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Infinitely many homoclinic solutions for sublinear and nonperiodic Schrödinger lattice systems

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Abstract
By using variational methods we obtain infinitely many nontrivial solutions for a class of nonperiodic Schrödinger lattice systems, where the nonlinearities are sublinear at both zero and infinity.

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1 Introduction and main results

Schrödinger lattice systems are a class of very important discrete models, ranging from biology and condensed matter physics to solid state physics [8, 10, 11]. In fact, most results are about the *periodic* Schrödinger lattice systems, such as [2, 4, 12–14, 18, 19, 24]. However, there are only few results about the *nonperiodic* Schrödinger lattice systems [5, 9, 15, 16, 22, 23]. In particular, in [3, 6, 7] the authors recently obtained the existence and multiplicity of homoclinic solutions for a class of Schrödinger lattice systems with perturbed terms.

In this paper, we investigate the nonperiodic Schrödinger lattice system

$$\begin{cases} -(\Delta u)_n + v_n u_n = \mu \chi_n |u_n|^{\mu-2} u_n, & n \in \mathbb{Z}, \\ \lim_{|n|\to\infty} u_n = 0, \end{cases} \tag{1.1}$$

where $\mu \in (1,2)$,

$$(\Delta u)_n := u_{n+1} + u_{n-1} - 2u_n, \tag{1.2}$$

$\{u_n\}$, $\{v_n\}$, and $\{\chi_n\}$ are real-valued sequences, and the discrete potential $V = \{v_n\}_{n\in\mathbb{Z}}$ and $\{\chi_n\}$ are nonperiodic. A solution $u = \{u_n\}_{n\in\mathbb{Z}}$ is said to be nontrivial if $u_n \not\equiv 0$. Problem (1.1) appears when we look for standing wave (or breather) solutions of the Schrödinger lattice system

$$i\dot{\psi}_n = -(\Delta\psi)_n + \widetilde{\nu}_n\psi_n - \mu\chi_n|\psi_n|^{\mu-2}\psi_n, \quad n \in \mathbb{Z}, \tag{1.3}$$

where $\{\psi_n\}$ is a real-valued sequence. Standing waves (or breathers) are the solutions for (1.3) of the form $\psi_n = u_n e^{-i\omega t}$, where $\omega \in \mathbb{R}$ is the temporal frequency, and u_n satisfies $\lim_{|n|\to\infty} u_n = 0$. By the standing wave ansatz $\psi_n = u_n e^{-i\omega t}$ we get that (1.3) reduces to (1.1) with $\nu_n \equiv \widetilde{\nu}_n - \omega$. Therefore we only need to study the existence of solutions of (1.1).

Let

$$\|u\|_{l^q} := \left(\sum_{n=-\infty}^{+\infty} |u_n|^q\right)^{1/q}, \qquad \|u\|_{l^\infty} := \sup_{n\in\mathbb{Z}} |u_n|, \qquad u = \{u_n\}_{n\in\mathbb{Z}},$$

be the norms of the real sequence spaces $l^q := l^q(\mathbb{Z})$ $(q \in [1,\infty))$. The following embedding between such spaces is well known:

$$l^q \subset l^p, \qquad \|u\|_{l^p} \le \|u\|_{l^q}, \quad 1 \le q \le p \le \infty.$$

We study solutions of (1.1) in l^2 since any $u = \{u_n\}_{n\in\mathbb{Z}} \in l^2$ satisfies $\lim_{|n|\to\infty} u_n = 0$.

Note that the domain of (1.1) is $\mathbb{Z}$, and thus, to overcome the loss of compactness caused by the unboundedness of the domain $\mathbb{Z}$, we need the following condition:

($\mathbf{V_1}$) $\lim_{|n|\to+\infty} \nu_n = +\infty$.

Then (V_1) implies that (see [21]) the spectrum $\sigma(-\triangle + V)$ is discrete and consists of simple eigenvalues accumulating to $+\infty$, that is, we can assume that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty$$

are all eigenvalues of $-\triangle + V$, where $((-\triangle + V)u)_n := -(\triangle u)_n + \nu_n u_n$ for $u = \{u_n\} \in l^2$.

Theorem 1.1 *System* (1.1) *has infinitely many nontrivial solutions if* (V_1), *and the following conditions hold*:

($\mathbf{W_1}$) $0 \notin \sigma(-\Delta + V)$.

($\mathbf{SG_1}$) $\chi := \{\chi_n > 0\}_{n\in\mathbb{Z}} \in l^{\frac{2}{2-\mu}}$, $\mu \in (1,2)$.

Clearly, condition (W_1) implies that we have the following two cases:

($\mathbf{W_1'}$) $0 \in (\lambda_{k_0}, \lambda_{k_0+1})$ for some $k_0 \ge 1$ (the indefinite case);

($\mathbf{W_1''}$) $0 < \lambda_1$ (the positive definite case).

Remark 1.1 To the best of our knowledge, there is no result published concerning the multiplicity of nontrivial solutions for (1.1) with sublinear nonlinearities at both zero and infinity. For the nonperiodic system (1.1), the main differences between our and known results [5, 9, 15, 16, 22, 23] are as follows:

(1) The nonlinearities $g_n(s)$ in [5, 15, 16, 22, 23] are *superlinear* as $|s| \to 0$ ($\lim_{|s|\to 0} \frac{g_n(s)}{s} = 0$, $\forall n \in \mathbb{Z}$), and the nonlinearities $g_n(s)$ in [9] are *superlinear* or *asymptotically linear* ($\lim_{|s|\to 0} \frac{g_n(s)}{s} = l_n \in (0,+\infty)$, $\forall n \in \mathbb{Z}$) as $|s| \to 0$. *However*, our nonlinearities $g_n(s) = \mu\chi_n|u_n|^{\mu-2}u_n$ are *sublinear* as $|s| \to 0$ ($\lim_{|s|\to 0} \frac{g_n(s)}{s} = +\infty$, $\forall n \in \mathbb{Z}$).

(2) The nonlinearities $g_n(s)$ in [5, 9, 15, 22, 23] are *superlinear* as $|s| \to \infty$ ($\lim_{|s|\to\infty} \frac{g_n(s)}{s} = +\infty$, $\forall n \in \mathbb{Z}$), and the nonlinearities $g_n(s)$ in [16] are *asymptotically linear* as $|s| \to \infty$ ($\lim_{|s|\to\infty} \frac{g_n(s)}{s} = c_n \in (0, +\infty)$, $\forall n \in \mathbb{Z}$). *However*, our nonlinearities $g_n(s) = \mu\chi_n|u_n|^{\mu-2}u_n$ are *sublinear* as $|s| \to \infty$ ($\lim_{|s|\to\infty} \frac{g_n(s)}{s} = 0$, $\forall n \in \mathbb{Z}$).

(3) Our method is based on the variant fountain theorem in [25], which is different from the methods used in the papers mentioned.

In Sect. 2, we give some lemmas and the proofs of our main result. In Appendix, we give the proofs of the conditions in the critical point theorem used in this paper.

2 Proof of the main result

The corresponding action functional Φ of (1.1) is defined as follows:

$$\Phi(u) = \frac{1}{2}(Lu, u)_{l^2} - \sum_{n=-\infty}^{+\infty} \chi_n|u_n|^{\mu}, \quad u \in E,$$

where $(\cdot,\cdot)_{l^2}$ is the inner product in l^2, $L := -\triangle + V$, $E := \mathcal{D}(|L|^{1/2})$ is the form domain of L (the domain of $|L|^{1/2}$). Since the operator $-\triangle$ is bounded in l^2, we easily see that

$$E = \{u \in l^2 : |V|^{1/2}u \in l^2\}$$

with the inner product and norm

$$(u, v) := (|L|^{1/2}u, |L|^{1/2}v)_{l^2} = (-\triangle u, v)_{l^2} + (|V|^{1/2}u, |V|^{1/2}v)_{l^2}, \qquad \|u\| := (u, u)^{1/2};$$

E is a Hilbert space, where $|V|^{1/2}u$ is defined by $(|V|^{1/2}u)_n := |v_n|^{1/2}u_n$ ($n \in \mathbb{Z}$). By (W_1). We have the orthogonal decomposition

$$E = E^- \oplus E^+$$

with respect to both inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{l^2}$, where $E^{\pm} := E \cap (l^2)^{\pm}$, and $(l^2)^{\pm}$ is the positive (negative) eigenspace of L.

Then the functional Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n=-\infty}^{+\infty} \chi_n|u_n|^{\mu}, \quad u \in E,$$

where $u = u^+ + u^- \in E = E^+ \oplus E^-$. Let $I(u) := \sum_{n=-\infty}^{+\infty} \chi_n|u_n|^{\mu}$. Under our assumptions, $I, \Phi \in C^1(E, \mathbb{R})$ with derivatives

$$\langle \Phi'(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \langle I'(u), v\rangle,$$

$$\langle I'(u), v\rangle = \sum_{n=-\infty}^{+\infty} \mu\chi_n|u_n|^{\mu-2}u_n v_n, \quad u, v \in E,$$

where $u = u^+ + u^-$, $v = v^+ + v^- \in E = E^+ \oplus E^-$. The standard argument shows that nonzero critical points of Φ are nontrivial solutions of (1.1). We will use the following critical point theorem.

Lemma 2.1 ([25]) *Let E be a Banach space with norm $\|\cdot\|$ and suppose $E = \overline{\bigoplus_{j=1}^{\infty} X_j}$ with $\dim X_j < \infty$, $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Assume that the functional $\Phi_\lambda = A(u) - \lambda B(u)$ ($\Phi_\lambda \in C^1$, $\Phi_\lambda : E \to \mathbb{R}$, $\lambda \in [1,2]$) satisfies*

- (F_1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$;*
- (F_2) *$B(u) \geq 0$, $\forall u \in E$; and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of E.*
- (F_3) *There exist $\rho_k > r_k > 0$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2],$$

and

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_j \to 1$ and $u^{\lambda_j} \in Y_j$ such that

$$\Phi'_{\lambda_j}|_{Y_j}\big(u^{\lambda_j}\big) = 0, \qquad \Phi_{\lambda_j}\big(u^{\lambda_j}\big) \to \eta_k \in \big[\xi_k(2), \beta_k(1)\big] \quad \text{as } j \to \infty.$$

Particularly, if $\{u^{\lambda_j}\}$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u^k\} \subset E \setminus \{0\}$ satisfying $\Phi_1(u^k) \to 0^-$ as $k \to \infty$.

From (V_1), (W_1), and [21] we have that the eigenvalues of L are as follows:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{k_0} < 0 < \lambda_{k_0+1} < \cdots \to +\infty.$$

Let $\{e_j\}_{j=1}^{k_0}$ and $\{e_j\}_{j=k_0+1}^{\infty}$ be the orthonormal bases of E^- and E^+, respectively ($E^- = \{0\}$ if $0 < \lambda_1$). Then $\{e_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of E. Let $X_j := \operatorname{span}\{e_j\}$ for $j \in \mathbb{N}$. Then Z_k and Y_k can be defined as in Lemma 2.1. Let

$$A(u) := \frac{1}{2}\big\|u^+\big\|^2, \qquad B(u) := \frac{1}{2}\big\|u^-\big\|^2 + \sum_{n=-\infty}^{+\infty} \chi_n |u_n|^\mu,$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\big\|u^+\big\|^2 - \lambda\left(\frac{1}{2}\big\|u^-\big\|^2 + \sum_{n=-\infty}^{+\infty} \chi_n |u_n|^\mu\right)$$

for all $u = u^+ + u^-$, $v = v^+ + v^- \in E = E^+ \oplus E^-$ and $\lambda \in [1,2]$. Obviously, $\Phi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1,2]$.

Proof of Theorem 1.1 Under our assumptions, the definition of Φ_λ implies that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Evidently, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$, and thus ($F_1$) of Lemma 2.1 holds. Besides, Ax 3.1 and Ax 3.2 in the Appendix show that (F_2) and (F_3) of Lemma 2.1 hold for all $k \geq k_1$. Therefore by Lemma 2.1, for each $k \geq k_1$, there exist $\lambda_j \to 1$ and $u^{\lambda_j} \in Y_j$ such that

$$\Phi'_{\lambda_j}|_{Y_j}\big(u^{\lambda_j}\big) = 0, \qquad \Phi_{\lambda_j}\big(u^{\lambda_j}\big) \to \eta_k \in \big[\xi_k(2), \beta_k(1)\big] \quad \text{as } j \to \infty. \tag{2.1}$$

Let

$$u^j := u^{\lambda_j}, \quad \forall j \in \mathbb{N}.$$

By (2.1), (SG_1), and the definition of Φ_{λ_j},

$$\begin{aligned} -\Phi_{\lambda_j}(u^j) &= \frac{1}{2}\langle \Phi'_{\lambda_j}|_{Y_j}(u^j), u^j\rangle - \Phi_{\lambda_j}(u^j) \\ &= \lambda_j \sum_{n=-\infty}^{+\infty} \left(1 - \frac{\mu}{2}\right)\chi_n |u_n^j|^\mu \\ &\geq \lambda_j\left(1 - \frac{\mu}{2}\right)\theta \sum_{n=-\infty}^{+\infty} |u_n^j|^\mu, \quad \forall j \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Relations (2.1), (2.2), and $\mu < 2$ imply that $\|u^j\|_{l^\mu} = (\sum_{n=-\infty}^{+\infty} |u_n^j|^\mu)^{1/\mu} < \infty$. It follows from the equivalence of any two norms on finite-dimensional space E^- and the Hölder inequality that

$$\|(u^j)^-\|_{l^2}^2 = ((u^j)^-, u_j)_{l^2} \leq \|u^j\|_{l^\mu} \cdot \|(u^j)^-\|_{l^{\mu'}} \leq C_1\|(u^j)^-\|_{l^2}$$

for some $C_1 > 0$, where μ' satisfies $1/\mu + 1/\mu' = 1$. Consequently, we have $\|(u^j)^-\|_{l^2} \leq C_1$, $\forall j \in \mathbb{N}$. In view of the equivalence of norms $\|\cdot\|_{l^2}$ and $\|\cdot\|$ on E^- again, there exists $C_2 > 0$ such that

$$\|(u^j)^-\| \leq C_2, \quad \forall j \in \mathbb{N}. \tag{2.3}$$

Obviously, the definition of Φ_{λ_j} implies

$$\|(u^j)^+\|^2 = 2\Phi_{\lambda_j}(u^j) + \lambda_j\|(u^j)^-\|^2 + 2\lambda_j \sum_{n=-\infty}^{+\infty} \chi_n |u_n^j|^\mu.$$

It follows from $\|u^j\|^2 = \|(u^j)^+\|^2 + \|(u^j)^-\|^2$ that

$$\|u^j\|^2 = 2\Phi_{\lambda_j}(u^j) + (\lambda_j + 1)\|(u^j)^-\|^2 + 2\lambda_j \sum_{n=-\infty}^{+\infty} \chi_n |u_n^j|^\mu,$$

which, together with (2.1), (2.3), (SG_1), and the fact E is compactly embedded into l^2 (see [21]), implies that

$$\begin{aligned} \|u^j\|^2 &\leq C_3 + 4\|\chi\|_{l^{\frac{2}{2-\mu}}}\|u^j\|_{l^2}^\mu \\ &\leq C_3 + C_4\|u^j\|^\mu \end{aligned}$$

for some $C_3, C_4 > 0$. This implies that $\{u^j\}$ is bounded in E since $\mu < 2$.

Thus, without loss of generality, we can assume that

$$u^j \rightharpoonup u \quad \text{as } j \to \infty \tag{2.4}$$

for some $u \in E$. By the Riesz representation theorem, $\Phi'_{\lambda_j}|_{Y_j} : Y_j \to Y_j^*$ and $I' : E \to E^*$ can be viewed as $\Phi'_{\lambda_j}|_{Y_j} : Y_j \to Y_j$ and $I' : E \to E$, respectively, where Y_j^* and E^* are the dual spaces of Y_j and E, respectively. Note that (2.1) implies that

$$0 = \Phi'_{\lambda_j}(u^j)|_{Y_j} = u^j - \lambda_j P_j I'(u^j), \quad \forall j \in \mathbb{N},$$

where $P_j : E \to Y_j$ is the orthogonal projection for all $j \in \mathbb{N}$, that is,

$$u^j = \lambda_j P_j I'(u^j), \quad \forall j \in \mathbb{N}. \tag{2.5}$$

By the standard argument (see [1, 17]) we know that $I' : E \to E^*$ is compact. Therefore $I' : E \to E$ is also compact. It follows from (2.4) that the right-hand side of (2.5) converges strongly in E, and hence $u^j \to u$ in E.

Therefore $\{u^{\lambda_j}\}$ has a convergent subsequence in E for every $k \ge k_1$, and then Lemma 2.1 implies that Φ has infinitely many nontrivial solutions. □

3 Conclusion

We obtain infinitely many nontrivial solutions for a class of non-periodic Schrödinger lattice systems with nonlinearities sublinear at both zero and infinity.

Appendix

Ax 3.1 $B(u) \ge 0$, $\forall u \in E$, $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite-dimensional subspace of* E.

Proof Obviously, $B(u) \ge 0$ for all $u \in E$ by (SG_1) and the definition of $B(u)$.

We claim that for any finite-dimensional subspace $H \subset E$, there exists a constant $\epsilon > 0$ such that

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n |u_n|^\mu \ge \epsilon \|u\|^\mu\right\}\right) \ge 1, \quad \forall u \in H \backslash \{0\}, \tag{A.1}$$

where $\sharp(\{n \in \mathbb{Z} : \chi_n |u_n|^\mu \ge \epsilon \|u\|^\mu\})$ denotes the number of integers n such that $\chi_n |u_n|^\mu \ge \epsilon \|u\|^\mu$. If not, then for any $j \in \mathbb{N}$, there exists $u^j \in H\backslash\{0\}$ such that

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n \left|u_n^j\right|^\mu \ge \left\|u^j\right\|^\mu / j\right\}\right) = 0.$$

Let $v^j := \frac{u^j}{\|u^j\|} \in H$. Then $\|v^j\| = 1$, and

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n \left|v_n^j\right|^\mu \ge 1/j\right\}\right) = 0, \quad \forall j \in \mathbb{N}. \tag{A.2}$$

Since $\{v^j\}$ is bounded, passing to a subsequence if necessary, we may assume that $v^j \to v$ in E for some $v \in H$ (H is finite dimensional). Evidently, $\|v\| = 1$. Since any two norms on H are equivalent, we have

$$\left\|v^j - v\right\|_{l^2} = \left(\sum_{n=-\infty}^{+\infty} \left|v_n^j - v_n\right|^2\right)^{\frac{1}{2}} \to 0 \quad \text{as } j \to \infty.$$

It follows by the Hölder inequality and $\chi \in l^{\frac{2}{2-\mu}}$ (see (SG_1)) that

$$\sum_{n=-\infty}^{+\infty} \chi_n \left|v_n^j - v_n\right|^\mu \le \|\chi\|_{l^{\frac{2}{2-\mu}}} \left\|v^j - v\right\|_{l^2}^\mu \to 0 \quad \text{as } j \to \infty. \tag{A.3}$$

In fact, since $\|v\| = 1$, there is a constant $\delta_0 > 0$ such that

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n |v_n|^\mu \ge \delta_0\right\}\right) \ge 1. \tag{A.4}$$

If not, then

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n |v_n|^\mu \ge 1/j\right\}\right) = 0, \quad \forall j \in \mathbb{N}.$$

It implies that

$$0 \le \sum_{n=-\infty}^{+\infty} \chi_n |v_n|^{\mu+2} = \sum_{n \in \{n \in \mathbb{Z} : \chi_n |v_n|^\mu < 1/j\}} \chi_n |v_n|^{\mu+2} \le \frac{\|v\|_{l^2}^2}{j} \to 0 \quad \text{as } j \to \infty,$$

which, together with (SG_1), implies that $v = 0$. It is a contradiction to $\|v\| = 1$. Thus (A.4) holds. For any $j \in \mathbb{N}$, let

$$\Lambda_j := \left\{n \in \mathbb{Z} : \chi_n \left|v_n^j\right|^\mu < 1/j\right\} \quad \text{and} \quad \Lambda_j^c := \mathbb{Z} \backslash \Lambda_j = \left\{n \in \mathbb{Z} : \chi_n \left|v_n^j\right|^\mu \ge 1/j\right\}.$$

Set $\Lambda_0 := \{n \in \mathbb{Z} : \chi_n |v_n|^\mu \ge \delta_0\}$. Then for j large enough, by (A.2), (A.4), and the definitions of Λ_0 and Λ_j^c we have

$$\sharp(\Lambda_j \cap \Lambda_0) \ge \sharp(\Lambda_0) - \sharp\left(\Lambda_j^c\right) \ge 1 - 0 = 1.$$

It follows from (SG_1) and the definitions of Λ_j and Λ_0 that for j large enough,

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \chi_n \left|v_n^j - v_n\right|^\mu &\ge \sum_{n \in \Lambda_j \cap \Lambda_0} \chi_n \left|v_n^j - v_n\right|^\mu \\
&\ge \sum_{n \in \Lambda_j \cap \Lambda_0} \left(\frac{1}{2^\mu} \chi_n |v_n|^\mu - \chi_n \left|v_n^j\right|^\mu\right) \\
&\ge \sharp(\Lambda_j \cap \Lambda_0) \left(\frac{\delta_0}{2^\mu} - 1/j\right) \\
&\ge \frac{\delta_0}{2^{\mu+1}} > 0.
\end{aligned}$$

This is a contradiction to (A.3). Therefore (A.1) holds.

For ϵ given in (A.1), let

$$\Lambda_u := \left\{n \in \mathbb{Z} : \chi_n |u_n|^\mu \ge \epsilon \|u\|^\mu\right\}, \quad \forall u \in H \backslash \{0\}.$$

It follows from (SG_1), (A.1), and the definition of Λ_u that

$$B(u) = \sum_{n=-\infty}^{+\infty} \chi_n |u_n|^\mu \ge \sum_{n \in \Lambda_u} \chi_n |u_n|^\mu \ge \epsilon \|u\|^\mu \cdot \sharp(\Lambda_u) \ge \epsilon \|u\|^\mu, \quad \forall u \in H \backslash \{0\}.$$

This implies that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace $H \subset E$. The proof is finished. □

Ax 3.2 *There exist a positive integer k_1 and two sequences $0 < r_k < \rho_k \to 0$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \tag{A.5}$$

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2], \tag{A.6}$$

and

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{A.7}$$

where $Y_k = \bigoplus_{m=1}^{k} X_m = \operatorname{span}\{e_1, \ldots, e_k\}$ *and* $Z_k = \overline{\bigoplus_{m=k}^{\infty} X_m} = \overline{\operatorname{span}\{e_k, \ldots\}}$ *for* $k \in \mathbb{N}$.

Proof (*a*) First, we show that (A.5) holds.

Note first that $Z_k \subset E^+$ for all $k \geq k_1 := k_0 + 1$, where k_0 is the integer defined in the paragraph just before the proof of Theorem 1.1. Thus by the definition of Φ_λ and and the Hölder inequality we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\sum_{n=-\infty}^{+\infty} \chi_n |u_n|^\mu \\ &\geq \frac{1}{2}\|u\|^2 - 2\|\chi\|_{l^{\frac{2}{2-\mu}}} \|u\|_{l^2}^\mu, \quad \forall (\lambda, u) \in [1,2] \times Z_k, \end{aligned} \tag{A.8}$$

for any $k \geq k_1$. Let

$$l(k) := \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_{l^2}}{\|u\|}, \quad \forall k \in \mathbb{N}. \tag{A.9}$$

From [20] and the fact that E is compactly embedded into l^2 (see [21]) we get

$$l(k) \to 0 \quad \text{as } k \to \infty. \tag{A.10}$$

By (A.8) and (A.9) we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2\|\chi\|_{l^{\frac{2}{2-\mu}}} l^\mu(k) \|u\|^\mu, \quad \forall (\lambda, u) \in [1,2] \times Z_k, \tag{A.11}$$

for any $k \geq k_1$. Let

$$\rho_k := \left(8\|\chi\|_{l^{\frac{2}{2-\mu}}} l^\mu(k)\right)^{\frac{1}{2-\mu}}, \quad \forall k \in \mathbb{N}. \tag{A.12}$$

By (A.10) and the fact that $1 < \mu < 2$ we have

$$\rho_k \to 0 \quad \text{as } k \to \infty. \tag{A.13}$$

Therefore by (A.11) and (A.12) we have

$$\alpha_k(\lambda) := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0, \quad \forall k \geq k_1.$$

(*b*) Second, we show that (A.6) holds.

By (A.11) we have

$$\Phi_\lambda(u) \geq -2\|\chi\|_{l^{\frac{2}{2-\mu}}} l^\mu(k)\|u\|^\mu \geq -2\|\chi\|_{l^{\frac{2}{2-\mu}}} l^\mu(k)\rho_k^\mu, \quad \forall\lambda \in [1,2],$$

for all $k \geq k_1$ and $u \in Z_k$ with $\|u\| \leq \rho_k$. Therefore we get

$$-2\|\chi\|_{l^{\frac{2}{2-\mu}}} l^\mu(k)\rho_k^\mu \leq \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \leq 0, \quad \forall\lambda \in [1,2], \forall k \geq k_1.$$

It follows from (A.10) and (A.13) that

$$\xi_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

(*c*) Finally, we show that (A.7) holds.

Note that Y_k is finite dimensional, and thus (A.1) implies that for any $k \in \mathbb{N}$, there exists a constant $\epsilon_k > 0$ such that

$$\sharp\left(\left\{n \in \mathbb{Z} : \chi_n|u_n|^\mu \geq \epsilon_k\|u\|^\mu\right\}\right) \geq 1, \quad \forall u \in Y_k\backslash\{0\}. \tag{A.14}$$

For any $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq \epsilon_k^{\frac{1}{2-\mu}}$, by the definition of Φ_λ and (A.14) we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\left\|u^+\right\|^2 - \sum_{n=-\infty}^{+\infty} \chi_n|u_n|^\mu \\
&\leq \frac{1}{2}\|u\|^2 - \sum_{n\in\{n\in\mathbb{Z}:\chi_n|u_n|^\mu\geq\epsilon_k\|u\|^\mu\}} \epsilon_k\|u\|^\mu \\
&\leq \frac{1}{2}\|u\|^2 - \epsilon_k\|u\|^\mu \cdot \sharp\left(\left\{n \in \mathbb{Z} : \chi_n|u_n|^\mu \geq \epsilon_k\|u\|^\mu\right\}\right) \\
&\leq \frac{1}{2}\|u\|^2 - \epsilon_k\|u\|^\mu \leq -\frac{1}{2}\|u\|^2, \quad \forall\lambda \in [1,2].
\end{aligned} \tag{A.15}$$

Now for any $k \in \mathbb{N}$, if we choose

$$0 < r_k < \min\left\{\rho_k, \epsilon_k^{\frac{1}{2-\mu}}\right\},$$

then (A.15) implies that

$$\beta_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u) \leq -r_k^2/2 < 0, \quad \forall k \in \mathbb{N}.$$

Therefore the proof is finished by (*a*), (*b*), and (*c*). □

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