w_m(\tau) w_m(\tau) dV d\tau \right]^{1/2} \\ &\leq Q_1 \left[\int_0^{t^*} \int_D \varrho f_m(\tau) f_m(\tau) dV d\tau \right]^{1/2}, \end{aligned}$$

where, to get the last row, we made the transformation of variable $2t - \tau \rightarrow \tau$.

The other integrals which appear in identity (16) in its right-hand part are approached in a similar manner. All the inequalities obtained in this way are added together, a new inequality that is integrated over $[0, \tau]$, $\tau \in [0, t^*/2]$ is obtained and, finally, we are led to inequality (31), and this ends the demonstration of Theorem 3. \square

Remark Inequality (31) has a double importance.

First, it proves the continuous dependence of the solutions of the mixed problem \mathcal{P} with respect to the loads.

Second, this inequality underlies the proof of the following theorem, which addresses another type of continuous dependence, namely with respect to the initial data.

To this aim, we consider two solutions of the mixed problem \mathcal{P} :

$$(v_m^{(1)}, \vartheta^{(1)}), \quad (v_m^{(1)} + w_m, \vartheta^{(1)} + \mu)$$

corresponding to the same boundary conditions and to the same heat supply and body force, but to different initial conditions, namely

$$(v_m^{(1)}(0), \dot{v}_m^{(1)}(0), \vartheta^{(1)}(0)), \quad (v_m^{(1)}(0) + \alpha_m^0, \dot{v}_m^{(1)}(0) + \beta_m^0, \vartheta^{(1)}(0) + \delta^0),$$

where the perturbations $(\alpha_m^0, \beta_m^0, \delta_m^0)$ satisfy the following conditions: there exist the constants M_3 and M_4 such that

$$\int_D \varrho (\alpha_m^0 \alpha_m^0 + \beta_m^0 \beta_m^0) dV \leq M_3^2, \quad \int_B \frac{T_0}{\varrho} \eta_0^2 dV \leq M_4^2,$$

where we denote by η_0 the following expression:

$$\eta_0 = \frac{a}{T_0} \delta^0 - \alpha_{mn} \alpha_{n,m}^0 - \beta_{mnk} \beta_{k,mn}^0.$$

With the help of the perturbation w_m and μ , we introduce the functions $V_m(t)$ and $\Theta(t)$ by means of the notations

$$V_m(t) = \int_0^t \int_0^s w_m(\tau) d\tau ds, \quad \Theta(t) = \int_0^t \int_0^s \mu(\tau) d\tau ds. \tag{32}$$

Theorem 4 *Suppose that symmetry relations (5) take place and the functions (V_m, Θ) satisfy restrictions (30). Then the following estimate takes place:*

$$\begin{aligned} & \int_D \varrho V_m(t) V_m(t) dV + \int_0^t \int_D \frac{1}{T_0} \kappa_{mn} \left(\int_0^s \Theta_{,m}(\xi) d\xi \right) \left(\int_0^s \Theta_{,n}(\xi) d\xi \right) dV ds \\ & \leq t^* Q_1 \left[\left(t^* + \frac{t^{*2}}{2} \right) \int_D \varrho \alpha_m^0 \alpha_m^0 dV + \left(\frac{t^{*2}}{2} + \frac{t^{*3}}{3} \right) \int_D \varrho \beta_m^1 \beta_m^1 dV \right]^{1/2} \\ & + t^{*7/2} Q_2 \frac{1}{\sqrt{20}} \left(\int_D \frac{T_0}{\varrho} \eta_0^2 dV \right)^{1/2}, \quad t \in \left[0, \frac{t^*}{2} \right]. \end{aligned} \tag{33}$$

Proof First, it is easy to obtain from (32) that

$$V_m(t) = \int_0^t (t - \tau) w_m(\tau) d\tau, \quad \Theta(t) = \int_0^t (t - \tau) \mu(\tau) d\tau$$

after integration by parts.

On the other hand, the difference functions (w_m, μ) verify the motion equations and the energy equation in the form from (8), but in the particular case of zero loads

$$f_m = 0, \quad r = 0.$$

The initial conditions satisfied by the difference functions have the form

$$w_m(0) = \alpha_m^0, \quad \dot{w}_m(0) = \beta_m^0, \quad \mu(0) = \delta^0.$$

It is not difficult to prove that the functions (V_m, Θ) , introduced in relations (32), satisfy the equations of motion and the energy equation exactly of the form of the equations from system (8), but the heat supply and the body force have the following form:

$$r(t) = \frac{T_0}{Q} t \left[\frac{a}{T_0} \delta^0 - \alpha_{mn} \alpha_{m,n}^0 - \beta_{mnk} \beta_{k,nn}^0 \right],$$

$$f_m(t) = \alpha_m^0 + t \beta_m^0.$$

In view of these clarifications, we can obtain estimate (32) directly from (31) such that the proof of Theorem 4 is finished. \square

4 Conclusions

It is worth noting that our main results with regards to the uniqueness of solution and with regards to the continuous dependence of solutions were obtained without recourse to any assumptions with respect to the boundedness of thermoelastic coefficients. We also did not resort to any conservation law to demonstrate these qualitative theorems. In many previous papers, any qualitative theorems regarding the solutions of some mixed problems of the form of \mathcal{P} , such as existence, uniqueness, continuous dependence, or stability of solutions, have been obtained based on some strong restrictions. One of the most used restrictions is the imposition of the internal energy to be positively defined, a condition that we have not used in any of the results mentioned.

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Authors' contributions

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References

1. Mindlin, R.D., Eshel, N.N.: On first strain-gradient theories in linear elasticity. *Int. J. Solids Struct.* **4**(1), 109–124 (1968)
2. Mindlin, R.D.: Second gradient of strain and surface-tension in linear elasticity. *Int. J. Solids Struct.* **1**(4), 417–438 (1965)
3. Lam, D.C., Yang, F., Chong, A.C.M., Wang, J., Tong, P.: Experiments and theory in strain gradient elasticity. *J. Mech. Phys. Solids* **51**(8), 1477–1508 (2003)
4. Yang, F., Chong, A.C.M., Lam, D.C., Tong, P.: Couple stress based strain gradient theory for elasticity. *Int. J. Solids Struct.* **39**(10), 2731–2743 (2002)
5. Pata, V., Quintanilla, R.: On the decay of solutions in nonsimple elastic solids with memory. *J. Math. Anal. Appl.* **363**, 19–28 (2010)
6. Martinez, F., Quintanilla, R.: On the incremental problem in thermoelasticity of nonsimple materials. *Z. Angew. Math. Mech.* **78**(10), 703–710 (1998)
7. Ciarletta, M.: Thermoelasticity of nonsimple materials with thermal relaxation. *J. Therm. Stresses* **19**(8), 731–748 (1996)
8. Ahmadi, G., Firoozbaksh, K.: First strain-gradient theory of thermoelasticity. *Int. J. Solids Struct.* **11**, 339–345 (1975)
9. Aouadi, M., El Dhaba, A.R., Ghaleb, A.F.: Stability aspects in strain gradient theory of thermoelasticity with mass diffusion. *Z. Angew. Math. Mech.* **98**(10), 1794–1812 (2018)
10. Grot, R.: Thermodynamics of a continuum with microstructure. *Int. J. Eng. Sci.* **7**, 801–814 (1969)
11. Toupin, R.A.: Theories of elasticity with couple-stress. *Arch. Ration. Mech. Anal.* **17**, 85–112 (1964)
12. Green, A.E., Rivlin, R.S.: Multipolar continuum mechanics. *Arch. Ration. Mech. Anal.* **17**, 113–147 (1964)
13. Mindlin, R.D.: Micro-structure in linear elasticity. *Arch. Ration. Mech. Anal.* **16**, 51–78 (1964)
14. Wilkes, N.S.: Continuous dependence and instability in linear thermoelasticity. *SIAM J. Appl. Math.* **11**, 292–299 (1980)
15. Green, A.E., Lindsay, K.A.: Thermoelasticity. *J. Elast.* **2**, 1–7 (1972)
16. Rionero, S., Chirita, S.: Lagrange identity method in linear thermoelasticity. *Int. J. Eng. Sci.* **25**, 935–946 (1987)
17. Chirita, S.: Some applications of the Lagrange identity in thermoelasticity with one relaxation time. *J. Therm. Stresses* **11**(3), 207–231 (1988)
18. Marin, M.: Some basic theorems in elastostatics of micropolar materials with voids. *J. Comput. Appl. Math.* **70**(1), 115–126 (1996)
19. Day, W.A.: Means and autocorrections in elastodynamics. *Arch. Ration. Mech. Anal.* **73**, 243–256 (1980)
20. Knops, R.J., Payne, L.E.: On uniqueness and continuous dependence in dynamical problems of linear thermoelasticity. *Int. J. Solids Struct.* **6**, 1173–1184 (1970)
21. Marin, M.: Cesaro means in thermoelasticity of dipolar bodies. *Acta Mech.* **122**(1–4), 155–168 (1997)
22. Marin, M.: On existence and uniqueness in thermoelasticity of micropolar bodies. *C. R. Acad. Sci. Paris, Ser. II, B* **321**(12), 375–480 (1995)
23. Marin, M., Stan, G.: Weak solutions in elasticity of dipolar bodies with stretch. *Carpath. J. Math.* **29**(1), 33–40 (2013)
24. Abd-Elaziz, E.M., Marin, M., Othman, M.I.A.: On the effect of Thomson and initial stress in a thermo-porous elastic solid under GN electromagnetic theory. *Symmetry* **11**(3), 413 (2019)
25. Itu, C., Öchsner, A., et al.: Improved rigidity of composite circular plates through radial ribs. *Proc. Inst. Mech. Eng., Part L J. Mater.-Des. Appl.* **233**(8), 1585–1593 (2019)
26. Marin, M.: On the minimum principle for dipolar materials with stretch. *Nonlinear Anal., Real World Appl.* **10**(3), 1572–1578 (2009)
27. Sharma, K., Marin, M.: Reflection and transmission of waves from imperfect boundary between two heat conducting micropolar thermoelastic solids. *An. Ştiinţ. Univ. 'Ovidius' Constanţa* **22**(2), 151–175 (2014)
28. Vlaşe, S., Marin, M., et al.: Motion equation for a flexible one-dimensional element used in the dynamical analysis of a multibody system. *Contin. Mech. Thermodyn.* **31**(3), 715–724 (2019)
29. Vlaşe, S., Teodorescu, P.P.: Elasto-dynamics of a solid with a general rigid motion using FEM model. Part I: theoretical approach. *Rom. J. Phys.* **58**(7–8), 872–881 (2013)
30. Singh, A., Das, S., Craciun, E.M.: The effect of thermo-mechanical loading on the edge crack of finite length in an infinite orthotropic strip. *Mech. Compos. Mater.* **55**(3), 285–296 (2019)
31. Kumar, S., Pandey, P., Das, S., Craciun, E.M.: Numerical solution of two dimensional reaction-diffusion equation using operational matrix method based on Genocchi polynomial. Part I: Genocchi polynomial and operational matrix. *Proc. Rom. Acad., Ser. A* **20**(4), 393–399 (2019)
32. Singh, A., Das, S., Craciun, E.M.: Thermal stress intensity factor for an edge crack in orthotropic composite media. *Composites, Part B* **153**(15), 130–136 (2018)
33. Iesan, D.: *Thermoelastic Models of Continua*. Kluwer Academic, Dordrecht (2004)

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