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Long-time behavior of solutions for a fractional diffusion problem

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Abstract

This paper deals with the asymptotic behavior of solutions to the initial-boundary value problem of the following fractional p -Kirchhoff equation:

$$u_t + M([u]_{s,p}^p)(-\Delta)_p^s u + f(x, u) = g(x) \quad \text{in } \Omega \times (0, \infty),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N > ps$, $0 < s < 1 < p$, $M : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing continuous function, $[u]_{s,p}$ is the Gagliardo seminorm of u , $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $g \in L^2(\Omega)$. With general assumptions on f and g , we prove the existence of global attractors in proper spaces. Then, we show that the fractal dimension of global attractors is infinite provided some conditions are satisfied.

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Keywords: Fractional p -Laplacian; Global attractors; Fractal dimension

1 Introduction and main results

Let $s \in (0, 1)$ and let Ω be a bounded domain in \mathbb{R}^N ($N > 2s$) with smooth boundary $\partial\Omega$. We consider the asymptotic behavior of solutions to the following fractional p -Kirchhoff problem:

$$\begin{cases} u_t + M([u]_{s,p}^p)(-\Delta)_p^s u + f(x, u) = g(x) & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $u_0 \in L^2(\Omega)$, $g \in L^2(\Omega)$, $[u]_{s,p}^p = \iint_Q \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{N+ps}} dx dy$ with $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$, $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is a continuous function, and $(-\Delta)_p^s$ is the fractional p -Laplacian, which (up to normalization factors) may be defined by

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{R \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} dy$$

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for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $1 < p < \frac{N}{s}$ and $s \in (0, 1)$. Henceforward $B_R(x)$ denotes the ball of \mathbb{R}^N centered at $x \in \mathbb{R}^N$ and radius $R > 0$. We assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying:

(f₁) there exists a positive constant λ such that

$$(f(x, \xi_1) - f(x, \xi_2))(\xi_1 - \xi_2) \geq -\lambda |\xi_1 - \xi_2|^2 \quad (1.2)$$

for any $(x, t) \in \Omega \times \mathbb{R}^+$ and $\xi_1, \xi_2 \in \mathbb{R}$;

(f₂) there exist positive constants c, c_1, c_2 such that

$$c_1 |\xi|^q - c \leq f(x, \xi) \leq c_2 |\xi|^q + c \quad \text{for any } (x, t) \in \Omega \times \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}, \quad (1.3)$$

where q satisfies $2 \leq q < \infty$.

A typical example of f is given by $f(x, \xi) = a(x)|\xi|^{q-2}\xi - b(x)|\xi|^{r-2}\xi$ for all $x \in \Omega$ and $\xi \in \mathbb{R}$, with $2 < r < q < \infty$. Here a, b are two bounded continuous functions in Ω .

Throughout this paper, we always assume that $s \in (0, 1)$, $N > 2s$, $u_0(x) \in L^2(\Omega) \cap W_0$ is a nonzero function. For the coefficient function M , we assume that

(M₁) $M : [0, \infty) \rightarrow (0, \infty)$ is a nondecreasing and continuous function and there exists $m_0 > 0$ such that $M(\tau) \geq m_0$ for all $\tau \geq 0$.

In particular, if $p = 2$, then $(-\Delta)_p^s$ reduces to the fractional Laplace operator $(-\Delta)^s$, see [5] and the references cited there. For the basic properties of fractional p -Laplacian, we refer the readers to [13, 16, 17]. Problem (1.1) is a class of nonlocal fractional diffusion problems. It is relevant in anomalous diffusion theory. More precisely, if $p = 2$ and $M \equiv 1$, as stated in [7], the function $u(x, t)$ is thought of as a density of population at the point x and time t , the term $(-\Delta)^s u(x, t)$ represents the diffusion of density, $f(x, u)$ is the source term, g is the perturbation term, and the coefficient $M : [0, \infty) \rightarrow (0, \infty)$ denotes the possible changes of total population in whole space. This implies that the behavior of individuals is subject to total population, such as the diffusion process of bacteria, see [21, 27]. It is worth mentioning that the interest in fractional problems goes beyond the mathematical curiosity. Actually, the study of problems like (1.1) has many significant applications, as explained by Caffarelli in [2, 3], Laskin in [15], and Vázquez in [31]. Very recently, Fiscella and Valdinoci [9] first proposed a stationary Kirchhoff variational equation which models the nonlocal aspect of the tension of the string. Indeed, the stationary problem (1.1) is a fractional version of a model, the so-called stationary Kirchhoff equation, introduced by Kirchhoff in [14]. The body of literature on elliptic type problems involving nonlocal operators is quite large, here we just collect a few works, see [1, 8, 9, 19–22, 26, 28, 33, 36] and the references cited there.

To the best of our knowledge, there are a few papers that deal with the asymptotic behavior of solutions for problems like (1.1). Pucci, Xiang, and Zhang [27] discussed the initial-boundary value problem of the following equation:

$$u_t + M([u]_{s,p}^p)(-\Delta)_p^s u = f(x, t) \quad \text{in } \Omega \times (0, \infty), \quad (1.4)$$

where $M : [0, \infty) \rightarrow (0, \infty)$ is a continuous and nondecreasing function. Using the sub-differential approach, the authors obtained the well-posedness of solutions for (1.4). Moreover, the large-time behavior and extinction property of solutions were given. In [25], Pan,

Zhang, and Cao studied the initial-boundary value problem of the following fractional p -Kirchhoff equation:

$$u_t + [u]_{s,p}^{(\lambda-1)p} (-\Delta)_p^s u = |u|^{q-2} u \quad \text{in } \Omega \times (0, \infty),$$

where $1 \leq \lambda < N/(N - sp)$ and $p < q < Np/(N - sp)$ with $1 < p < N/s$. The existence of a global solution was obtained by employing the Galerkin method and the potential well theory. Xiang, Rădulescu, and Zhang [21] studied the initial-boundary value problem of the following fractional Kirchhoff equation:

$$u_t + M([u]_{s,2}^2) (-\Delta)^s u = |u|^{p-2} u \quad \text{in } \Omega \times (0, \infty),$$

where $M : [0, \infty) \rightarrow (0, \infty)$ is continuous, and there exist $m_0 > 0$ and $\theta > 1$ such that $M(\sigma) \geq m_0 \sigma^{\theta-1}$ for all $\sigma \geq 0$, and M also satisfies the following:

(M) There exists $\theta \in [1, p_s^*/p)$ such that $tM(t) \leq \theta \mathcal{M}(t)$ for all $t \geq 0$, where $\mathcal{M}(t) = \int_0^t M(\tau) d\tau$.

Under suitable assumptions, the authors obtained the local existence of nonnegative solutions by applying the Galerkin method and proved that the local nonnegative solutions blow up in finite time. In [34], Xiang and Yang studied the initial-boundary value of the following equation:

$$u_t + M([u]_{s,p}^p) (-\Delta)_p^s u = \lambda |u|^{q-2} u - \mu |u|^{r-2} u.$$

The authors gave some sufficient conditions such that the solutions of the above equation vanish in finite time. Moreover, the non-extinction of solutions was also investigated.

Very recently, Wang and Huang [32] considered a weakly damped fractional nonlinear Schrödinger equation on the real line \mathbb{R}

$$u_t - i(-\Delta)u^s + i|u|^2 u + \gamma u = f(x), \quad u(x, 0) = u_0(x), \quad (1.5)$$

where $s \in (1/2, 1)$, $\gamma > 0$, and $f \in L^2(\mathbb{R})$. The authors proved that (1.5) possesses a finite dimensionality of global attractor. In [12], Hurtado studied the following initial-boundary value problem of fractional Laplacian equation with variable exponents:

$$u_t + (-\Delta)_{p(\cdot)}^s u = f(x, u), \quad (1.6)$$

where $(-\Delta)_{p(\cdot)}^s$ is the fractional $p(\cdot)$ -Laplacian defined as

$$(-\Delta)_{p(\cdot)}^s u(x) = 2 \lim_{R \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x - y|^{N+p(x,y)s}} dy,$$

and $p \in C(\overline{\Omega} \times \overline{\Omega})$. The author established the well-posedness of solutions by using the techniques of monotone operators. Moreover, the author obtained the existence of global attractors under suitable assumptions.

When $s = 1$ and $M \equiv 1$, the equation in (1.1) reduces to the following equation:

$$u_t - \Delta_p u + f(x, u) = g(x) \quad \text{in } \Omega, \quad (1.7)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian. In [37], Yang et al. obtained the existence of global attractors for (1.7). Indeed, the existence of global attractors is an important asymptotic property of solutions for parabolic equations which have been studied extensively by many researchers, see for example [6, 10, 23, 24, 38].

In this paper, inspired by the above-mentioned papers, we discuss the existence of global attractors and fractal dimension of solutions for problem (1.1) involving the fractional p -Laplacian and nonlocal diffusion coefficient. Motivated by [23, 37], we first study the existence and uniqueness of solutions by using the Galerkin method. Then we give the existence of global attractors in proper spaces and obtain the fractal dimension of the global attractors. Since problem (1.1) contains a nonlocal coefficient M , in order to get the uniqueness of solutions, we impose a monotonicity assumption on M . However, to the best of our knowledge, there are no results on the existence of global attractors for problem (1.1) in the literature.

To introduce our main results, we first give the definition of (weak) solutions, see [25, 27].

Definition 1.1 For any fixed $T > 0$, a function $u \in L^p(0, T; W_0) \cap C(0, T; L^2(\Omega)) \cap L^q(\Omega \times (0, T))$ is said to be a (weak) solution of problem (1.1) if $u_t \in L^2(0, T; L^2(\Omega))$ and, for a.e. $t \in (0, T)$,

$$\int_{\Omega} u_t \varphi \, dx + M([u]_{s,p}^p) \langle u, \varphi \rangle_{W_0} + \int_{\Omega} f(x, u) \varphi \, dx = \int_{\Omega} g(x) \varphi \, dx \quad (1.8)$$

for all $\varphi \in W_0$, where

$$\langle u, \varphi \rangle_{W_0} = \iint_{\mathcal{Q}} \frac{|u(x, t) - u(y, t)|^{p-2} [u(x, t) - u(y, t)] [\varphi(x) - \varphi(y)]}{|x - y|^{N+sp}} \, dx \, dy,$$

and W_0 is the fractional Sobolev space which will be introduced in Sect. 2.

Theorem 1.1 Assume that $2 \leq p < N/s$, M satisfies (M_1) , f fulfils (f_1) – (f_2) with $2 \leq q < Np/(N - sp)$, and $g \in L^2(\Omega)$. Then the semigroup associated with problem (1.1) admits a global attractor \mathcal{A}_q in $L^q(\Omega)$, i.e., \mathcal{A}_q is compact, invariant in $L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the norm topology of $L^q(\Omega)$.

Remark 1.1 The monotonicity assumption on M is just for getting the uniqueness of solutions.

Finally, we consider the following problem:

$$\begin{cases} u_t + M([u]_{s,p}^p)(-\Delta)_p^s u + |u|^{q-2} u = |u|^{r-2} u & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+, \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (1.9)$$

Theorem 1.2 Assume that $0 < s < 1$, $2 < r < p < N/s$, $r < q < Np/(N - sp)$, and M satisfies (M_1) . Then the semigroup associated with problem (1.9) admits a symmetric global attractor in $L^q(\Omega)$ and its fractal dimension is infinite.

The rest of the paper is organized as follows. In Sect. 2, we give some preliminary lemmas. The existence of global attractors is proved in Sect. 3. The existence of infinite dimensional attractors is obtained in Sect. 4.

2 Preliminaries

In this section, we first recall some necessary definitions and properties of the fractional Sobolev spaces, see [5, 8, 9, 13, 35] for further details. From now on, we shortly denote by $\|\cdot\|_v$ the norm of $L^v(\Omega)$ ($v \geq 1$). Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp}, & sp < N, \\ \infty, & \text{otherwise.} \end{cases}$$

In the following, we set $\mathcal{Q} = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, where $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$. W is a linear space of Lebesgue measurable functions from \mathbb{R}^N to \mathbb{R} such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

It is easy to get that $\|\cdot\|_W$ is a norm on W . We shall work in the closed linear subspace

$$W_0 = \{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

By the result in [35], one can deduce that

$$[u]_{s,p} = \left(\iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}$$

is an equivalent norm of W_0 .

Definition 2.1 (see [23, 24]) Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space X . A subset $\mathcal{A} \subset X$ is called a global attractor for the semigroup if \mathcal{A} is compact in X and satisfies the following properties:

- (1) \mathcal{A} is invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;
- (2) \mathcal{A} attracts all bounded subset of X , that is, for any bounded subset $B \subset X$,

$$\text{dist}(S(t)B, \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where $\text{dist}(A, B)$ is the Hausdorff semidistance of two sets A and B given by

$$\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_X.$$

Definition 2.2 (see [23, 24]) Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space X . A subset $B_0 \subset X$ is called an absorbing set for the semigroup $\{S(t)\}_{t \geq 0}$ if B_0 satisfies that, for any bounded subset $B \subset X$, there exists a positive constant $T = T(B)$ such that

$$S(t)B \subset B_0 \quad \text{for any } t \geq T.$$

Definition 2.3 (see [4]) Let X be a metric space. Assume that M is a compact subset in X . The fractal dimension $\dim_f M$ of M is defined by

$$\dim_f M = \limsup_{\varepsilon \rightarrow 0} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(M, \varepsilon)$ denotes the minimal number of closed balls of the radius ε which cover the set M .

3 Global attractors in $L^q(\Omega)$

In this section, we provide the existence results for problem (1.1), and then we show the existence of a global attractor in $L^q(\Omega)$.

Theorem 3.1 Assume that $2 \leq p < N/s$, M satisfies (M_1) , g satisfies (g_1) , and f fulfils $(f_1) - (f_2)$ with $2 \leq q < Np/(N - sp)$. Then problem (1.1) admits a unique solution

$$u \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0) \cap L^q(0, T; L^q(\Omega)).$$

Moreover, the mapping $u_0 \rightarrow u(t)$ is continuous in $L^2(\Omega)$.

Proof The existence of solutions for problem (1.1) can be obtained by using the Galerkin method, see for example [21, 25]. For completeness, we give a sketch of the proof.

Choose a sequence of functions $\{e_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ which is an orthonormal basis in $L^2(\Omega)$. We shall find the approximate solutions as follows:

$$u_n(x, t) = \sum_{j=1}^n (\eta_n(t))_j e_j(x) \quad \text{for all } n \in \mathbb{N},$$

where the unknown functions $(\eta_n(t))_j$ are determined by the following ODEs:

$$\begin{cases} \eta'_n(t) = -I_n(t, \eta_n(t)), & t \in \mathbb{R}^+, \\ \eta_n(0) = U_{0n}. \end{cases} \quad (3.1)$$

Here $U_{0n} = (\int_\Omega u_{0n}(x) e_1(x) dx, \dots, \int_\Omega u_{0n}(x) e_n(x) dx)$, $u_{0n} \rightarrow u_0$ in W_0 ,

$$\begin{aligned} (I_n(t, \eta_n))_j &= M([u_n]_{s,p}^p) \langle u_n, e_j \rangle_{W_0} + \int_\Omega f(x, u_n(x, t)) e_j(x) dx \\ &\quad - \int_\Omega g(x) e_j(x) dx, \quad j = 1, 2, \dots, n. \end{aligned}$$

The definition of $\langle u_n, e_j \rangle_{W_0}$ is given by

$$\langle u_n, e_j \rangle_{W_0} = \iint_Q \frac{|u_n(x, t) - u_n(y, t)|^{p-2} [u_n(x, t) - u_n(y, t)] [e_j(x) - e_j(y)]}{|x - y|^{N+sp}} dx dy.$$

By the continuity of M and the definition of I_n , we know that I_n is continuous on $\mathbb{R}_0^+ \times \mathbb{R}^n$. Then the Peano theorem (see [11]) yields that there exists a local solution of problem (3.1) on $(0, T_n)$ ($0 < T_n < \infty$). The following a priori estimate implies that the local solution can be extended to $(0, \infty)$.

Multiplying (3.1) by $\eta_n(t)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_n(x, t)|^2 dx + M([u_n]_{s,p}^p) \iint_Q \frac{|u_n(x, t) - u_n(y, t)|^p}{|x - y|^{N+sp}} dx dy \\ & + \int_{\Omega} f(x, u_n) u_n dx = \int_{\Omega} g(x) u_n dx. \end{aligned} \quad (3.2)$$

Let $u_0 \in W_0 \cap L^q(\Omega)$. Then, multiplying (3.1) by $\eta'_n(t)$, we get

$$\int_{\Omega} \left| \frac{\partial u_n(x, t)}{\partial t} \right|^2 dx + \frac{d}{dt} \left[\mathcal{M}([u_n]_{s,p}^p) + \int_{\Omega} F(x, u_n) dx \right] = \int_{\Omega} g(x) \frac{\partial u_n(x, t)}{\partial t} dx. \quad (3.3)$$

It follows from (M_1) , (1.3), (3.2), and the Hölder inequality that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_n(x, t)|^2 dx + C_1 \int_{\Omega} |u_n(x, t)|^q dx \leq C|\Omega| + \|g(x)\|_2 \|u_n(x, t)\|_2.$$

Further, by $q \geq 2$ and the Young inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u_n(x, t)|^2 dx + \frac{C_1}{2} \int_{\Omega} |u_n(x, t)|^2 dx \leq \left[(C + C_1)|\Omega| + \frac{2}{C_1} \|g(x)\|_2^2 \right],$$

which implies that

$$\begin{aligned} & \int_{\Omega} |u_n(x, t)|^2 dx \\ & \leq \int_{\Omega} |u_{0n}(x, 0)|^2 dx e^{-C_1 t} + \frac{2(C + C_1)|\Omega|}{c_1} (1 - e^{-C_1 t}) + \frac{4}{C_1} \int_0^t \|g\|_2^2 e^{C_1(\tau-t)} d\tau \\ & \leq C, \end{aligned} \quad (3.4)$$

where $C > 0$ denotes various constants independent of n and t . This together with (3.2) deduces that the local solution u_n can be extended to $(0, \infty)$.

Then, using a similar discussion as that in [21], we can obtain that the limit of $\{u_n\}$ is a solution of problem (1.1).

Next we prove that problem (1.1) only has one solution. Assume that u and v are two solutions of problem (1.1). Taking $\varphi = u - v$ as a test function in Definition 1.1, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u - v|^2 dx + M([u]_{s,p}^p) \langle u, u - v \rangle_{W_0} - M([v]_{s,p}^p) \langle u, u - v \rangle_{W_0} \\ & + \int_{\Omega} (f(x, u) - f(x, v))(u - v) dx = 0. \end{aligned}$$

Note that

$$\langle u, u - v \rangle_{W_0} = [u]_{s,p}^p - \langle u, v \rangle_{W_0}.$$

By the Young inequality, we have

$$\begin{aligned} \langle u, v \rangle_{W_0} &= \iint_Q \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+sp}} dx dy \\ &\leq \left(1 - \frac{1}{p}\right) [u]_{s,p}^p + \frac{1}{p} [v]_{s,p}^p. \end{aligned}$$

Thus,

$$\langle u, u - v \rangle_{W_0} \geq \frac{1}{p} ([u]_{s,p}^p - [v]_{s,p}^p).$$

Similarly,

$$\langle v, u - v \rangle_{W_0} \leq \frac{1}{p} ([u]_{s,p}^p - [v]_{s,p}^p).$$

Using the above inequalities and assumption (1.2), we arrive at the inequality

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u - v|^2 dx + \frac{1}{p} [M([u]_{s,p}^p) - M([v]_{s,p}^p)] ([u]_{s,p}^p - [v]_{s,p}^p) \leq \lambda \int_{\Omega} |u - v|^2 dx.$$

Since M is a nondecreasing function, we deduce that

$$\frac{d}{dt} \int_{\Omega} |u - v|^2 dx \leq 2\lambda \int_{\Omega} |u - v|^2 dx,$$

which implies that $u - v = 0$ a.e. in $\Omega \times (0, \infty)$. Hence the solution is unique. With a similar discussion as the uniqueness of solution, we can obtain the continuity of the mapping $u_0 \rightarrow u(t)$ in $L^2(\Omega)$. \square

Now we define a functional $E : W_0 \rightarrow \mathbb{R}$ by

$$E(u) = \frac{1}{p} \mathcal{M}([u]_{s,p}^p) + \int_{\Omega} F(x, u) dx - \int_{\Omega} g(x) u dx$$

for all $u \in W_0$, where $F(x, u) = \int_0^u f(x, \xi) d\xi$. Then we have the following.

Lemma 3.1 Assume that $u_0 \in W_0 \cap L^q(\Omega)$. Let u be a solution of problem (1.1), then

$$E(u(x, t)) = E(u_0) - \int_0^t \int_{\Omega} |u_{\tau}(x, \tau)|^2 dx d\tau, \quad t > 0. \quad (3.5)$$

Proof Let us recall that the solution of problem (1.1) can be obtained as the limit of the following sequence of Galerkin's approximation (see [25]):

$$u_n(x, t) = \sum_{j=1}^n (g_n(t))_j e_j(x), \quad n = 1, 2, \dots,$$

where $g_n(t) \in C^1[0, T]$ and $\{e_j\} \subset C_0^\infty(\Omega)$ is an orthonormal basis in $L^2(\Omega)$. Let u be a sufficiently smooth solution of problem (1.1) (or the approximate solution u_n). Choosing $\varphi = u_t$ in Definition 1.1 and using the fact that

$$\langle u, u_t \rangle_{W_0} = \frac{1}{p} \frac{d}{dt} \mathcal{M}([u]_{s,p}^p),$$

we have

$$\int_{\Omega} |u_t|^2 dx + \frac{d}{dt} E(u(x, t)) = 0,$$

which implies that the function $E(u(x, t))$ is nonincreasing with respect to t . Moreover, integrating the above equality with respect to t from 0 to t , we arrive at the equality

$$\int_0^t \int_{\Omega} |u_{\tau}|^2 dx d\tau + E(u(x, t)) - E(u_0) = 0.$$

This ends the proof. \square

By Theorem 3.1, the solution of problem (1.1) generates a semigroup $\{S(t)\}_{t \geq 0}$ in $L^2(\Omega)$. Next, we show that the semigroup possesses a global attractor in $L^q(\Omega)$.

Theorem 3.2 *Under the assumptions of Theorem 3.1, the semigroup $\{S(t)\}_{t \geq 0}$ associated with problem (1.1) possesses an absorbing set in $L^2(\Omega)$ and $W_0 \cap L^q(\Omega)$, respectively.*

Proof Taking $\varphi = u$ in Definition 1.1, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + M([u]_{s,p}^p) [u]_{s,p}^p + \int_{\Omega} f(x, u) u dx = \int_{\Omega} g u dx.$$

Note that assumption (1.3) implies that

$$\int_{\Omega} f(x, u) u dx \geq c_1 \int_{\Omega} |u|^q dx - c |\Omega|.$$

This together with the Young inequality and assumption (M_1) yields that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + m_0 [u]_{s,p}^p + c_1 \int_{\Omega} |u|^q dx \leq C_{\varepsilon} \int_{\Omega} |g|^2 dx + \varepsilon \int_{\Omega} |u|^2 dx + c |\Omega|. \quad (3.6)$$

Using the Young inequality, one can deduce that

$$\begin{aligned} \int_{\Omega} |u|^2 dx &\leq \int_{\Omega} \frac{2}{q} |u|^q dx + \int_{\Omega} \frac{q-2}{q} dx \\ &= \frac{2}{q} \int_{\Omega} |u|^q dx + \frac{q-2}{q} |\Omega|. \end{aligned}$$

Thus,

$$\frac{q}{2} c_1 \int_{\Omega} |u|^2 dx \leq c_1 \int_{\Omega} |u|^q dx + \frac{q-1}{2} c_1 |\Omega|.$$

Inserting this inequality into (3.6), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + m_0 [u]_{s,p}^p + \frac{q}{2} c_1 \int_{\Omega} |u|^2 dx \\ & \leq \varepsilon^{-1} \int_{\Omega} |g|^2 dx + \varepsilon \int_{\Omega} |u|^2 dx + \left(c + \frac{q-1}{2} c_1 \right) |\Omega|. \end{aligned}$$

Choose $\varepsilon = \frac{qc_1}{4}$. Then

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + \frac{qc_1}{4} \int_{\Omega} |u|^2 dx \\ & \leq \frac{4}{qc_1} \int_{\Omega} |g|^2 dx + \left(c + \frac{q-1}{2} c_1 \right) |\Omega|. \end{aligned} \quad (3.7)$$

Then, using a similar discussion as (3.4), we get that there exists $t_0 > 0$ such that

$$\|u(x, t)\|_2 \leq C \quad \text{for any } t \geq t_0.$$

Thus, the semigroup has an absorbing set in $L^2(\Omega)$. Integrating (3.6) with respect to t over $[t, t+1]$, $t \geq t_0$, we obtain

$$\begin{aligned} & \int_t^{t+1} (m_0 [u(x, \tau)]_{s,p}^p + c_1 \|u(x, \tau)\|_q^q) d\tau \\ & \leq C_\varepsilon \|g\|_2^2 + \|u(x, t)\|_2^2 + C|\Omega| \leq C \quad \text{for } t \geq t_0, \end{aligned}$$

which implies that

$$\int_t^{t+1} ([u(x, \tau)]_{s,p}^p + \|u(x, \tau)\|_q^q) d\tau \leq C \quad \text{for } t \geq t_0. \quad (3.8)$$

On the other hand, using Lemma 3.1 and the Young inequality, we deduce

$$\frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{d}{dt} \mathcal{M}([u]_{s,p}^p) + \frac{d}{dt} \int_{\Omega} F(x, u) dx \leq \frac{1}{2} \|g\|_2^2. \quad (3.9)$$

By assumption (1.3), we have

$$c_1 |u|^q - c \leq F(x, u) \leq c_2 |u|^q + c. \quad (3.10)$$

Integrating (3.9) over $[\tau, t+1]$, $t_0 \leq t < \tau < t+1$, one can deduce

$$\begin{aligned} & \mathcal{M}([u(x, t)]_{s,p}^p) + \int_{\Omega} F(x, u(x, t)) dx \\ & \leq C \|g\|_2^2 + \left(\mathcal{M}([u(x, \tau)]_{s,p}^p) + \int_{\Omega} F(x, u(x, \tau)) dx \right). \end{aligned}$$

Integrating the above inequality with respect to τ between t and $t + 1$, we obtain

$$\begin{aligned} & \mathcal{M}([u(x, t + 1)]_{s,p}^p) + \int_{\Omega} F(x, u(x, t + 1)) dx \\ & \leq C \|g\|_2^2 + \int_t^{t+1} \left(\mathcal{M}([u(x, \tau)]_{s,p}^p) + \int_{\Omega} F(x, u(x, \tau)) dx \right) d\tau. \end{aligned}$$

Gathering (3.8) and (3.10), we get

$$[u(x, t)]_{s,p}^p + \|u(x, t)\|_q^q \leq C \quad \text{for all } t \geq t_0 + 1.$$

The proof is complete. \square

By the compact imbedding results in [35] and [5, Theorem 6.7], we are now in a position to obtain the global attractor in $L^q(\Omega)$.

Proof of Theorem 1.1 The proof is inspired by [30]. Let B_0 be an absorbing set in $L^q(\Omega)$. We define the ω -limit set of B_0 as

$$\omega(B_0) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)B_0}^{L^q(\Omega)}.$$

Here $A^{L^q(\Omega)}$ denotes the closure of A in the topology of $L^q(\Omega)$. Note that $\varphi \in \omega(B_0)$ if and only if there exist sequences $\{\varphi_n\} \subset B_0$ and $t_n \rightarrow \infty$ such that

$$S(t_n)\varphi_n \rightarrow \varphi \quad \text{as } n \rightarrow \infty.$$

Set $\mathcal{A} = \omega(B_0)$. Next we verify that \mathcal{A} is a global attractor of the semigroup $S(t)$ in $L^q(\Omega)$.

(1) \mathcal{A} is compact. Clearly, by the compact imbedding results in [35] and [5, Theorem 6.7], one can obtain that \mathcal{A} is compact in $L^q(\Omega)$ being $q \in (2, Np/(N - sp))$.

(2) \mathcal{A} is invariant. If $v \in S(t)\mathcal{A}$, then $v = S(t)\varphi$, $\varphi \in \mathcal{A}$. Thus, there exist φ_n and t_n such that

$$S(t)S(t_n)\varphi_n = S(t + t_n)\varphi_n \rightarrow S(t)\varphi = v,$$

which implies that $v \in \mathcal{A}$. If $v \in \mathcal{A}$, then there exist $\varphi_n \in B_0$ and $t_n \rightarrow \infty$ such that $S(t_n)\varphi_n \rightarrow v$. Observe that, for $t_n \geq t$, the sequence $S(t_n - t)\varphi_n$ is compact in $L^q(\Omega)$. Thus, there exist a subsequence $t_{n_k} \rightarrow \infty$ and $\varphi \in L^q(\Omega)$ such that $S(t_{n_k} - t)\varphi_{n_k} \rightarrow \varphi$. It follows that $\varphi \in \mathcal{A}$. By the continuity of $S(t)$, we deduce

$$S(t_{n_k})\varphi_{n_k} = S(t)S(t_{n_k} - t)\varphi_{n_k} \rightarrow S(t)\varphi = v.$$

It yields that $v \in S(t)\mathcal{A}$. Consequently, we obtain that $S(t)\mathcal{A} = \mathcal{A}$.

(3) \mathcal{A} attracts any bounded sets in $L^q(\Omega)$. Arguing by contradiction, we assume that for some bounded set B_1 of $L^2(\Omega)$, $\text{dist}(S(t)B_1, \mathcal{A})$ does not limit to 0 as $t \rightarrow \infty$. Hence there exist $\delta > 0$ and a sequence $t_n \rightarrow \infty$ such that, for all n ,

$$\text{dist}(S(t_n)B_1, \mathcal{A}) \geq \delta > 0.$$

For each $n \geq 1$, there exists $\varphi_n \in B_1$ such that

$$\text{dist}(S(t_n)\varphi_n, \mathcal{A}) \geq \frac{\delta}{2} > 0. \quad (3.11)$$

Recall that B_0 is an absorbing set. Then $S(t_n)\varphi_n \subset B_0$ for $t_n \geq t_0 := t_0(B)$. Since $S(t_n)\varphi_n$ is compact, there exist $\varphi \in L^q(\Omega)$ and a subsequence of t_n denoted by t_{n_k} such that

$$\varphi = \lim_{n_k \rightarrow \infty} S(t_{n_k})\varphi_{n_k} = \lim_{n_k \rightarrow \infty} S(t_{n_k} - t_0)S(t_0)\varphi_{n_k}.$$

It follows from $S(t_0)\varphi_n \in B_0$ that $\varphi \in \mathcal{A}_q$, which contradicts (3.11). \square

4 Infinite dimensional global attractors

In this section, we study the fractal dimension of the global attractor. First, we prove that the Z_2 index of the global attractor is infinite. Then, by the Mané projection theorem [18], we obtain the infinite dimensionality of the global attractor.

Let X be a Banach space. Denote by $\Sigma = \{A \subset V : A \text{ is closed, } A = -A\}$ the class of closed symmetric subsets of X . For any $A \in \Sigma$, the Z_2 index of A is defined as follows:

$$\gamma(A) = \begin{cases} \inf\{m : \exists h \in C^0(A; \mathbb{R}^m \setminus \{0\}), h(-u) = -h(u)\}; \\ \infty & \text{if } \{\dots\} = \emptyset, \text{ in particular if } 0 \in A; \\ 0 & A = \emptyset. \end{cases}$$

Now we list some properties of Z_2 which will be used later, for more details see [29].

Lemma 4.1 *The Z_2 index defined on Σ satisfies the following properties:*

- (1) $\gamma(A) = 0 \Leftrightarrow A = \emptyset$.
- (2) If $A \subset B \subset \Sigma$, then $\gamma(A) \leq \gamma(B)$.
- (3) For any $A, B \subset \Sigma$, $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
- (4) If $A \in \Sigma$ is a compact set, then $\exists \delta > 0$ such that $\gamma(\overline{U_\delta(A)}) = \gamma(A)$, where $U_\delta(A)$ is a symmetric δ -neighborhood of A .
- (5) $\gamma(A) \leq \gamma(h(A))$, $\forall A \in \Sigma$, and $h : V \rightarrow V$ is an odd and continuous function.

To prove Theorem 1.2, we need the following lemma.

Lemma 4.2 *Let $\{S(t)\}_{t \geq 0}$ be an odd semigroup on a complete metric space X , which possesses a symmetric global attractor \mathcal{A} . Then, for any $m \in \mathbb{N}$, there exists a neighborhood $U(0)$ of 0 such that the Z_2 index of the set $\mathcal{A} \setminus U(0)$ satisfies $\gamma(\mathcal{A} \setminus U(0)) \geq m$.*

Proof We first show that, for any integer $m > 0$, there exists a symmetric set B_m such that

$$\gamma(B_m) \geq m \quad \text{and} \quad \omega(B_m) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)B_m} \subset \mathcal{A} \setminus \{0\}. \quad (4.1)$$

For any $k \in \mathbb{N}^+$, let V_k be a k -dimensional subspace of $W_0 \cap L^q(\Omega)$. Set $A_k = \{u \in V_k : [u]_{s,p} = 1\}$, then A_k is compact in $W_0 \cap L^q(\Omega)$ and $L^r(\Omega)$. Since all norms are equivalent on a finite dimensional Banach space, there exists $C > 0$ such that

$$\|u\|_r \geq C[u]_{s,p} = C \quad \text{for all } u \in A_m.$$

Set $\epsilon A_m = \{\epsilon u : u \in A_m\}$, $0 < \epsilon < 1$. Then $\gamma(\epsilon A_m) = \gamma(A_m) = m$. For $v = \epsilon u \in \epsilon A_m$, we have

$$\begin{aligned} E(v) &= \frac{1}{p} \mathcal{M}(\epsilon^p [u]_{s,p}^p) + \epsilon^q \frac{1}{q} \int_{\Omega} |u|^q dx - \epsilon^r \int_{\Omega} |u|^r dx \\ &\leq \frac{1}{p} \epsilon^p \max_{\tau \in [0,1]} M(\tau) + C \epsilon^q - C \epsilon^r. \end{aligned}$$

Since $2 \leq r < p$ and $r < q$, for ϵ small enough such that $E(v) < 0$ for all $v \in \epsilon A_m$. Now fix $\epsilon > 0$ such that $E(v) < 0$. Since $E(0) = 0$ and the function $t \rightarrow E(u(t))$ is nonincreasing, we have $\omega(\epsilon A_m) \subset \mathcal{A} \setminus \{0\}$. Thus, (4.1) holds true by taking $B_m = \epsilon A_m$.

Since $B_m \subset \mathcal{A} \setminus \{0\}$ and B_m is closed and compact, there exist open neighborhoods of 0 and $\omega(B)$, denoted respectively by $U(0)$ and $\mathcal{N}(B_m)$, such that

$$U(0) \cap \mathcal{N}(B_m) = \emptyset.$$

Since $S(t)B_m \subset \mathcal{N}(B_m)$ for t large enough, we have $S(t)B_m \subset \mathcal{N}(\mathcal{A})$ for t large enough. Therefore, there exists T such that, for $t > T$,

$$S(t)B_m \subset \mathcal{N}(B_m) \subset \mathcal{N}(\mathcal{A}) \setminus U(0) \subset \mathcal{N}(\mathcal{A} \setminus U(0)).$$

Note that $\mathcal{A} \setminus U(0)$ is compact. Choosing a proper neighborhood $\mathcal{N}(\mathcal{A} \setminus U(0))$, by (4) in Lemma 4.1, we have

$$\gamma(\mathcal{A} \setminus U(0)) = \gamma(\overline{\mathcal{N}(\omega(A) \setminus U(0))}) \geq \gamma(\overline{S(t)B_m}) \geq \gamma(B_m) \geq m, \quad t \text{ large enough.}$$

The proof is complete. \square

Proof of Theorem 1.2 By our assumptions, one can obtain that the semigroup of problem (1.9) is odd. Indeed, for any $u_0 \in L^2(\Omega)$, clearly, $-u_0 \in L^2(\Omega)$. Let u be the unique solution of problem (1.9) with initial data u_0 . Since f is odd, $-u$ is the unique solution of problem (1.9) corresponding to initial data $-u_0$. Thus, $S(t)(-u_0) = -u = -S(t)u_0$, i.e., $S(t)$ is odd.

Let B be a symmetric absorbing set. Then the symmetry of the global attractor follows from the fact that

$$\mathcal{A} = \omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}.$$

Finally, we may take a linear (and thus odd) projection in the Mané projection theorem (see [18]). If there exists $m \in \mathbb{N}$ such that the fractal dimension of \mathcal{A} is less than m , then every symmetric closed subset of the attractor (not containing zero) has a Z_2 index less than $2m + 1$, which contradicts Lemma 4.2. Thus the fractal dimension of the global attractor is infinite. \square

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Availability of data and materials

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