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L^∞ decay estimates of solutions of nonlinear parabolic equation

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Abstract
In this paper, we are interested in L^∞ decay estimates of weak solutions for the doubly nonlinear parabolic equation and the degenerate evolution m-Laplacian equation not in the divergence form. By a modified Moser's technique we obtain L^∞ decay estimates of weak solutiona.

MSC: 35K59; 35K65; 35K92

Keywords: Doubly nonlinear equation; Degenerate evolution m-Laplacian equation; L^∞ decay estimates of solution

1 Introduction

In this paper, we are interested in the L^∞ decay estimate of the solution for the initial-boundary-value problem of the nonlinear parabolic equation in the divergence form

$$\begin{cases} u_t = \operatorname{div}(a(x,t,u,\nabla u)) & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

and the degenerate evolution m-Laplacian equation

$$\begin{cases} u_t = |u|^k \operatorname{div}(|\nabla u|^{m-2}\nabla u) + b(u)\cdot\nabla u & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $k > 0$, Ω is a open set of $\mathbb{R}^N$ (not necessary bounded) with smooth boundary $\partial\Omega$, and $a(x,t,u,\xi)$ is a Carathéodory function in $\Omega \times \mathbb{R}^+ \times \mathbb{R}^1 \times \mathbb{R}^N$, where $\mathbb{R}^+ = [0,+\infty)$.

The model problem for (1.1) is the so-called doubly nonlinear equation

$$\begin{cases} u_t = \operatorname{div}(|u|^r|\nabla u|^{m-2}\nabla u) & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

with $r > 0$ and $1 < m < N$.

The interest in parabolic equations (1.1) and (1.2) comes from their mathematical structure. Many results concerning the global existence, blowup, and asymptotic behavior of solutions have been established; see [1–3, 8, 9, 13, 19, 20, 22, 23].

It is well-known that the solution $u(t)$ of the initial value problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{1.4}$$

satisfies the L^∞ decay estimate

$$\|u(t)\|_{L^\infty(R^N)} \le C\|u_0\|_{L^q(R^N)} t^{-N/2q}, \quad t > 0, \tag{1.5}$$

with $u_0 \in L^q(\mathbb{R}^N)$, $q \ge 1$. Estimate (1.5) remains true for the solution of heat equation in a general open set Ω of $\mathbb{R}^N$ with zero Dirichlet boundary condition

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Estimate (1.5), or more general estimates

$$\|u(t)\|_{L^\infty(\Omega)} \le C\|u_0\|^{\alpha}_{L^q(\Omega)} t^{-\lambda}, \quad t > 0, \tag{1.7}$$

where α and λ are suitable positive constants, is known in the literature as L^∞ decay estimates or ultracontractive estimates; see [6, 7, 11, 13, 17, 19].

These estimates have been proved not only for the heat equation but also for various differential problems, linear or nonlinear, degenerate or singular, for example, the evolution m-Laplacian equation, the porous media equation, the fast equation, and the doubly nonlinear equation; see [1–3, 8, 9, 11, 15, 17–19] and the references therein. The importance of estimate (1.7) describes the behavior of solution as $t \to 0$ and $t \to +\infty$.

The proofs of these estimates vary from problem to problem. In many cases, suitable families of logarithmic Sobolev inequalities are derived. These inequalities are similar to the well-known Gross logarithmic Sobolev inequalities [11].

Porzio [17] investigated the solution of the Leray–Lions-type problem

$$\begin{cases} u_t = \operatorname{div}(a(x,t,u,\nabla u)) & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $a(x,t,s,\xi)$ is a Carathéodory function satisfying the following structure condition:

$$a(x,t,s,\xi)\xi \ge \theta|\xi|^m, \quad \forall (x,t,s,\xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^1 \times \mathbb{R}^N, \tag{1.9}$$

with $\theta > 0$. By the integral inequalities method Porzio derived the L^∞ decay estimate of the form (1.7) with $C = C(N,q,m,\theta)$, $\alpha = \frac{mq}{N(m-2)+mq}$, and $\lambda = \frac{N}{N(m-2)+mq}$. We see that the equation in problem (1.8) is in the divergence form.

Recently, Ghoul et al. [10] studied the Cauchy problem of the parabolic equation

$$\begin{cases} u_t = -(-\Delta)^m u + u|u|^{p-1}, & (x,t) \in \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

and derived an estimate for $\|u(t)\|_{L^\infty(\mathbb{R}^N)}$ with $u_0 \in L^\infty(\mathbb{R}^N)$ by a formal approach based on spectral analysis. Similar consideration can been found in [12, 21].

In this paper, we derive the L^∞ decay estimate like (1.7) for the solutions of problems (1.1) and (1.2). Our method is different from that in [17], and we will use a modified Moser technique as in [4, 5, 15] to get an L^∞ decay estimate. Since the equation in (1.2) is not in the divergence form, it seems difficult to derive estimate (1.7) by the integral inequalities method in [17].

This paper is organized as follows. In Sect. 2, we state the main results and present some needed lemmas. In Sect. 3, we use these lemmas to derive L^∞ decay estimates for the solutions of (1.1). The L^∞ decay estimates for the solutions of (1.2) are established in Sect. 4.

2 Preliminaries and main results

We first make the following assumptions.

(H_1) $a(x,t,u,\xi)$ is a Carathéodory function and satisfies the structure condition

$$a(x,t,u,\xi)\xi \geq \alpha_0|u|^r|\xi|^m, \quad \forall(x,t,u,\xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^1 \times \mathbb{R}^N, \tag{2.1}$$

for some $\alpha_0 > 0$ and $r \geq 0$, where $1+\beta < m < N$ and $0 < \beta = (m-1)(r+m-1)^{-1} \leq 1$.

(H_2) the initial data $u_0 \in L^q(\Omega)$, $q \geq 1$.

As in [20], we introduce a new independent variable $u = |v|^{\beta-1}v$. Then from (2.1) it follows that the principal part of the equation in (1.1) satisfies

$$a(x,t,u,\nabla u)\nabla v \geq \alpha_0\beta^{m-1}|\nabla v|^m. \tag{2.2}$$

Instead of (1.1), we consider the initial-boundary-value problem

$$\begin{cases} (|v|^{\beta-1}v)_t = \operatorname{div}(a(x,t,|v|^{\beta-1}v,\nabla(|v|^{\beta-1}v))) & \text{in } \Omega \times (0,+\infty), \\ v(x,t) = 0, \quad \text{on } \partial\Omega \times (0,+\infty), & v(x,0) = v_0(x), \quad \text{in } \Omega, \end{cases} \tag{2.3}$$

with $v_0(x) = |u_0(x)|^{-1+1/\beta}u_0(x)$.

Let $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote the norms in the Banach spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, $1 \leq p \leq \infty$. We often drop the letter Ω in these notations. In the following, we will consider (2.3) instead of (1.1), with v replaced by u in (2.3) for convenience.

Definition 1 A measurable function $u(x,t)$ on $\Omega \times (0,\infty)$ is said to be a global weak solution of problem (2.3) if $u(x,t) \in L^\beta_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$, $a(x,t,|u|^{\beta-1}u,\nabla(|u|^{\beta-1}u)) \in L^1_{\mathrm{loc}}(\mathbb{R}^+;L^1(\Omega))$, and the equality

$$\begin{aligned} &\int_0^t \int_\Omega \bigl\{-|u|^{\beta-1}u\varphi_t - a\bigl(x,\tau,|u|^{\beta-1}u,\nabla\bigl(|u|^{\beta-1}u\bigr)\bigr)\nabla\varphi\bigr\}\,dx\,d\tau \\ &\quad = \int_\Omega \bigl|u_0(x)\bigr|^{\beta-1}u_0(x)\varphi(x,0) - \bigl|u(x,t)\bigr|^{\beta-1}u(x,t)\varphi(x,t)\,dx \end{aligned} \tag{2.4}$$

is valid for any $\varphi \in C^1(\mathbb{R}^+, C^1_0(\Omega))$ and $t > 0$.

Our first main result reads as follows.

Theorem 1 *Assume* (H_1)–(H_2). *If* $u(t)$ *is a global weak solution of* (2.3), *then it satisfies*

$$u(t) \in L^\infty\bigl(\mathbb{R}^+; L^q(\Omega)\bigr) \cap L^{m-1}_{\mathrm{loc}}\bigl((0,\infty); W_0^{m-1}(\Omega)\bigr) \tag{2.5}$$

and the L^∞ *decay estimate*

$$\bigl\|u(t)\bigr\|_q \le \|u_0\|_q, \quad t > 0, \tag{2.6}$$

$$\bigl\|u(t)\bigr\|_\infty \le C_0\|u_0\|_q^\mu t^{-\lambda}, \quad t > 0, \tag{2.7}$$

with $\mu = \frac{mq}{MN+mq}$, $\lambda = \frac{N}{MN+mq}$, $M = m - 1 - \beta > 0$, *and* $C_0 = C_0(N, m, q)$.

Remark 1 The existence of a global weak solution for (2.3) can be established similarly as in [4, 15, 20].

For the degenerate evolution m-Laplacian problem (1.2), Passo and Luckhaus [16] considered the global existence and blowup of solution for $m = 2$, $k = 1$ by the lower and upper solution method. For $m = 2$, $k > 1$, blowup and asymptotic behavior of solution have been established by Wiegner [22] and Winkler [23]. Here we derive an L^∞ decay estimate for the solution of (1.2) with $k > 0$, $1 < m < N$.

For problem (1.2), we assume:

(H_3) Let $B(u) = (B_1(u), B_2(u), \dots, B_N(u))$, $B'(u) = (B_1'(u), B_2'(u), \dots, B_N'(u))$, where $B'(u) = b(u) = (b_1(u), b_2(u), \dots, b_N(u))$, $b_i(u) \in C^1(\mathbb{R}^1)$, $i = 1, 2, \dots, N$. There exist $k_1, \gamma \ge 0$, such that

$$\bigl|B(u)\bigr| \le k_1|u|^{1+\gamma}, \qquad \bigl|B'(u)\bigr| \le k_1|u|^\gamma, \quad \forall u \in \mathbb{R}^1; \tag{2.8}$$

(H_4) $u_0 \in L^q(\Omega)$, $q \ge 1$.

Definition 2 ([16, 22, 23]) A measurable function $u(t) = u(x,t)$ on $\Omega \times (0, +\infty)$ is said to be a global weak solution of problem (1.2) if $u(t) \in X = L^\infty(\mathbb{R}^+, L^q(\Omega))$, $|u|^{(k-1)/m}u \in L^m_{\mathrm{loc}}((0,+\infty); W_0^{1,m}(\Omega))$, $|u|^{(k-1)/(m-1)}u \in L^{m-1}_{\mathrm{loc}}((0,\infty); W_0^{1,m-1}(\Omega))$,

$$\begin{aligned} &\int_0^t \int_\Omega \bigl\{-u\phi_t + |\nabla u|^{m-2}\nabla u \cdot \nabla\bigl(|u|^k\phi\bigr) + B(u)\cdot\nabla\phi\bigr\}\,dx\,d\tau \\ &\quad = \int_\Omega u(x,t)\phi(x,t)\,dx - \int_\Omega u_0(x)\phi(x,0)\,dx \end{aligned} \tag{2.9}$$

for all $\varphi \in C^1(R^+, C_0^1(\Omega))$ and $t > 0$.

Our second main result is the following:

Theorem 2 *Suppose that* (H_3)–(H_4) *hold and* $k \ge 0$. *If* $u(t)$ *is a global weak solution of* (1.2), *then* $u(t)$ *satisfies the following* L^∞ *estimates*:

$$\bigl\|u(t)\bigr\|_q \le \|u_0\|_q, \quad t > 0, \tag{2.10}$$

$$\left\|u(t)\right\|_\infty \le C_0\|u_0\|_q^\alpha t^{-\lambda}, \quad t>0, \tag{2.11}$$

with $\alpha = \frac{qm}{MN+mq}$, $\lambda = \frac{N}{MN+mq}$, $M = k+m-2>0$, *and* $C_0 = C_0(N,m,q)$.

To derive above results, we will use the following lemmas.

Lemma 1 *Let* $y(t)$ *be a nonnegative differentiable function on* $(0,\infty)$ *satisfying*

$$y'(t) + At^\mu y^{1+\theta}(t) \le 0, \quad t \ge 0,$$

with $A,\theta>0$, $\mu\ge 0$. *Then we have*

$$y(t) \le \big(A\theta/(1+\mu)\big)^{-1/\theta} t^{-(1+\mu)/\theta}, \quad t>0.$$

Lemma 2 (Gagliardo–Nirenberg-type inequality) *Let* Ω *be a domain (not necessary bounded) in* $\mathbb{R}^N$ *with smooth boundary* $\partial\Omega$. *Let* $\beta\ge 0$, $N>m\ge 1$, $q\ge 1+\beta$, *and* $1\le r\le q\le(1+\beta)Nm/(N-m)$. *Then for* $|u|^\beta u \in W_0^{1,m}(\Omega)$, *we have*

$$\|u\|_q \le C_0^{1/(\beta+1)}\|u\|_r^{1-\theta}\big\|\nabla\big(|u|^\beta u\big)\big\|_m^{\theta/(\beta+1)}$$

with $\theta = (1+\beta)(r^{-1}-q^{-1})/(N^{-1}-m^{-1}+(1+\beta)r^{-1})$, *where the constant* C_0 *depends only on* m, N.

The proof of Lemma 2 can be obtained from the well-known Gagliardo–Nirenberg–Sobolev inequality and the interpolation inequality, and we omit it here.

3 Proof of Theorem 1

In this section, we assume that all assumptions in Theorem 1 are satisfied. As in [4, 5, 15], we derive a priori estimates of the smooth approximate solutions $u(t)$, and our argument will be justified through such an approximate procedure.

Proof of Theorem 1 First, we take $f_n(s)$ $(n=1,2,\dots)$ such that $f_n(s)\to f(s)=|s|^{q-2}s$ uniformly in $\mathbb{R}^1$ as $n\to\infty$.

For $1<q<2$, we choose $f_n^+(s) = a_n s^2 + b_n s$ if $0\le ns\le 1$ and $f_n^+(s) = s^{q-1}$ if $ns\ge 1$, where $a_n = (q-2)n^{3-q}$, $b_n = (3-q)n^{2-q}$. Further, let $f_n(s)$ be the odd extension of $f_n^+(s)$ in $\mathbb{R}^1$.

If $q\ge 2$, then we take $f_n(s) = |s|^{q-2}s$. For $q=1$, we let

$$f_n(s) = \begin{cases} 1, & s\ge 1/n, \\ ns(2-ns), & 0\le s\le 1/n, \\ -ns(2+ns), & -1/n\le s\le 0, \\ -1, & s<-1/n. \end{cases} \tag{3.1}$$

Then we easily verify that $f_n(s)\in C^1(\mathbb{R}^1)$, $f_n(s)\to f(s)=|s|^{q-2}s$ uniformly in $\mathbb{R}^1$ as $n\to\infty$.

Let $\varphi_n^+(s) = s^{\beta-1}$ if $ns\ge 1$, $\varphi_n^+(s) = A_n s + B_n$ if $0\le ns\le 1$, where $A_n = (\beta-1)n^{2-\beta}$, $B_n = (2-\beta)n^{1-\beta}$. Further, let $\varphi_n(s)$ be the even extension of $\varphi_n^+(s)$ in $\mathbb{R}^1$. Obviously, $\varphi_n(s)\in C^1(\mathbb{R}^1)$, and $\varphi_n(s)\to\varphi(s)=|s|^{\beta-1}$ uniformly in $\mathbb{R}^1$ as $n\to\infty$.

Let $u_{0,n} \in C_0^2(\Omega)$ and $u_{0,n} \to u_0$ in $L^q(\Omega)$ as $n \to \infty$. We take the approximate problem of (2.3) of the form

$$\begin{cases} \varphi_i(u)u_t = \operatorname{div}(a(x,t,|u|^{\beta-1}u, \nabla(|u|^{\beta-1}u))) & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_{0,i}(x) & \text{in } \Omega, \end{cases} \tag{3.2}$$

for $i = 1, 2, \dots$.

Then problem (3.2) has a unique smooth solution $u_i(x,t)$; see [14]. We further always write u instead of u_i and u^p for $|u|^{p-1}u$ when $p > 0$.

Multiplying the equation in (3.2) by $f_k(u)\varphi_i^{-1}(u)$, we obtain

$$\begin{aligned} &\int_\Omega f_k(u)u_t\,dx \\ &\quad = -\int_\Omega a\bigl(x,t,|u|^{\beta-1}u, \nabla\bigl(|u|^{\beta-1}u\bigr)\bigr)\nabla u\bigl(f_k'(u)\varphi_i(u) - \varphi_i'(u)f_k(u)\bigr)\varphi_i^{-2}(u)\,dx, \end{aligned} \tag{3.3}$$

where

$$f_k'(u)\varphi_i(u) - \varphi_i'(u)f_k(u) \ge 0.$$

By (H_1) we have

$$\begin{aligned} &a\bigl(x,t,|u|^{\beta-1}u, \nabla\bigl(|u|^{\beta-1}u\bigr)\bigr)\nabla u \\ &\quad = \beta^{-1}a\bigl(x,t,|u|^{\beta-1}u, \nabla\bigl(|u|^{\beta-1}u\bigr)\bigr)\nabla\bigl(|u|^{\beta-1}u\bigr)|u|^{1-\beta} \\ &\quad \ge \alpha_0\beta^{-1}|u|^{\beta r}\bigl|\nabla\bigl(|u|^{\beta-1}u\bigr)\bigr|^m|u|^{1-\beta} \ge 0. \end{aligned} \tag{3.4}$$

Hence from (3.3) and (3.4) it follows that

$$\int_\Omega f_k(u)u_t\,dx \le 0. \tag{3.5}$$

Letting $k \to \infty$ in (3.5) gives

$$\bigl\|u(t)\bigr\|_q \le \|u_0\|_q, \quad t \ge 0. \tag{3.6}$$

We now derive an L^∞ decay estimate for the solution $u_i(t)$ of (3.2). Multiplying the equation in (3.2) by $\varphi_i^{-1}(u)|u|^{p-2}u$, $p \ge 2$, we have

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + \int_\Omega a\bigl(x,t,|u|^{\beta-1}u, \nabla\bigl(|u|^{\beta-1}u\bigr)\bigr)\nabla u E_i[u]\,dx = 0, \tag{3.7}$$

where

$$E_i[u] = \bigl((p-1)|u|^{p-2}\varphi_i(u) - \varphi_i^{-1}(u)|u|^{p-2}u\bigr)\varphi_i^{-2}(u) \ge \frac{p-\beta}{4}|u|^{p-\beta-1}. \tag{3.8}$$

Noting that $\beta = (m-1)/(r+m-1)$, from (3.4) we get that

$$a\big(x,t,|u|^{\beta-1}u,\nabla\big(|u|^{\beta-1}u\big)\big)\nabla u \geq \beta^{-1}\alpha_0|u|^{\beta r}\big|\nabla\big(|u|^{\beta-1}u\big)\big|^m|u|^{1-\beta} = \alpha_0\beta^{m-1}|\nabla u|^m. \tag{3.9}$$

Hence from (3.7)–(3.9) it follows that

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + C_1p\left(\frac{m}{p+M}\right)^m\int_\Omega\big|\nabla u^{\frac{p+M}{m}}\big|^m\,dx \leq 0, \tag{3.10}$$

where $M = m-1-\beta > 0$. Then (3.10) implies that

$$\frac{d}{dt}\big\|u(t)\big\|_p^p + C_1p^{2-m}\big\|\nabla u^{\frac{p+M}{m}}\big\|_m^m \leq 0, \quad \forall t > 0. \tag{3.11}$$

Let C, C_j be general constants independent of p, i, n changeable from line to line. We now employ Moser's technique as in [4, 5, 15]. Set $R > 1 + M/q$, $p_1 = q$, $p_n = Rp_{n-1} - M$, $\theta_n = RN(1-p_{n-1}p_n^{-1})(m+N(R-1))^{-1}$, $\beta_n = (p_n+M)\theta_n^{-1}$, $n = 2,3,\ldots$.

From Lemma 2 we see that

$$\big\|u(t)\big\|_{p_n} \leq C^{\frac{m}{p_n+M}}\|u\|_{p_{n-1}}^{1-\theta_n}\big\|\nabla u^{\frac{p_n+M}{m}}\big\|_m^{m\theta_n/(p_n+M)}. \tag{3.12}$$

Inserting this into (3.11) ($p = p_n$) yields

$$\frac{d}{dt}\big\|u(t)\big\|_{p_n} + C_1C^{\frac{-m}{\theta_n}}p_n^{2-m}\|u\|_{p_{n-1}}^{M-\beta_n}\|u\|_{p_n}^{1+\beta_n} \leq 0, \quad \forall t > 0. \tag{3.13}$$

We claim that there exist bounded sequences $\{\xi_n\}$ and $\{\lambda_n\}$ such that

$$\big\|u(t)\big\|_{p_n} \leq \xi_n t^{-\lambda_n}, \quad \forall t > 0, \tag{3.14}$$

where $\lambda_n = (1+\lambda_{n-1}(\beta_n-M))/\beta_n$. It is not difficult to show that $\lambda_n \to \lambda = \frac{N}{MN+mq}$ as $n \to \infty$.

In fact, let $\xi_1 = \|u_0\|_q$ and $\lambda_1 = 0$. If (3.14) is true for $n-1$, the from (3.13) it follows that

$$\frac{d}{dt}\big\|u(t)\big\|_{p_n} + C_1C^{\frac{-m}{\theta_n}}p_n^{1-m}\xi_n^{M-\beta_n}t^{\lambda_{n-1}(\beta_n-M)}\|u\|_{p_n}^{1+\beta_n} \leq 0, \quad \forall t > 0. \tag{3.15}$$

An application of Lemma 1 to (3.15) yields

$$\begin{aligned}\big\|u(t)\big\|_{p_n} &\leq \big(C_1C^{\frac{-m}{\theta_n}}p_n^{1-m}\xi_{n-1}^{M-\beta_n}\beta_n/\big(1+\lambda_{n-1}(\beta_n-M)\big)\big)^{-1/\beta_n}t^{-(1+\lambda_{n-1}(\beta_n-\mu))/\beta_n} \\ &= \big(C_1C^{\frac{-m}{\theta_n}}\big)^{-1/\beta_n}\lambda_n^{1/\beta_n}p_n^{(m-1)/\beta_n}\xi_{n-1}^{(\beta_n-M)/\beta_n}t^{-\lambda_n}.\end{aligned} \tag{3.16}$$

Since

$$\lim_{n\to\infty}\frac{p_n}{\beta_n} = \frac{M+2}{N(M+1)},$$

we see that there exists a constant $\lambda_0 > 0$, independent of n, such that

$$\big\|u(t)\big\|_{p_n} \leq (\lambda_0p_n)^{\lambda_0/p_n}\xi_{n-1}^{1-M/\beta_n}t^{-\lambda_n}, \quad t > 0. \tag{3.17}$$

Hence we define ξ_n inductively by

$$\xi_n = (\lambda_0 p_n)^{\lambda_0/p_n} \xi_{n-1}^{1-M/\beta_n} \tag{3.18}$$

for $n = 2, 3, \ldots$ with $\xi_1 = \|u_0\|_q$. Here, setting $\omega_n = mp_n + MN$, $p_1 = q$, and $p_n = Rp_{n-1} - M$, by direct calculation we get

$$\frac{\beta_n - M}{\beta_n} = \frac{\omega_n}{p_n} \cdot \frac{p_{n-1}}{\omega_{n-1}} \tag{3.19}$$

and

$$\prod_{k=2}^{n} \frac{\beta_k - M}{\beta_k} = \frac{\omega_n}{p_n} \cdot \frac{p_1}{\omega_1} = \frac{MN + p_n m}{p_n} \cdot \frac{q}{mq + MN}. \tag{3.20}$$

It is easy to show that

$$\lim_{n\to\infty} \prod_{k=2}^{n} \frac{\beta_k - M}{\beta_k} = \mu = \frac{mq}{mq + MN}. \tag{3.21}$$

On the other hand, the definition of ξ_n gives

$$\begin{aligned}
\log \xi_n &= \frac{\lambda_0}{p_n}(\log \lambda_0 + \log p_n) + \left(1 - \frac{M}{\beta_n}\right) \log \xi_{n-1} \\
&= \frac{\lambda_0}{p_n}(\log \lambda_0 + \log p_n) + \left(1 - \frac{M}{\beta_n}\right)\left(\frac{\lambda_0}{p_n}(\log \lambda_0 + \log p_{n-1})\right. \\
&\quad \left. + \left(1 - \frac{M}{\beta_{n-1}}\right) \log \xi_{n-2}\right) \\
&\le \lambda_0 \sum_{k=2}^{n} \frac{\log \lambda_0 + \log p_k}{p_k} + \prod_{k=2}^{n} \left(1 - \frac{M}{\beta_k}\right) \log \xi_1.
\end{aligned} \tag{3.22}$$

Hence

$$\log \xi_n \le C_0 + \frac{MN + p_n m}{p_n} \cdot \frac{q}{mq + MN} \log \xi_1 \tag{3.23}$$

with some $C_0 > 0$ independent of n. Then

$$\log \xi_n \le C_0 + \mu \log \xi_1 \tag{3.24}$$

and

$$\xi_n \le e^{C_0} \xi_1^{\mu} = C_1 \|u_0\|_q^{\mu} \quad t > 0. \tag{3.25}$$

Then, letting $n \to \infty$ in (3.14), we obtain (2.7) and finish the proof of Theorem 1. □

4 Proof of Theorem 2

In this section, we derive L^∞ decay estimates of solutions for the degenerate evolution m-Laplacian problem (1.2).

Similarly as in the proof of Theorem 1, we take $u_{0,n} \in C_0^2(\Omega)$ such that $u_{0,n} \to u_0$ in $L^q(\Omega)$. Further, we choose $\phi_n(s) \in C^1(\mathbb{R}^1)$, $\phi_n(s) \to \phi(s)$ uniformly in $\mathbb{R}^1$.

In fact, for $n = 1, 2, \dots$, we define $\phi_n(s) = |s|^k + n^{-k}$ if $k > 1$ and

$$\phi_n(s) = \begin{cases} |s|^k + n^{-k} & \text{for } |s| \geq n^{-1}, \\ s^2 n^{2-k}(3 - k + (k-2)n|s|) + n^{-k} & \text{for } |s| \leq n^{-1} \end{cases} \tag{4.1}$$

if $0 < k \leq 1$.

We now consider the following approximate problem for (1.2):

$$\begin{cases} u_t = \phi_i(u) \operatorname{div}((|\nabla u|^2 + i^{-1})^{m/2} \nabla u) + b(u)\nabla u & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_{0,i} & \text{in } \Omega, \end{cases} \tag{4.2}$$

for $i = 1, 2, \dots$.

Problem (4.2) is a standard quasilinear parabolic equation and admits a unique smooth solution $u_i(x,t)$ for each i; see [4, 5, 14, 15]. For convenience, we denote u_i by u and $|u|^{p-1}u$ by u^p if $p > 0$.

Multiplying the equation in (4.2) by $|u|^{q-2}u$ (if $q > 1$), we obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |u|^q \, dx + \int_\Omega |\nabla u|^m \big(\phi_i'(u)|u|^{q-2}u + (q-1)\phi_i(u)|u|^{q-2}\big)\, dx \leq 0. \tag{4.3}$$

Note that

$$\phi_i'(u)|u|^{q-2}u + (q-1)\phi_i(u)|u|^{q-2}\, dx \geq 0. \tag{4.4}$$

Then

$$\frac{1}{q}\frac{d}{dt}\int_\Omega |u|^q \, dx \leq 0. \tag{4.5}$$

This implies that

$$\big\|u(t)\big\|_q \leq \|u_0\|_q, \quad \forall t \geq 0. \tag{4.6}$$

If $q = 1$, then we multiply the equation in (4.2) by $f_n(u)$, where $f_n(u)$ is defined by (3.1). Similarly, we can get estimate (4.6).

To derive an L^∞ decay estimate of solutions for (4.2), we multiply the equation in (4.2) by $|u|^{p-2}u(p \geq q)$ and obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \, dx + \int_\Omega |\nabla u|^m \big(\phi_i'(u)|u|^{p-2}u + (p-1)\phi_i(u)|u|^{p-2}\big)\, dx \leq 0. \tag{4.7}$$

Note that

$$\phi_i'(u)|u|^{p-2}u + (p-1)\phi_i(u)|u|^{p-2}\,dx \geq \frac{k+p-1}{4}|u|^{k+p-2}. \tag{4.8}$$

Hence from (4.7) and (4.8) it follows that

$$\frac{d}{dt}\left\|u(t)\right\|_p^p + C_1 p^{2-m}\left\|\nabla u^{\frac{p+M}{m}}\right\|_m^m \leq 0, \quad \forall t > 0, \tag{4.9}$$

where $M = k + m - 2 > 0$.

Set $R > 1 + M/q$, $p_1 = q$, $p_n = Rp_{n-1} - M$, $\theta_n = RN(1 - p_{n-1}p_n^{-1})(m + N(R-1))^{-1}$, $\beta_n = (p_n + M)\theta_n^{-1}$, $n = 2, 3, \ldots$. From Lemma 2 we see that

$$\left\|u(t)\right\|_{p_n} \leq C^{m/(p_n+M)}\|u\|_{p_{n-1}}^{1-\theta_n}\left\|\nabla u^{\frac{p_n+M}{m}}\right\|_m^{m\theta_n/(p_n+M)}. \tag{4.10}$$

Inserting this into (4.9) ($p = p_n$) yields

$$\frac{d}{dt}\left\|u(t)\right\|_{p_n} + C_2 C^{\frac{-m}{\theta_n}} p_n^{2-m}\|u\|_{p_{n-1}}^{M-\beta_n}\|u\|_{p_n}^{1+\beta_n} \leq 0 \quad t > 0. \tag{4.11}$$

As in the proof of Theorem 1, we can show that there exist bounded sequences $\{\xi_n\}$ and $\{\lambda_n\}$ such that

$$\left\|u(t)\right\|_{p_n} \leq \xi_n t^{-\lambda_n} \quad t > 0, \tag{4.12}$$

in which $\lambda_n \to \lambda$ and $\xi_n \leq C_0\|u_0\|_q^{\mu}$ with

$$\lambda = \frac{N}{mq + MN}, \qquad \mu = \frac{qm}{qm + MN}, \qquad M = k + m - 2 > 0. \tag{4.13}$$

Letting $n \to \infty$ in (4.12), we have

$$\left\|u(t)\right\|_\infty \leq C_0\|u_0\|_q^{\mu} t^{-\lambda}, \quad \forall t \geq 0. \tag{4.14}$$

This finishes the proof of Theorem 2.

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Not applicable.

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Authors' contributions
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