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How far does logistic dampening influence the global solvability of a high-dimensional chemotaxis system?

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Abstract

This paper deals with the homogeneous Neumann boundary value problem for chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + \kappa u - \mu u^{\alpha}, & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N (N \geq 2)$, where $\alpha > 1$ and $\kappa \in \mathbb{R}, \mu > 0$ for suitably regular positive initial data.

When $\alpha \geq 2$, it has been proved in the existing literature that, for any $\mu > 0$, there exists a weak solution to this system. We shall concentrate on the weaker degradation case: $\alpha < 2$. It will be shown that when $N < 6$, any sublinear degradation is enough to guarantee the global existence of weak solutions. In the case of $N \geq 6$, global solvability can be proved whenever $\alpha > \frac{4}{3}$. It is interesting to see that once the space dimension $N \geq 6$, the qualified value of α no longer changes with the increase of N.

MSC: 35Q30; 35Q92; 92C17; 35K55; 35B65

Keywords: Chemotaxis; Logistic source; Global weak solution

1 Introduction

Chemotaxis is a characteristic of organisms that move toward an environment conducive to their own growth. A pioneering mathematical model for chemotaxis was proposed by *Keller* and *Segel* in 1970s [1]. The so-called Keller–Segel minimal model consists of two equations of the form:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1}$$

where u and v denote the cell density and chemosignal concentration, respectively. It has been shown that all the solutions to the homogenous Neumann initial-boundary value problem associated with (1) in $\Omega \subset \mathbb{R}^N (N \in \mathbb{N})$ are bounded when either $N = 1$ or $N = 2$

and total mass $\int_\Omega u_0$ is small [2, 3], while some finite/infinite-time blow-up may occur when $N \geq 3$ or $N = 2$ and $\int_\Omega u_0 = m > 0$ is large [4–6].

The classical Keller–Segel model only considers the diffusion and chemotaxis of cells. However, in the actual biological context, the reproduction and death of cells or population themselves need to be considered. A prototypical choice to achieve this is the logistic type source $\kappa u - \mu u^\alpha$ with birth and death rates κ and μ, respectively. In the past decades, the homogeneous Neumann initial boundary value problem of the following Keller–Segel system with logistic source has been widely investigated:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \kappa u - \mu u^\alpha, & x \in \Omega, t > 0, \\ v_t = \Delta v + u - v, & x \in \Omega, t > 0, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded smooth domain, $\kappa \geq 0$, $\mu > 0$, $\alpha > 1$. The presence of logistic source has been shown to have an effect of blow-up prevention. When $\alpha = 2$, if the spatial dimension $N \leq 2$, the system with nonnegative regular initial data only allows for global and uniformly bounded solutions even for arbitrarily small $\mu > 0$ [2, 7, 8]. In the higher dimension setting, it has been proved in [9] that if μ is large enough, for all sufficiently smooth and nonnegative initial data, the problem possesses a unique bounded and global classical solution (we also refer to [10–14] and the references therein for more recent and specific results). Some recent studies show that the presence of weaker degradation terms may also exhibit significant relaxation effects in comparison with (1). Actually, by resorting to some weaker notions of solvability, global solutions have recently been constructed for systems merely containing certain subquadratic degradation terms, where it is required that $\alpha \geq 2 - \frac{1}{N}$ when $N \geq 2$ [15, 16] and even the wider ranges $\alpha > \frac{2N+4}{N+4}$ [17]. However, despite the presence of superlinear degradation, some unboundedness phenomena have been detected in the literature for problems of type (2) and certain parabolic-elliptic versions. Even in some situations in which solutions are known to remain bounded globally, such as e.g. in the quadratic case $\alpha = 2$, certain results on spontaneous emergence of arbitrarily large densities have been reported when there is a diffusion coefficient before Δu in the first equation of (2) and the diffusion coefficient is small. See [18–20] for parabolic-elliptic case and [21] for parabolic-parabolic case, respectively. On the other hand, it has been proved in [22] that the solutions of parabolic-elliptic model, in which the second equation of (2) is replaced by $0 = \Delta v - v + u$, exhibits a finite-time blow-up phenomenon under the condition $\alpha < \frac{7}{6}$ when $N \in \{3, 4\}$ or $\alpha < 1 + \frac{1}{2(N+1)}$ when $N \geq 5$. Some similar results were also derived for the simplified version by replacing the second equation with $0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega u(x,t)\,dx + u$ in [23, 24].

Apart from (2), another typical chemotaxis system is the following consumption-type model:

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) + \kappa u - \mu u^\alpha, & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0. \end{cases} \tag{3}$$

The reaction term $-uv$ in the second equation indicates that the cells consume chemicals during the overall chemotaxis process. In contrast to the models with production mechanism (1), (2), the chemoattractant consumption mechanism in this system is more prone to the global existence of solutions due to the fact that the second equation immediately provides an L^{∞}-bound for v. However, such a bound is not sufficient for dealing with the chemotaxis term. Actually, global existence and boundedness of solutions to (3) with $\kappa = \mu = 0$ are only known under the smallness condition $\|v(\cdot,t)\|_{L^{\infty}(\Omega)} \le \frac{1}{6(N+1)}$ [25] or in a two-dimensional setting [26, 27]. The three-dimensional version admits a global weak solution which after some waiting time eventually becomes classical [28].

When the logistic dampening is in consideration, in the high-dimensional setting, the global solvability seems still restricted to the quadratic degradation case ($\alpha = 2$) [29]. When the logistic-type degradation is weaker, that is, $\alpha < 2$, the global existence result obtained so far concentrates on the small-data solutions [30]. *As for the global existence of arbitrarily large initial data solutions to (4) in the sub-qudratic case* $1 < \alpha < 2$, *to the best of our knowledge, it still remains unknown.* In this short paper, we shall do some work and give a definite answer in this respect. We shall include the case of any high-dimensional domain.

Precisely, we will consider the problem

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) + \kappa u - \mu u^{\alpha}, & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where ν is the unit outer normal vector on the boundary. We are interested which α can guarantee the global solvability of this system even in the high-dimensional case.

To formulate our main results, we assume throughout that the initial data satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) & \text{is nonnegative with } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative.} \end{cases} \tag{5}$$

Our main result can be read as follows.

Theorem 1.1 *Let $N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that $\kappa \in \mathbb{R}$, $\mu > 0$ and that*

$$\begin{cases} \alpha > 1 & \text{if } N < 6, \\ \alpha > \frac{4}{3} & \text{if } N \ge 6. \end{cases} \tag{6}$$

Then, for any choice of initial data u_0, v_0 fulfilling (5), *one can find functions $u \in L^1_{\mathrm{loc}}([0,\infty); L^1(\Omega))$, $v \in L^1_{\mathrm{loc}}([0,\infty); W^{1,1}(\Omega))$ such that $u \ge 0$ and $v \ge 0$ a.e. in $\Omega \times (0,\infty)$, and that (u,v) forms a global weak solution of* (4) *in the sense of Definition* 1.2.

Our result shows that when the spatial dimension is smaller than 6, any superlinear degradation is enough to guarantee the global existence of solutions to system (4), and that if the domain dimension is no less than 6, the qualified value of α is bigger than $\frac{4}{3}$.

We define the weak solution in the natural way as follows.

Definition 1.2 A pair of functions (u,v) is called a global weak solution of (4) if $u \in L^1_{\mathrm{loc}}([0,\infty);L^1(\Omega))$, $v \in L^1_{\mathrm{loc}}([0,\infty);W^{1,1}(\Omega))$ such that uv and $u\nabla v \in L^1_{\mathrm{loc}}([0,\infty);L^1(\Omega))$ and the following integral equalities hold:

$$-\int_0^\infty \int_\Omega u\xi_t - \int_\Omega u_0\xi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u\cdot\nabla\xi + \int_0^\infty \int_\Omega u\nabla v\cdot\nabla\xi + \kappa\int_0^\infty \int_\Omega \xi u - \mu\int_0^\infty \int_\Omega \xi u^\alpha, \tag{7}$$

$$-\int_0^\infty \int_\Omega v\xi_t - \int_\Omega v_0\xi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v\cdot\nabla\xi - \int_0^\infty \int_\Omega uv\xi \tag{8}$$

for all $\xi \in C_0^\infty(\Omega\times[0,\infty))$.

The rest of this paper is organized as follows. In Sect. 2, we establish the global existence of a family of approximate problems to system (4). Section 3 is devoted to the estimates of the time derivatives for the approximate problems. Finally, we obtain the global weak solution of our problem by an approximate procedure in Sect. 4.

2 Global existence in the approximate systems

In order to suitably regularize the original problem (4) for $\varepsilon \in (0,1)$, let us firstly consider a family of approximate systems:

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla\cdot(\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\nabla v_\varepsilon) + \kappa u_\varepsilon - \mu u_\varepsilon^\alpha, & x\in\Omega, t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}, & x\in\Omega, t>0,\\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u_\varepsilon(x,0) = u_0(x), \qquad v_\varepsilon(x,0) = v_0(x), & x\in\Omega. \end{cases} \tag{9}$$

All the above approximate problems admit local-in-time smooth solutions.

Lemma 2.1 *Let u_0, v_0 satisfy (5), let $\kappa \in \mathbb{R}$, $\mu > 0$, and $q > N$. Then, for each $\varepsilon \in (0,1)$, there exist $0 < T_{\max,\varepsilon} \le +\infty$ and uniquely determined functions*

$$\begin{cases} u_\varepsilon \in C^0([\overline{\Omega}\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\varepsilon})),\\ v_\varepsilon \in \bigcap_{q>N} C^0([0,T_{\max,\varepsilon});W^{1,q}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max,\varepsilon})), \end{cases} \tag{10}$$

which are such that $u_\varepsilon \ge 0$ and $v_\varepsilon \ge 0$ in $\overline{\Omega}\times(0,T_{\max,\varepsilon})$, and the pair $(u_\varepsilon, v_\varepsilon)$ solves (9) classically in $\Omega\times(0,T_{\max,\varepsilon})$. Moreover, if $T_{\max,\varepsilon} < \infty$, then

$$\limsup_{t\nearrow T_{\max,\varepsilon}}\left(\left\|u_\varepsilon(\cdot,t)\right\|_{L^\infty(\Omega)} + \left\|v_\varepsilon(\cdot,t)\right\|_{W^{1,q}(\Omega)}\right) = \infty. \tag{11}$$

Proof The proof follows the reasoning of [31], Lemma 3.1 (see also [26], Lemma 2.1). □

In contrast to the situation without source terms, we cannot hope for mass conservation in the first component. Nevertheless, the following result still holds (see also other works involving logistic source e.g. [32–34]).

Lemma 2.2 *Let u_0, v_0 satisfy* (5), *let $\kappa \in \mathbb{R}$, $\mu > 0$. Then, for any $\varepsilon \in (0,1)$, the solution of* (9) *satisfies*

$$\int_\Omega u_\varepsilon(t) \le \max\left\{\int_\Omega u_0(x), \left(\frac{\kappa_+}{\mu}\right)^{\frac{1}{\alpha-1}}|\Omega|\right\} \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{12}$$

$$\|v_\varepsilon(t)\|_{L^\infty(\Omega)} \le \|v_0(x)\|_{L^\infty(\Omega)} =: v_\infty \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{13}$$

where $\kappa_+ = \max\{0, \kappa\}$.

Proof By Hölder's inequality, $(\int_\Omega u_\varepsilon)^\alpha \le (\int_\Omega u_\varepsilon^\alpha)|\Omega|^{\alpha-1}$. An integration of the first equation in (9) yields

$$\begin{aligned}\left(\int_\Omega u_\varepsilon\right)_t &\le \kappa_+ \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^\alpha \\ &\le \kappa_+ \int_\Omega u_\varepsilon - \mu|\Omega|^{1-\alpha}\left(\int_\Omega u_\varepsilon\right)^\alpha.\end{aligned} \tag{14}$$

We can obtain (12) by an ODE comparison argument. Estimate (13) is a consequence of the parabolic comparison principle. □

Lemma 2.3 *Let u_0, v_0 satisfy* (5), *let $\kappa \in \mathbb{R}$, $\mu > 0$. There is $C(\tau) > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\int_t^{t+\tau}\int_\Omega u_\varepsilon^\alpha \le C(\tau) \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau), \tag{15}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max,\varepsilon}\}$.

Proof Estimate (15) results from (14) after time-integration. □

Next we want to derive a (quasi-)energy inequality for the functional

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}$$

to get some essential estimates of $(u_\varepsilon, v_\varepsilon)$. The method used here is from [35].

Lemma 2.4 *Let u_0, v_0 satisfy* (5), *let $\kappa \in \mathbb{R}$, $\mu > 0$. Then there exist $K, C > 0$ such that*

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\right) + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + K\int_\Omega v_\varepsilon\left|D^2 \ln v_\varepsilon\right|^2 \\ &\quad + \frac{\mu}{2}\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon + K\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le C\end{aligned} \tag{16}$$

on $(0, T_{\max,\varepsilon})$ *for all* $\varepsilon \in (0,1)$.

Proof From integration by parts, we obtain that on $(0, T_{\max,\varepsilon})$

$$\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon = \int_\Omega u_{\varepsilon t} + \int_\Omega u_{\varepsilon t}\ln u_\varepsilon$$

$$
\begin{aligned}
&= \kappa \int_\Omega u_\varepsilon - \mu \int_\Omega u_\varepsilon^\alpha - \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \\
&\quad + \kappa \int_\Omega u_\varepsilon \ln u_\varepsilon - \mu \int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon.
\end{aligned} \tag{17}
$$

As $\alpha > 1$, we can see that $s \mapsto \kappa s - \mu s^\alpha$, $s \in [0,\infty)$ and $s \mapsto (\kappa s - \frac{\mu}{2} s^\alpha)\ln s$, $s \in [0,\infty)$ are bounded from above by some constant C_1. We thus can estimate

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\mu}{2}\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} + 2C_1 \\
&\quad \text{for all } t \in (0, T_{\max,\varepsilon}).
\end{aligned} \tag{18}
$$

Next we compute $\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}$. From the second equation of (9) we know that on $(0, T_{\max,\varepsilon})$

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &= \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla v_{\varepsilon t}}{v_\varepsilon} - \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2 \cdot v_{\varepsilon t}}{v_\varepsilon^2} \\
&= \int_\Omega \frac{\nabla v_\varepsilon}{v_\varepsilon} \nabla \cdot \left(\Delta v_\varepsilon - v_\varepsilon \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\right) \\
&\quad - \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\left(\Delta v_\varepsilon - v_\varepsilon \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\right) \\
&= \int_\Omega \frac{\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon}{v_\varepsilon} - \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} - \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \\
&\quad - \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \\
&\le -\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon - \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}.
\end{aligned} \tag{19}
$$

We know from Lemma 2.7 of [35] that there exist ε-independent positive constants $K > 0$, $K_1 > 0$ such that

$$
-\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\Delta v_\varepsilon \le -K\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 - K\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + K_1\int_\Omega v_\varepsilon
$$

on $(0, T_{\max,\varepsilon})$. Thereupon, we derive that

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le K\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 - K\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + K_1\int_\Omega v_\varepsilon - \int_\Omega \frac{\nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \tag{20}
$$

on $(0, T_{\max,\varepsilon})$. Combining (18) and (20), we obtain that, for any $\varepsilon \in (0,1)$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\right) + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + K\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 \\
&\quad + \frac{\mu}{2}\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon + K\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\le K_1\int_\Omega v_\varepsilon + 2C_1 \le C \quad \text{on } t \in (0, T_{\max,\varepsilon})
\end{aligned}
$$

with $C := K_1 v_\infty + 2C_1$. □

Lemma 2.4 immediately entails the following boundedness estimates.

Lemma 2.5 *Let u_0, v_0 satisfy (5), let $\kappa \in \mathbb{R}$, $\mu > 0$. Then there exists $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\int_\Omega u_\varepsilon \ln u_\varepsilon + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le C \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{21}$$

and such that

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_t^{t+\tau}\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon \le C, \tag{22}$$

$$\int_t^{t+\tau}\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 + \int_t^{t+\tau}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \le C, \tag{23}$$

$$\int_\Omega |\nabla v_\varepsilon|^2 \le C \tag{24}$$

for all $\varepsilon \in (0,1)$ and any $t \in [0, T_{\max,\varepsilon} - \tau)$, where $\tau := \min\{1, \frac{1}{2}T_{\max,\varepsilon}\}$.

Proof Fix $p \in (1, 1 + \frac{2}{N})$ and observe that

$$\xi \ln \xi \le \frac{1}{p(p-1)}\xi^p \quad \text{for all } \xi > 0.$$

An application of the Gagliardo–Nirenberg inequality yields ε-independent positive constant C_1 such that

$$\begin{aligned}\int_\Omega u_\varepsilon \ln u_\varepsilon &\le \frac{1}{p(p-1)}\int_\Omega u_\varepsilon^p \\ &= \frac{1}{p(p-1)}\left\|u_\varepsilon^{\frac{1}{2}}\right\|_{L^{2p}(\Omega)}^{2p} \\ &\le C_1\left(\left\|\nabla u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{2p\cdot\theta} \cdot \left\|u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{2p(1-\theta)} + \left\|u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{2p}\right) \quad \text{for all } t \in (0, T_{\max,\varepsilon}),\end{aligned}$$

where $\theta := \frac{(p-1)N}{2p} \in (0,1)$ and $2p \cdot \theta < 2$ due to $p \in (1, 1 + \frac{2}{N})$. Thereupon, we can find some ε-independent positive constants C_2, C_3 such that

$$\int_\Omega u_\varepsilon \ln u_\varepsilon \le C_2\left(\left\|\nabla u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^2 + 1\right) \le C_3\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + C_2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{25}$$

by making use of (12). On the other hand, making use of the boundedness of $\|v_\varepsilon\|_{L^\infty(\Omega)}$ and the Young inequality, we know there exist ε-independent positive constants C_4, C_5 fulfilling

$$\begin{aligned}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} &\le C_4\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{4C_4}\int_\Omega v_\varepsilon \\ &\le C_4\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + C_5 \quad \text{for all } t \in (0, T_{\max,\varepsilon}).\end{aligned} \tag{26}$$

Substituting (25), (26) into (16), we conclude that there exist positive constants C_6 and C_7 such that, for any $\varepsilon > 0$,

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\right) + C_6\left(\int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\right) \\ &\quad + K\int_\Omega v_\varepsilon \left|D^2 \ln v_\varepsilon\right|^2 + \frac{\mu}{2}\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon \\ &\le C_7 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \end{aligned} \tag{27}$$

with K as given by Lemma 2.4. We can conclude the validity of (21). The result of (22) and (23) can be obtained by an integration of (16) and the fact $\int_\Omega u_\varepsilon \ln u_\varepsilon \ge -\frac{|\Omega|}{e}$. Furthermore, for (24), by Lemma 2.2, for any $\varepsilon > 0$,

$$\int_\Omega \left|\nabla v_\varepsilon(t)\right|^2 \le v_\infty \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \quad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

which is bounded due to (21). □

By using interpolation inequalities, we can derive some further estimates from Lemma 2.5.

Lemma 2.6 *Let u_0, v_0 satisfy (5), let $\kappa \in \mathbb{R}$, $\mu > 0$. Then there exists $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\int_t^{t+\tau}\int_\Omega u_\varepsilon^{\frac{N+2}{N}} \le C, \tag{28}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^{\frac{N+2}{N+1}} \le C \tag{29}$$

for any $t \in [0, T_{\max,\varepsilon} - \tau)$, where $\tau := \min\{1, \frac{1}{2}T_{\max,\varepsilon}\}$.

Proof From (22), we know that $\int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon^{\frac{1}{2}}|^2 \le C_1$ with some ε-independent constant $C_1 > 0$. Then with the aid of the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} \int_t^{t+\tau}\int_\Omega u_\varepsilon^{\frac{N+2}{N}} &= \int_t^{t+\tau}\int_\Omega \left\| u_\varepsilon^{\frac{1}{2}}\right\|_{L^{\frac{2(N+2)}{N}}}^{\frac{2(N+2)}{N}} \\ &\le \int_t^{t+\tau} C_2\left(\left\|\nabla u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^2 \cdot \left\|u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{\frac{4}{N}} + \left\|u_\varepsilon^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{\frac{2(N+2)}{N}}\right) \\ &\le C_3 \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau) \end{aligned}$$

with some ε-independent positive constants C_2, C_3.

Furthermore, we can make use of the Young inequality to obtain that

$$\begin{aligned} \int_t^{t+\tau}\int_\Omega |\nabla u_\varepsilon|^{\frac{N+2}{N+1}} &= \int_t^{t+\tau}\int_\Omega \left(\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\right)^{\frac{N+2}{2N+2}} \cdot u_\varepsilon^{\frac{N+2}{2N+2}} \\ &\le C_4\int_t^{t+\tau}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + C_5\int_t^{t+\tau}\int_\Omega u_\varepsilon^{\frac{N+2}{N}} \end{aligned}$$

$$\le C_6 \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau)$$

with some ε-independent positive constants C_i $(i = 4, 5, 6)$. □

Lemma 2.7 *Let u_0, v_0 satisfy (5), let $\kappa \in \mathbb{R}$, $\mu > 0$. There exists $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\int_t^{t+\tau} \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{4(N+2)}{5N+2}} \le C \tag{30}$$

for any $t \in [0, T_{\max,\varepsilon} - \tau)$, where $\tau := \min\{1, \frac{1}{2} T_{\max,\varepsilon}\}$.

Proof As $\|v_\varepsilon\|_{L^\infty(\Omega)} \le v_\infty$, the time-spatial estimate (23) implies that there exists ε-independent constant $C_1 > 0$ such that

$$\int_t^{t+\tau} \int_\Omega |\nabla v_\varepsilon|^4 \le C_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau). \tag{31}$$

Noticing (28), we can use the Young inequality to estimate

$$\begin{aligned} \int_t^{t+\tau} \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{4(N+2)}{5N+2}} &\le C_2 \int_t^{t+\tau} \int_\Omega |u_\varepsilon|^{\frac{N+2}{N}} + C_3 \int_t^{t+\tau} \int_\Omega |\nabla v_\varepsilon|^4 \\ &\le C_4 \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau), \end{aligned}$$

where $C_i > 0$ $(i = 2, 3, 4)$ are all independent of ε. □

Lemma 2.8 *Let u_0, v_0 satisfy (5), let $\kappa \in \mathbb{R}$, $\mu > 0$. Then there exists $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\int_t^{t+\tau} \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{4\alpha}{4+\alpha}} \le C \tag{32}$$

for any $t \in [0, T_{\max,\varepsilon} - \tau)$, where $\tau := \min\{1, \frac{1}{2} T_{\max,\varepsilon}\}$.

Proof We can use the Young inequality, (31), and (15) to estimate

$$\begin{aligned} &\int_t^{t+\tau} \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{4\alpha}{4+\alpha}} \\ &\quad \le C_1 \int_t^{t+\tau} \int_\Omega |u_\varepsilon|^\alpha + C_2 \int_t^{t+\tau} \int_\Omega |\nabla v_\varepsilon|^4 \le C_3 \quad \text{for all } t \in (0, T_{\max,\varepsilon} - \tau), \end{aligned}$$

with some ε-independent positive constants C_i $(i = 1, 2, 3)$. □

We are now in the position to prove that the classical solution $(u_\varepsilon, v_\varepsilon)$ to the approximate systems (9) is global for each $\varepsilon \in (0,1)$.

Lemma 2.9 *Let $\kappa \in \mathbb{R}$, $\mu > 0$ and assume that u_0, v_0 satisfy (5). For any $\varepsilon \in (0,1)$, $T_{\max,\varepsilon} = \infty$.*

Proof Assume that $T_{\max,\varepsilon}$ is finite for some $\varepsilon \in (0,1)$. To deduce a contradiction from this, we fix a suitably large $q \geq N+1$ and use the standard estimate for the Neumann heat semigroup (see e.g. [36]) together with Lemma 2.2 and the fact that $\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \leq \frac{1}{\varepsilon}$ to obtain $C_i > 0$ ($i = 1,2,3,4$) such that

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L^q} &= \left\| \nabla e^{t\Delta} v_\varepsilon(\cdot,0) - \int_0^t \nabla \left(e^{(t-s)\Delta} \left(v_\varepsilon \cdot \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \right) \right) ds \right\|_{L^q} \\
&\leq \left\| \nabla e^{t\Delta} v_0 \right\|_{L^q} + C_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) e^{-\lambda t} \left\| v_\varepsilon \cdot \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \right\|_{L^q} ds \\
&\leq C_2 \|\nabla v_0\|_{L^q} + \frac{C_3}{\varepsilon} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) ds \\
&\leq C_4(\varepsilon, T) \quad \text{for all } t \in (0, T_{\max,\varepsilon}).
\end{aligned} \tag{33}$$

Similarly, according to the fact $\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \leq \frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \leq \frac{1}{\varepsilon}$ and $\kappa u_\varepsilon - \mu u_\varepsilon^\alpha$ is bounded, there exist $C_i > 0$ ($i = 5,6,7,8$) such that

$$\begin{aligned}
\|u_\varepsilon\|_{L^\infty} &= \left\| e^{t\Delta} u_\varepsilon(\cdot,0) + \int_0^t e^{(t-s)\Delta} \left(-\nabla \left(\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \cdot \nabla v_\varepsilon \right) + \kappa u_\varepsilon - \mu u_\varepsilon^\alpha \right) ds \right\|_{L^\infty} \\
&\leq \left\| \nabla e^{t\Delta} u_0 \right\|_{L^\infty} + C_5 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{N}{2N+2}}\right) e^{-\lambda t} \left\| \frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2} \cdot \nabla v_\varepsilon \right\|_{L^{N+1}} ds \\
&\quad + \int_0^t \left\| e^{(t-s)\Delta} \kappa u_\varepsilon - \mu u_\varepsilon^\alpha \right\|_{L^\infty} ds \\
&\leq \left\| \nabla e^{t\Delta} u_0 \right\|_{L^\infty} + \frac{C_6}{\varepsilon} \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{N}{2N+2}}\right) \|\nabla v_\varepsilon\|_{L^{N+1}} ds + C_7 \\
&\leq C_8(\varepsilon, T) \quad \text{for all } t \in (0, T_{\max,\varepsilon}).
\end{aligned}$$

Together with (33), this contradicts criterion (11) in Lemma 2.1 and thereby entails that actually $T_{\max,\varepsilon} = \infty$, as claimed. □

3 Time regularity

In preparation of an Aubin–Lions type compactness argument, we shall supplement the estimates obtained in Sect. 2 with bounds on time-derivatives, since in Lemma 4.1 these will be used to warrant pointwise convergence.

Lemma 3.1 *Let $\kappa \in \mathbb{R}$, $\mu > 0$ and assume that u_0, v_0 satisfy (5). Suppose $N < 6$, $\alpha > 1$ or $N \geq 6$, $\alpha > \frac{4}{3}$. Then, for any $T > 0$, there is $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\|u_{\varepsilon t}\|_{L^1((0,T);(W^{1,\infty}(\Omega))^*)} \leq C. \tag{34}$$

Proof When $N < 6$, we pick $\varphi \in C_0^\infty(\Omega)$ having norm $\|\varphi\|_{W^{1,\infty}(\Omega)} \leq 1$ and integrate by parts in (9) to obtain that, for any $T > 0$,

$$\begin{aligned}
&\int_0^T \sup_{\|\varphi\|_{W^{1,\infty}(\Omega)} \leq 1} \left| \int_\Omega u_{\varepsilon t} \cdot \varphi \right| \\
&\quad \leq \int_0^T \sup_{\|\varphi\|_{W^{1,\infty}(\Omega)} \leq 1} \left(\left| \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi \right| + \left| \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \varphi \right| + \left| \kappa \int_\Omega u_\varepsilon \varphi \right| + \mu \left| \int_\Omega u_\varepsilon^\alpha \varphi \right| \right),
\end{aligned}$$

where we can use $\|\varphi\|_{W^{1,\infty}(\Omega)} \leq 1$ and the Young inequality to see that there exists some ε-independent constant $C_1 > 0$ such that

$$\begin{aligned}
&\|u_{\varepsilon t}\|_{L^1((0,T);(W^{1,\infty}(\Omega))^*)} \\
&\quad \leq \int_0^T \left(\int_\Omega |\nabla u_\varepsilon| + \int_\Omega |u_\varepsilon \nabla v_\varepsilon| + \kappa_+ \int_\Omega u_\varepsilon + \mu \int_\Omega u_\varepsilon^\alpha \right) \\
&\quad \leq \int_0^T \left(\int_\Omega |\nabla u_\varepsilon| + \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{4(N+2)}{5N+2}} + \kappa_+ \int_\Omega u_\varepsilon + \mu \int_\Omega u_\varepsilon^\alpha + C_1 \right)
\end{aligned}$$

and infer boundedness of this norm, independent of ε, from Lemmas 2.3, 2.6, and 2.7. When $N \geq 6$, $\alpha > \frac{4}{3}$ implies $\frac{4\alpha}{4+\alpha} > 1$. We can obtain the same conclusion from Lemmas 2.3, 2.6, and 2.8 in quite a similar way. □

Lemma 3.2 *Let $\kappa \in \mathbb{R}$, $\mu > 0$ and assume that u_0, v_0 satisfy* (5). *For any $T > 0$, there is $C > 0$ such that, for any $\varepsilon \in (0,1)$,*

$$\|v_{\varepsilon t}\|_{L^1((0,T);(W_0^{N,2}(\Omega))^*)} \leq C. \tag{35}$$

Proof Take an arbitrary $\psi \in L^\infty((0,T); W_0^{N,2}(\Omega))$, then from the second equation in (9) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\left| \int_0^T \int_\Omega v_{\varepsilon t} \cdot \psi \right| &= \left| -\int_0^T \int_\Omega \nabla v_\varepsilon \cdot \nabla \psi - v_\varepsilon \frac{u_\varepsilon}{(1 + \varepsilon u_\varepsilon)} \psi \right| \\
&\leq \|\nabla \psi\|_{L^2(\Omega \times (0,T))} \cdot \int_0^T \|\nabla v_\varepsilon\|_{L^2(\Omega)} \\
&\quad + \|v_\varepsilon\|_{L^\infty(\Omega \times (0,T))} \|\psi\|_{L^\infty((0,T);L^\infty(\Omega))} \int_0^T \|u_\varepsilon\|_{L^1(\Omega)}.
\end{aligned}$$

Since $W_0^{N,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, we can obtain our conclusion from the boundedness of $\|\nabla v_\varepsilon\|_{L^2(\Omega)}$ and $\|u_\varepsilon\|_{L^1(\Omega)}$ stated in (24) and (12), respectively. □

4 Construction of a limit (u, v) and the proof of our main result

With the above compactness properties at hand, by means of a standard extraction procedure, we can now derive the following lemma.

Lemma 4.1 *Let $\kappa \in \mathbb{R}$, $\mu > 0$ and assume that u_0, v_0 satisfy* (5). *There exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions u and v such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$ we have*

$$u_\varepsilon \to u \quad \text{in } L_{\mathrm{loc}}^{\frac{N+2}{N+1}}\big(\overline{\Omega} \times [0,\infty)\big) \text{ and a.e. in } \Omega \times (0,\infty), \tag{36}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L_{\mathrm{loc}}^{\frac{N+2}{N+1}}\big((0,\infty); L^{\frac{N+2}{N+1}}(\Omega)\big), \tag{37}$$

$$u_\varepsilon^\alpha \to u^\alpha \quad \text{in } L_{\mathrm{loc}}^1\big(\overline{\Omega} \times [0,\infty)\big), \tag{38}$$

$$v_\varepsilon \to v \quad \text{in } L_{\mathrm{loc}}^2\big(\bar{\Omega} \times [0,\infty)\big) \text{ and a.e. in } \Omega \times (0,\infty), \tag{39}$$

$$v_\varepsilon \overset{\star}{\rightharpoonup} v \quad \text{in } L^\infty\big((0,\infty); L^p(\Omega)\big) \text{ for all } p \in [1,\infty], \tag{40}$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \quad \textit{in } L^4_{\mathrm{loc}}\big([0,\infty);L^4(\Omega)\big), \tag{41}$$

$$\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\nabla v_\varepsilon \rightharpoonup u\nabla v \quad \textit{in } L^{\frac{4(N+2)}{5N+2}}_{\mathrm{loc}}\big(\overline{\Omega}\times[0,\infty)\big), \tag{42}$$

$$\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\nabla v_\varepsilon \rightharpoonup u\nabla v \quad \textit{in } L^{\frac{4\alpha}{4+\alpha}}_{\mathrm{loc}}\big(\overline{\Omega}\times[0,\infty)\big). \tag{43}$$

Proof Lemma 2.6 shows that $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{N+2}{N+1}}((0,T);W^{1,\frac{N+2}{N+1}}(\Omega))$, which together with Lemma 3.1 enables us to employ an Aubin–Lions lemma to show that u_ε is relatively compact in $L^{\frac{N+2}{N+1}}((0,T);L^{\frac{N+2}{N+1}}(\Omega))$ with respect to the strong topology and thus $u_\varepsilon \to u$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_j \searrow 0$, and (36) holds. Similarly, the boundedness of $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^2((0,T);W^{1,2}(\Omega))$ deduced from (13) and (24), Lemma 3.2, and the Aubin–Lions lemma entail that $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively precompact in $L^2((0,T);W^{1,2}(\Omega))$, which enables us to find a further subsequence such that (39) holds. From (29), we can obtain the convergence for u_ε along a suitable subsequence in (37). From Lemma 2.5, we know $\int_0^T\int_\Omega u_\varepsilon^\alpha \ln u_\varepsilon$ is bounded, this implies $\{u_\varepsilon^\alpha;\varepsilon\in(0,1)\}$ is uniformly integrable (see e.g. [29], Lemma 6.4). By (36) and the Vitali convergence theorem, we can extract subsequence such that (38) holds. Also (40) along a subsequence is immediately obtained from (13) with $(0,T)\ni t\mapsto \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ is monotone decreasing for any $T>0$. Convergence of the gradient along further subsequence, as asserted in (41), is implied by (31). Noting that $|\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\nabla v_\varepsilon| \le |u_\varepsilon \nabla v_\varepsilon|$, we can deduce the convergence properties (42) and (43) along a suitable subsequence from Lemma 2.7 and Lemma 2.8, respectively. □

We now prove that the limit function obtained in Lemma 4.1 is indeed the weak solution to system (4).

Lemma 4.2 *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ and (u,v) be as provided by Lemma 4.1. Then the identities (7) and (8) are satisfied for all $\xi\in C_0^\infty(\Omega\times[0,\infty))$.*

Proof Testing the first and second equation of (9) against an arbitrary test function $\xi\in C_0^\infty(\Omega\times[0,\infty))$, we obtain

$$\begin{aligned}-\int_0^\infty\int_\Omega u_\varepsilon\xi_t-\int_\Omega u_0\xi(\cdot,0) = {}&-\int_0^\infty\int_\Omega \nabla u_\varepsilon\cdot\nabla\xi+\int_0^\infty\int_\Omega\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)^2}\nabla v_\varepsilon\cdot\nabla\xi\\ &+\kappa\int_0^\infty\int_\Omega \xi u_\varepsilon-\mu\int_0^\infty\int_\Omega \xi u_\varepsilon^\alpha\end{aligned}$$

and

$$-\int_0^\infty\int_\Omega v_\varepsilon\xi_t-\int_\Omega v_0\xi(\cdot,0)=-\int_0^\infty\int_\Omega\nabla v_\varepsilon\cdot\nabla\xi-\int_0^\infty\int_\Omega v_\varepsilon\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon}\xi,$$

respectively, for all $\varepsilon\in(0,1)$. Note the fact that when $N<6$, $\frac{4(N+2)}{5N+2}>1$ and $\frac{4\alpha}{4+\alpha}>1$ if $\alpha>\frac{4}{3}$. According to the convergence properties in Lemma 4.1, we may pass to the limit in each of the integrals above along the subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ and readily achieve (7) and (8). □

Proof of Theorem 1.1 The statement is evidently implied by Lemma 4.2. □

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