Boundary Value Problems
a SpringerOpen Journal

RESEARCH

On Neumann problem for the degenerate Monge–Ampère type equations

Juhua Shi[1] and Feida Jiang[2*]

*Correspondence:
jfd2001@163.com
[2] School of Mathematics and
Shing-Tung Yau Center of Southeast
University, Southeast University,
211189 Nanjing, P.R. China
Full list of author information is
available at the end of the article

Abstract

In this paper, we study the global $C^{1,1}$ regularity for viscosity solution of the degenerate Monge–Ampère type equation $\det[D^2u - A(x, Du)] = B(x, u, Du)$ with the Neumann boundary value condition $D_\nu u = \varphi(x)$, where the matrix A is under the regular condition and some structure conditions, and the right-hand term B is nonnegative.

Keywords: Degenerate equations; Monge–Ampère type equations; Neumann problem; Global second order derivative estimates; Existence

1 Introduction

In this paper, we study the degenerate Monge–Ampère type equation

$$\det[D^2u - A(\cdot, Du)] = B(\cdot, u, Du) \quad \text{in } \Omega, \tag{1}$$

associated with the Neumann boundary value condition

$$D_\nu u = \varphi(x) \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded domain, Du and D^2u denote the gradient and Hessian matrix of second order derivatives of the unknown function $u : \Omega \to \mathbb{R}$, respectively, $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a symmetric $n \times n$ matrix-valued function and $A \in C^2(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$, $B : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ is a nonnegative scalar function, ν denotes the unit inner normal on $\partial\Omega$ and φ is a scalar-valued function defined on $\partial\Omega$. We shall use x, z, and p to denote the points in Ω, $\mathbb{R}$, and $\mathbb{R}^n$, respectively.

We introduce the definitions of nondegenerate and degenerate Monge–Ampère type equation respectively. The Monge–Ampère type equation (1) is called nondegenerate or degenerate if $D^2u - A$ is positive definite or nonnegative definite, namely the right-hand side term $B \geq B_0 > 0$ or $B \geq 0$, respectively, where B_0 is a constant. Accordingly, a solution u of the Neumann boundary value problem (1)–(2) is called elliptic (degenerate elliptic) if $D^2u - A > 0$ (≥ 0).

We say that A is strictly regular in Ω, if

$$\sum_{i,j,k,l=1}^{n} A_{ij}^{kl}(x,p)\xi_i\xi_j\eta_k\eta_l \geq c_0|\xi|^2|\eta|^2 \tag{3}$$

holds for all $(x,p) \in \Omega \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$ with $\xi \cdot \eta = 0$, and some positive constant c_0, where $A_{ij}^{kl} = \frac{\partial^2 A_{ij}}{\partial p_k \partial p_l}$. If c_0 on the right-hand side in (3) is replaced with 0, we say that A is regular in Ω. As usual, the strictly regular condition and regular condition are also said to be the A3 condition and the A3w condition, respectively, see [1, 2].

For the Dirichlet boundary value problem, the regularity for degenerate Monge–Ampère equation has been extensively studied, see [3–8]. When $A \equiv 0$, equation (1) reduces to the classical Monge–Ampère equation. The global $C^{1,\frac{1}{3}}$ regularity was obtained by Bao in [4] under the condition $B^{\frac{1}{n}}(x) \in C^{1,1}(\bar{\Omega})$. Guan, Trudinger, and Wang showed the global $C^{1,1}$ regularity in [7] with the relaxed condition $B^{\frac{1}{n-1}}(x) \in C^{1,1}(\bar{\Omega})$. Using the Pogorelov estimates independent of the lower bound B, the interior regularity was established by Błocki [5] and the authors [8]. When $A \not\equiv 0$, Andriyanova [3] proved the second order derivative estimates under the A3w condition with the right-hand term $B(x,Du) = q(x)\xi(x,Du)$, where $q^{\frac{1}{n-1}}(x) \in C^{1,1}(\bar{\Omega})$ is nonnegative and $\xi^{\frac{1}{n-1}}(x,Du) \in C^{1,1}(\bar{\Omega} \times \mathbb{R}^n)$ is positive.

When $A \neq 0$, the nondegenerate Monge–Ampère type equation (1) arises in various aspects such as optimal mass transportation problems, geometric optics, and conformal geometry (see, for instance, [1, 9–11]). Ma, Trudinger, and Wang [1] obtained *a priori* interior second order estimates under the A3 condition. The boundary $C^{2,\alpha}$ estimate for the Dirichlet problem was established by Huang et al. [12] under the A3 condition. Jiang et al. [13] obtained the global smooth solutions under the A3w condition. Furthermore, Leoper [2] proved that the A3w condition is necessary for C^0 regularity of the optimal map in the optimal transportation problem.

For the Neumann boundary value problem, the global regularity of solutions for the classical Monge–Ampère equation was established by Lions, Trudinger, and Urbas in [14]. Subsequently, Jiang, Trudinger, and Xiang [15] proved the second order derivative estimates of solutions for the Monge–Ampère type equation (1) under the A3w condition. However, as far as we know, there is no regularity result for the degenerate Monge–Ampère type equation with Neumann boundary value condition, which is the motivation of this work.

Our global second derivative estimate is obtained by mixing the Bernstein type argument and the barrier argument. In order to construct the global barrier function under the regularity assumption of A, we need to assume the existence of a subsolution (strict subsolution) $\underline{u}$ to equation (1) satisfying

$$\det\left[D^2\underline{u} - A(x,D\underline{u})\right] \geq (>) \, B(x,\underline{u},D\underline{u}) \quad \text{in } \Omega \text{ (on } \bar{\Omega}), \tag{4}$$

together with the Neumann boundary condition

$$D_\nu\underline{u} = \varphi(x) \quad \text{on } \partial\Omega. \tag{5}$$

In order to treat the degenerate case for equation (1), we assume that B satisfies the following conditions:

$$|B_i|, |B_z|, |B_{p_i}| \leq C_1 B^{1-\frac{1}{2(n-1)}}, \tag{6}$$

$$|B_{ij}|, |B_{iz}|, |B_{ip_j}|, |B_{zz}|, |B_{zp_i}|, |B_{p_ip_j}| \leq C_2 B^{1-\frac{1}{n-1}}, \tag{7}$$

and

$$\{B_{p_ip_j} - B_{p_i}B_{p_j}\}_{n \times n} \geq -C_3 B^2 I, \tag{8}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ for $i, j = 1, \ldots, n$, where C_1, C_2, and C_3 are positive constants, I denotes the unit matrix, B_i, B_z, and B_{p_i} denote the partial derivatives of B with respect to x_i, z, and p_i, respectively. In fact, we can have the following relaxed version of condition (6) if $|B| \leq C$:

$$|B_i|, |B_z|, |B_{p_i}| \leq C_4 B^{1-\frac{1}{n-1}}, \tag{9}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ for $i = 1, \ldots, n$, where C_4 depends on n, $\sup B$, and C_1.

We define the domain Ω to be uniformly A-convex with respect to $u \in C^1(\bar{\Omega})$ if

$$\sum_{i,j=1}^{n} \left(D_i \nu_j(x) - A_{ij}^k(x, Du)\nu_k \right) \tau_i \tau_j \leq -\delta_0 \quad \text{on } \partial\Omega \tag{10}$$

for all vectors $\tau = \tau(x)$ tangent to $\partial\Omega$ and some positive constant δ_0. For the Neumann problem, the definition of uniform A-convexity of Ω with respect to u is introduced in [15]. We refer the readers to [15] for more descriptions of this definition. Based on (10), we can define a barrier function in the neighborhood of the boundary $\partial\Omega$ by using the defining function of the domain Ω, which will be used in the double normal derivative estimate of u on $\partial\Omega$ in Sect. 3.

We now formulate the main result.

Theorem 1.1 *Let Ω be a bounded $C^{3,1}$ domain in $\mathbb{R}^n$ and $u \in C^4(\Omega) \cap C^{1,1}(\bar{\Omega})$ be an elliptic solution of Neumann (1)–(2). Assume that $\varphi \in C^{2,1}(\partial\Omega)$, $A \in C^2(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$ satisfies the A3w condition and Ω is uniformly A-convex with respect to u. Suppose that $B \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying conditions (6)–(8) and $B_z > 0$. Assume that either of the following conditions holds*:
  (a) *$A_{ij} = f_{ij}(x, u_i)\delta_{ij}$, where f_{ij} $(i, j = 1, \ldots, n)$ are given functions and δ_{ij} is the usual Kronecker delta*;
  (b) *There exists a sufficiently small constant δ depending on n, A, B, $\sup_\Omega |u|$ and $\sup_\Omega |Du|$ such that if $|D_{p_k} A_{ij}| < \delta$ for all $i, j, k = 1, \ldots, n$.*
Assume further that there exists a subsolution $\underline{u} \in C^2(\bar{\Omega})$ of Neumann problem (1)–(2) satisfying (4)–(5). Then we have the estimate

$$\sup_\Omega |D^2 u| \leq C, \tag{11}$$

where the constant C depends on n, Ω, $|A|_{2;\bar{\Omega}}$, $|u|_{1;\bar{\Omega}}$, $\underline{u}$, C_1, C_2, C_3, $\sup_{\bar{\Omega}} |B|$, $\|\varphi\|_{C^{2,1}(\partial\Omega)}$, and δ_0.

We remark that the alternative conditions (a) and (b) for the matrix A are used in the treatment of the third derivative terms in the proof, see the details in Sect. 3.

Since the global second order derivative estimates are independent of the lower bound of B, we have the following existence result by approximation. Before stating the theorem, we introduce the definition of viscosity solution of equation (1). An upper (lower)-semicontinuous function u is called a viscosity subsolution (supersolution) of equation (1) if, for any function $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum (minimum) at some point $x_0 \in \Omega$, there holds

$$\det\{D^2\phi(x_0) - [A(x_0, D\phi(x_0))]\} \geq (\leq) B(x_0, u(x_0), D\phi(x_0)). \tag{12}$$

A continuous function u is a viscosity solution of equation (1) if it is both a viscosity subsolution and a viscosity supersolution of equation (1).

Based on the *a priori* second derivative estimate, we now establish the existence result in the following theorem.

Theorem 1.2 *Under the assumption of Theorem 1.1, assume that "B is a nonnegative function" and "a strict subsolution $\underline{u} \in C^2(\bar{\Omega})$" in place of "B is a positive function" and "a subsolution $\underline{u} \in C^2(\bar{\Omega})$", respectively. Assume further that there exists a supersolution $\bar{u}$ of equation (1) satisfying $D_v\bar{u} \leq \varphi(x)$ on $\partial\Omega$, and A satisfies the quadratic structure condition from below, namely*

$$A(x, p) \geq -\mu_0(1 + |p|^2)I, \tag{13}$$

where μ_0 is a positive constant depending on $\sup|u|$ and I denotes the $n \times n$ identity matrix. Then there exists a degenerate elliptic viscosity solution $u \in C^{1,1}(\bar{\Omega})$ for the Neumann problem (1)–(2).

Remark 1 Note that in Theorem 1.2, if B is a positive function and $\underline{u} \in C^2(\bar{\Omega})$ is merely a subsolution, we can get the existence and uniqueness of the elliptic solution $u \in C^{3,\alpha}(\bar{\Omega})$ of problem (1)–(2) for some $\alpha \in (0,1)$. We can prove this by following the proof of Theorem 1.2 in [15] and replacing $\bar{u}$ in its original proof with $\underline{u}$. The reason why we can replace $\bar{u}$ with $\underline{u}$ is that both A in (1) and φ in (2) have no u dependence. Indeed, when A in (1) is independent of u, we can obtain Lemma 2.2, which can replace Lemma 2.1 of [15]. Meanwhile, since φ in (2) has no u dependence, we are still able to get the key inequality (47) in our proof. Then we can follow the proof of Theorem 1.2 in [15] to establish $u \in C^{3,\alpha}(\bar{\Omega})$.

Remark 2 It is clear that Theorem 1.2 can be applied to the Monge–Ampère equation with homogenous right-hand side as follows:

$$\det[D^2u - A(x, Du)] = c_0|x|^\alpha \quad \text{in } \Omega, \tag{14}$$

where c_0 is a nonnegative constant, $\alpha \in [2(n-1), +\infty)$ is a constant. Thus, there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ of equation (14) under the Neumann boundary value condition (2). For the Dirichlet problem of the totally degenerate Monge–Ampère equation (14) when $c_0 = 0$ and $A \equiv 0$, we refer the reader to [16–18].

Remark 3 Condition (a) in Theorems 1.1 and 1.2 can include $A = \text{diag}(u_1^2, \ldots, u_n^2)$ as an example, which satisfies both the A3w condition and the structure condition (13). Both conditions (a) and (b) in Theorems 1.1 and 1.2 can cover the special case when the matrix A is independent of p, so that the standard Monge–Ampère case is already involved. It would be interesting to relax restrictions (a) or (b) of the matrix A in Theorems 1.1 and 1.2.

The paper is organized as follows. In Sect. 2, we introduce two lemmas which will be used to deal with the degenerate term and in the construction of the global barrier function, respectively. In Sect. 3, we give the proof of Theorem 1.1 which constitutes the core of this paper. In this section, we reduce the global estimates to the boundary by using a global auxiliary function, where the computations of the third order derivatives are carried out in detail under the two alternative conditions of Theorem 1.1. For the boundary estimates, we construct a suitable barrier function which can be used to control the degenerate term, originating from the differentiation of equation (1). In this part, we mix the techniques of dealing with the Neumann problems for nondegenerate Monge–Ampère type equations in [14, 15] and the Dirichlet problems for degenerate Monge–Ampère type equations in [3, 5–7]. In Sect. 4, we complete the proof of Theorem 1.2.

2 Preliminaries

In this section, we introduce two lemmas. One lemma is used to deal with the degenerate term which arises in differentiating equation (1). The other lemma is used to construct the global barrier function. In equation (1), we first suppose $B > 0$ in Ω,

$$\{\tilde{u}_{ij}\} := \{u_{ij} - A_{ij}\} \quad \text{and} \quad \{\tilde{u}^{ij}\} := \{\tilde{u}_{ij}\}^{-1}.$$

Then both the matrices $\{\tilde{u}_{ij}\}$ and $\{\tilde{u}^{ij}\}$ are positive definite. We can rewrite equation (1) in the form

$$\log\left[\det(\tilde{u}_{ij})\right] = \tilde{B} \quad \text{in } \Omega, \tag{15}$$

where $\tilde{B} := \log B$. By differentiating equation (15) in the direction $\xi \in \mathbb{R}^n$ once and twice respectively, we have

$$\tilde{u}^{ij}\left(D_\xi u_{ij} - A_{ij,k}\xi_k - A_{ij}^k D_\xi u_k\right) = D_\xi \tilde{B} \tag{16}$$

and

$$\tilde{u}^{ij}\left(D_{\xi\xi} u_{ij} - A_{ij,kl}\xi_k\xi_l - A_{ij}^{kl} D_\xi u_k D_\xi u_l - A_{ij}^k D_{\xi\xi} u_k - 2A_{ij,l}^k \xi_l D_\xi u_k\right) \tag{17}$$
$$= \tilde{u}^{is}\tilde{u}^{jt} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{st} + D_{\xi\xi}\tilde{B},$$

where

$$D_\xi \tilde{B} = \frac{B_\xi + B_z D_\xi u + B_{p_k} D_{k\xi} u}{B} \tag{18}$$

and

$$
\begin{aligned}
D_{\xi\xi}\tilde{B} = {}& \frac{B_{\xi\xi} + B_{zz}(D_\xi u)^2 + B_z D_{\xi\xi} u + B_{p_k} D_{\xi\xi} u_k + B_{p_k p_l}(D_\xi u_l)(D_\xi u_k)}{B} \\
& + \frac{2B_{\xi z} D_\xi u + 2B_{\xi p_l} D_\xi u_l + 2B_{z p_k}(D_\xi u)(D_\xi u_k)}{B} \\
& - \frac{B_\xi^2 + B_z^2(D_\xi u)^2 + B_{p_k} B_{p_l}(D_\xi u_k)(D_\xi u_l)}{B^2} \\
& - \frac{2B_\xi B_z u_\xi + 2B_\xi B_{p_l} D_\xi u_l + 2B_z B_{p_l}(D_\xi u)(D_\xi u_l)}{B^2},
\end{aligned}
\tag{19}
$$

where $B_i = \frac{\partial B}{\partial x_i}$, $B_z = \frac{\partial B}{\partial z}$, $B_{p_k} = \frac{\partial B}{\partial p_k}$, $A_{ij,k} = \frac{\partial A_{ij}}{\partial x_k}$, $A_{ij}^k = \frac{\partial A_{ij}}{\partial p_k}$,

$$
D_i v = \frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial z} D_i u + \frac{\partial v}{\partial p_k} D_{ki} u = v_i + v_z D_i u + v_{p_k} D_{ki} u,
$$

and

$$
D_\xi v = D_i v \xi_i = v_i \xi_i + v_z D_i u \xi_i + v_{p_k} D_{ki} u \xi_i
$$

for a function $v = v(x, z, p)$. Note that we use the standard summation convention in the context that repeated indices indicate summation from 1 to n unless otherwise specified.

Next, we point out several facts that will be used in the proof of Theorem 1.1.

Lemma 2.1 *Let $u \in C^3(\Omega)$ be a solution of equation (1) in a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that B satisfies (6), $B > 0$, and $\tilde{B} = \log B$. Then we have the following properties:*

(i)

$$
|D_i \tilde{B}| \leq C\Big[1 + \max_j\big(|\tilde{u}_{ij}|\big)\Big] B^{-\frac{1}{n-1}}
\tag{20}
$$

holds for $i = 1, \ldots, n$, where the constant C depends on n, C_1, A, and $\|u\|_{C^1(\bar{\Omega})}$.

(ii) *If B also satisfies conditions (7)–(8), then*

$$
D_{ii}\tilde{B} \geq -C\Big[1 + \max_j\big(|\tilde{u}_{ij}|\big)\Big] B^{-\frac{1}{n-1}} - C'\Big[1 + \max_j\big(|\tilde{u}_{ij}|\big)\Big]^2 + \sum_{k=1}^n \tilde{B}_{p_k} D_{ii} u_k
\tag{21}
$$

holds for $i = 1, \ldots, n$, where the constant C depends on n, C_1, C_2, A, and $\|u\|_{C^1(\bar{\Omega})}$, and the constant C' depends on C_3 and A.

Lemma 2.1 can be proved by direct calculations using (6) (or (9)), (7), and (8). We briefly prove it here. A more detailed proof can be found in [19].

Proof of Lemma 2.1 By choosing $\xi = e_i$ in (18) and using $\tilde{u}_{ij} = u_{ij} - A_{ij}$ and (9) (the relaxed version of (6)), we get (20) and finish the proof of conclusion (i).

Next, we turn to proving (ii). By condition (8), we have

$$
\begin{aligned}
\frac{B_{p_k p_l} B - B_{p_k} B_{p_l}}{B^2} u_{il} u_{ik} &\geq -C_3 \delta_{kl}(\tilde{u}_{il} + A_{il})(\tilde{u}_{ik} + A_{ik}) \\
&\geq -C'\Big[1 + \max_j\big(|\tilde{u}_{ij}|\big)\Big]^2,
\end{aligned}
\tag{22}
$$

where δ_{kl} denotes the usual Kronecker delta, the constant C' depends on C_3 and A. It follows from (6) and (7) that the sum of the other terms on the right-hand side of (7) has a lower bound $-C[1 + \max_j(|\tilde{u}_{ij}|)]B^{-\frac{1}{n-1}}$, where the constant C depends on n, C_1, C_2, $\sup |B|$, A, and $\sup_\Omega |Du|$.

Using the lower bounds $-C[1 + \max_j(|\tilde{u}_{ij}|)]B^{-\frac{1}{n-1}}$ and (22), and taking $\xi = e_1$ in (19), we get (21) and finish the proof of conclusion (ii). □

Remark 4 We remark that $\tilde{B} = \log B$ satisfies condition (8) if it is semi-convex in p. The term $\sum_{k=1}^n \tilde{B}_{p_k} D_{ii}u_k$ on the right-hand side of (21) can also be dealt with directly. We shall explain the treatment in the later discussion.

We introduce the following linearized operator of equation (1):

$$L = \sum_{i,j,k=1}^n \left[\tilde{u}^{ij}\left(D_{ij} - A_{ij}^k(x, Du)D_k\right) - \tilde{B}_{p_k}D_k \right]. \tag{23}$$

We introduce a fundamental lemma, which is important for constructing the global barrier function in the next section.

Lemma 2.2 *Let $u \in C^2(\bar{\Omega})$ be an elliptic solution of* (1), *and let $\underline{u} \in C^2(\bar{\Omega})$ be an elliptic function of equation* (1) *satisfying $D^2\underline{u} - A(x, D\underline{u}) \geq \delta I > 0$ in $\bar{\Omega}$ for some positive constant δ, A is regular and $B \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function satisfying* (6). *Then*

$$L\left(e^{K(\underline{u}-u)}\right) \geq \epsilon_1 \mathcal{T} - C\left(B^{-\frac{1}{2(n-1)}} + 1\right) \tag{24}$$

holds in Ω for sufficiently large positive constant K and uniform positive constants ϵ_1, C depending on $\|A\|_{C^2}$, $\|B\|_{C^{1,1}}$, Ω, $\|u\|_{C^1}$, $\|\underline{u}\|_{C^1}$, δ, and C_1 in (6), *where L is the operator in* (23), *and $\mathcal{T} = \sum_i^n \tilde{u}^{ii}$.*

Proof Since A is regular, similar to Lemma 2.1(ii) in [20], we have

$$L\left(e^{K(\underline{u}-u)}\right) \geq \epsilon_1(\mathcal{T}+1) + C\left\{ \log \frac{\det[D^2\underline{u} - A(x, D\underline{u})]}{\det[D^2u - A(x, Du)]} \right.$$

$$\left. - \sum_{k=1}^n \tilde{B}_{p_k}(x, u, Du)D_k(\underline{u} - u) \right\} \tag{25}$$

$$\geq \epsilon_1 \mathcal{T} + \epsilon_1 + C\left[\log \frac{\delta^n}{M_1} - B^{-\frac{1}{2(n-1)}} \right],$$

where $D^2\underline{u} - A(x, D\underline{u}) \geq \delta I > 0$ and condition (6) are used in the second inequality of (25), $M_1 = \sup_{x \in \Omega} B(x, u, Du)$, and the constant C changes from line to line. Then estimate (24) follows by further adjusting the constant C. □

Remark 5 Note that the barrier inequality in Lemma 2.2 is different from that in [15]. Here the constants ϵ_1 and C in estimate (24) of Lemma 2.2 are independent of the positive lower bound of B. Note also that estimate (24) is proved under condition (6), while in [19] a similar estimate is proved under condition (8).

3 Global second order derivative estimates

In this section, by constructing a global auxiliary function, we can reduce the global estimates to the boundary. On the boundary, in order to get the mixed tangential normal second order derivatives of u, we apply the tangential operator to the boundary value condition (2). A bound of double normal derivative of u on $\partial\Omega$ is obtained by constructing a suitable auxiliary function. We use the key trick of [14] to establish the global second order derivative estimates of solutions for the degenerate Monge–Ampère type equations with the Neumann boundary value condition. For the argument below, we assume that the functions φ, ν are smoothly extended to $\bar{\Omega}$. The constant C in this section changes from line to line.

Mixed tangential normal derivative estimate on $\partial\Omega$. We introduce the tangential gradient operator $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, where $\delta_i = \sum_{j=1}^{n}(\delta_{ij} - \nu_i\nu_j)D_j$ for $i = 1,\ldots,n$. Applying this tangential operator to boundary condition (2), we have

$$(D_k u)(\delta_i \nu_k) + \nu_k(\delta_i D_k u) = \delta_i \varphi \quad \text{on } \partial\Omega. \tag{26}$$

If τ is a direction tangential to $\partial\Omega$ at any point $y \in \partial\Omega$, we have

$$
\begin{aligned}
D_{\tau\nu}u(y) &= \sum_{i,k=1}^{n} \tau_i \nu_k D_{ik}u = \sum_{i,k=1}^{n} \tau_i \nu_k D_{ik}u - \sum_{i,j,k=1}^{n} \tau_i \nu_k \nu_i \nu_j D_{jk}u \\
&= \sum_{i,j,k=1}^{n} \tau_i \nu_k (\delta_{ij} - \nu_i\nu_j) D_{jk}u = \sum_{i,k=1}^{n} \tau_i \nu_k \delta_i D_k u \\
&= \sum_{i,k=1}^{n} \tau_i \delta_i \varphi - \tau_i(\delta_i \nu_k)D_k u = \sum_{i,k=1}^{n} \tau_i D_i \varphi - \tau_i(D_i \nu_k)D_k u,
\end{aligned}
\tag{27}
$$

where the second equality and the last equality are both valid using the fact that $\tau \cdot \nu = 0$, and the fifth equality holds by (26). Hence, from (27) we obtain

$$|D_{\tau\nu}u| \leq C \tag{28}$$

on $\partial\Omega$, where the constant C depends on $\sup_{\bar{\Omega}}|Du|$ and Ω.

Double normal derivative estimate on $\partial\Omega$. We may consider any boundary point. Without loss of generality, we may take it to be the origin and the x_n-axis in the direction of the interior normal. From the uniform A-convexity of Ω and the regularity of A, there exists a C^2 defining function in $\bar{\Omega}$ satisfying

$$\phi = 0 \quad \text{on } \partial\Omega, \qquad D_\nu \phi = -1 \quad \text{on } \partial\Omega, \quad \text{and} \quad \phi < 0 \quad \text{in } \Omega, \tag{29}$$

together with the inequality

$$\left\{ D_{ij}\phi - A_{ij}^{k}(x, Du)D_k\phi \right\}_{n \times n} \geq \delta_0 I, \tag{30}$$

in a neighborhood $\mathcal{N}$ of $\partial\Omega$, whenever $D_\nu u \geq \varphi(x)$, where δ_0 is a positive constant and I denotes the identity matrix. We employ the auxiliary function

$$w = \pm\left(D_\nu u - \varphi(x)\right) + \beta\phi \tag{31}$$

in

$$\Omega_{d_0} := \big\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < d_0 \big\},$$

where β and d_0 are two positive constants to be determined. By the continuity of $D_\nu\phi$ and $D_\nu\phi = -1$ on $\partial\Omega$, there exists a small constant d_0 such that $D_\nu\phi \le -\frac{1}{2}$ in Ω_{d_0}.

It follows from (16) that

$$
\begin{aligned}
Lw &= \pm\tilde{u}^{ij}\big[D_{ij}(\nu_k D_k u) - A_{ij}^l D_l(\nu_k D_k u)\big] \mp L\varphi + \beta L\phi \\
&= \pm\tilde{u}^{ij}\big[(D_{ij}\nu_k)D_k u + 2D_j\nu_k D_{ik}u - A_{ij}^l(D_l\nu_k)D_k u + \nu_k A_{ij,k}\big] \\
&\quad + \nu_k \tilde{B}_{p_k} \mp L\varphi + \beta L\phi \\
&\ge \beta L\phi - C\mathcal{T} - C - CB^{-\frac{1}{n-1}} \\
&\ge \beta L\phi - C\mathcal{T} - CB^{-\frac{1}{n-1}},
\end{aligned}
\tag{32}
$$

where $\mathcal{T} = \sum_{i=1}^n \tilde{u}^{ii}$, the first inequality is obtained by using (20) in Lemma 2.1, the second inequality is obtained by using the fact that

$$\frac{1}{n}\mathcal{T} \ge \left(\prod_{i=1}^n \tilde{u}^{ii}\right)^{\frac{1}{n}} = \left(\prod_{i=1}^n \tilde{u}_{ii}\right)^{-\frac{1}{n}} = B^{-\frac{1}{n}} \ge C. \tag{33}$$

By a direct calculation, we obtain

$$
\begin{aligned}
L\phi &= \tilde{u}^{ij}\big[D_{ij}\phi - A_{ij}^k(x, Du)D_k\phi\big] - \tilde{B}_{p_k} D_k\phi \\
&\ge \delta_0 \mathcal{T} - \tilde{B}_{p_k} D_k\phi \\
&\ge \delta_0 \mathcal{T} - CB^{-\frac{1}{2(n-1)}},
\end{aligned}
\tag{34}
$$

where the first and second inequalities are established by using (34) and (6), respectively. Therefore, from (32), (33), and (34), we have

$$Lw \ge (\beta\delta_0 - C)\mathcal{T} - CB^{-\frac{1}{n-1}} - C\beta B^{-\frac{1}{2(n-1)}}. \tag{35}$$

We decompose $\Omega_{d_0} = \Omega_{d_0}^1 \cup \Omega_{d_0}^2$, where

$$\Omega_{d_0}^1 = \Omega_{d_0} \cap \left\{\sum_{i=1}^n \tilde{u}_{ii} \ge n\right\}$$

and

$$\Omega_{d_0}^2 = \Omega_{d_0} \cap \left\{\sum_{i=1}^n \tilde{u}_{ii} < n\right\}.$$

If $\max_{\bar{\Omega}_{d_0}} w = w(\bar{x})$ for some $\bar{x} \in \bar{\Omega}_{d_0}$. The proof of the double normal derivative estimate of u on $\partial\Omega$ splits into three stages according to whether $\bar{x} \in \Omega_{d_0}^1$ or $\bar{x} \in \Omega_{d_0}^2$, or neither.

Case 1. $\bar{x} \in \Omega^1_{d_0}$. Since $\max_{\bar{\Omega}_{d_0}} w = w(\bar{x})$ for $\bar{x} \in \Omega^1_{d_0}$, then we have

$$Lw(\bar{x}) \leq 0. \tag{36}$$

We may assume that $\tilde{u}_{ij}(\bar{x})$ is diagonal. Without loss of generality, we may assume that $\tilde{u}_{11}(\bar{x}) \geq 1$. At the point $\bar{x}$, we have

$$
\begin{aligned}
\mathcal{T} &\geq \sum_{i=2}^{n} \tilde{u}^{ii} \\
&\geq (n-1)\left(\prod_{i=2}^{n} \tilde{u}^{ii}\right)^{\frac{1}{n-1}} \\
&= (n-1)\left(\prod_{i=1}^{n} \tilde{u}^{ii}\right)^{\frac{1}{n-1}} (\tilde{u}_{11})^{\frac{1}{n-1}} \\
&\geq (n-1) B^{-\frac{1}{n-1}},
\end{aligned}
\tag{37}
$$

where $\tilde{u}_{11}(\bar{x}) \geq 1$ is used in the last inequality. At $\bar{x}$, we can assume that $\sqrt{\mathcal{T}} \geq \frac{3C}{\delta_0\sqrt{n-1}}$, otherwise we have already got the global second order derivative estimates. Therefore at $\bar{x}$, by (35), we have

$$
\begin{aligned}
Lw &\geq \left(\frac{\beta\delta_0}{3} - C\right)\mathcal{T} + \frac{\beta\delta_0}{3}\mathcal{T} - CB^{-\frac{1}{n-1}} + \frac{\beta\delta_0}{3}\mathcal{T} - C\beta B^{-\frac{1}{2(n-1)}} \\
&\geq \left(\frac{\beta\delta_0}{3} - C\right)\mathcal{T} + \left(\frac{\beta\delta_0}{3} - \frac{C}{n-1}\right)\mathcal{T} + \left(\frac{\beta\delta_0}{3}\sqrt{\mathcal{T}} - \frac{C\beta}{\sqrt{n-1}}\right)\sqrt{\mathcal{T}} \\
&> 0,
\end{aligned}
\tag{38}
$$

where we choose $\beta \geq \frac{3C}{\delta_0} + 1$ such that the last inequality holds. It is a contradiction with (36). Therefore, $\bar{x} \notin \Omega^1_{d_0}$.

Case 2. $\bar{x} \in \Omega^2_{d_0}$. For the unit inner normal vector $\nu = (\nu_1, \ldots, \nu_n)$, we must have

$$0 \leq |\nu_i| \leq 1 \quad \text{for } i = 1, \ldots, n. \tag{39}$$

Since $\max_{\bar{\Omega}^2_{d_0}} w = w(\bar{x})$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \Omega^2_{d_0}$, we obtain

$$w_\nu(\bar{x}) = 0. \tag{40}$$

Therefore, we have at $\bar{x}$

$$
\begin{aligned}
0 = w_\nu &= \pm(D_{\nu\nu}u - D_\nu\varphi) + \beta D_\nu\phi \\
&= \pm\left(\sum_{i,j}^{n} \nu_i\nu_j D_{ij}u\right) \mp D_\nu\varphi + \beta D_\nu\phi \\
&\leq n^2 \sum_{i=1}^{n} \tilde{u}_{ii} \mp D_\nu\varphi + \beta D_\nu\phi.
\end{aligned}
\tag{41}
$$

Choose $\beta \geq 2 \max |D\varphi| + 4n^3$ such that $\sum_{i=1}^{n} \tilde{u}_{ii} \geq 2n$, which is a contradiction with $\sum_{i=1}^{n} \tilde{u}_{ii} < n$. Therefore, $\bar{x} \notin \Omega_{d_0}^2$.

Case 3. $\bar{x} \in \partial\Omega_{d_0}$. On $\partial\Omega_{d_0} \cup \Omega$, we have

$$w = \pm(D_\nu u - \varphi) + \beta\phi \leq 0, \tag{42}$$

where β is chosen large enough such that the last inequality holds. On $\partial\Omega$, we have

$$w = \pm\big(D_\nu u - \varphi(x)\big) + \beta\phi = 0. \tag{43}$$

Therefore, for all $x \in \bar{\Omega}_{d_0}$, we have

$$w(x) \leq 0. \tag{44}$$

Since $w(0) = 0$, we have

$$D_\nu w(0) \leq 0, \tag{45}$$

which implies

$$|D_{\nu\nu}u| \leq C \quad \text{on } \partial\Omega. \tag{46}$$

Remark 6 In the above proof, we only use (30) in a neighborhood $\mathcal{N}$ of $\partial\Omega$ whenever $D_\nu u \geq \varphi(x)$. If $D_\nu u < \varphi(x)$ in a neighborhood $\mathcal{N}_0$ of some boundary point $x_0 \in \partial\Omega$, from the boundary condition, we directly get an upper bound $D_{\nu\nu}u(x_0) \leq D_\nu\varphi(x_0)$. While the lower bound $D_{\nu\nu}u \geq \sum_{i,j}^{n} A_{ij}\nu_i\nu_j$ at x_0 can be derived from the ellipticity.

Based on the mixed tangential normal derivative estimate and the double normal derivative estimate on $\partial\Omega$, we now use the method in [15] to prove Theorem 1.1, which is a modification of the original method in [14].

Proof of Theorem 1.1 We modify the elliptic subsolution $\underline{u}$ by adding a perturbation function $a\phi$, where a is a small positive constant and ϕ is the defining function of the domain Ω satisfying (29). Note that the function $\underset{\sim}{u} := \underline{u} - a\phi$ is still uniformly elliptic in Ω if a is a sufficiently small. By a direct computation, we have

$$\begin{aligned} D_\nu(\underset{\sim}{u} - u) &= D_\nu\underline{u} - D_\nu u - aD_\nu\phi \\ &= -aD_\nu\phi \\ &= a \end{aligned} \tag{47}$$

on $\partial\Omega$. We define a function with the form $\Phi = \exp[K(\underset{\sim}{u} - u)]$ with a positive constant K. Replacing $\underline{u}$ in Lemma 2.2 with $\underset{\sim}{u}$, from (24), we have

$$L\Phi \geq \epsilon_1 \mathcal{T} - C\big(B^{-\frac{1}{2(n-1)}} + 1\big) \quad \text{in } \Omega. \tag{48}$$

By a direct calculation, we also have

$$D_v \Phi \geq Ka \inf\left(e^{K(\underline{u}-u)}\right) > 0 \quad \text{on } \partial\Omega. \tag{49}$$

We employ the auxiliary function

$$G(x,\xi) = \left(\tilde{u}_{\xi\xi} - v'(x,\xi)\right)e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi} \tag{50}$$

in $\bar{\Omega} \times \mathbb{R}^n$, where $|\xi| = 1$, α and κ are positive constants to be determined, v' is given by

$$v'(x,\xi) = 2(\xi \cdot v)\xi_i'\left(D_i\varphi(x) - D_k u D_i v_k - A_{ij}v_j\right), \tag{51}$$

and ξ' is given by

$$\xi' = \xi - (\xi \cdot v)v. \tag{52}$$

Assume that G attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. Let

$$H(x,\xi) = \log G(x,\xi) = \log\left(\tilde{u}_{\xi\xi} - v'\right) + \frac{\alpha}{2}|Du|^2 + \kappa\Phi, \tag{53}$$

then the function H attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. From now on, all the calculations will be made at the point $x = x_0$ and $\xi = \xi_0$ unless otherwise specified. At x_0, we obtain

$$0 = D_i H = \frac{D_i(\tilde{u}_{\xi\xi} - v')}{\tilde{u}_{\xi\xi} - v'} + \alpha D_k u D_{ik} u + \kappa D_i \Phi, \quad \text{for } i = 1, \ldots, n, \tag{54}$$

$$\begin{aligned}
0 &\geq D_{ij} H \\
&= \frac{D_{ij}(\tilde{u}_{\xi\xi} - v')}{\tilde{u}_{\xi\xi} - v'} - \frac{D_i(\tilde{u}_{\xi\xi} - v')D_j(\tilde{u}_{\xi\xi} - v')}{(\tilde{u}_{\xi\xi} - v')^2} \\
&\quad + \alpha(D_{ik}u D_{jk}u + D_k u D_{ijk}u) + \kappa D_{ij}\Phi.
\end{aligned} \tag{55}$$

Therefore, at x_0 we have

$$\begin{aligned}
0 &\geq LH \\
&= \frac{1}{\tilde{u}_{\xi\xi} - v'}L\left(\tilde{u}_{\xi\xi} - v'\right) - \frac{1}{(\tilde{u}_{\xi\xi} - v')^2}\tilde{u}^{ij}D_i\left(\tilde{u}_{\xi\xi} - v'\right)D_j\left(\tilde{u}_{\xi\xi} - v'\right) \\
&\quad + \alpha\tilde{u}^{ij}D_{ik}u D_{jk}u + \alpha D_k u L u_k + \kappa L\Phi.
\end{aligned} \tag{56}$$

By a direct computation and using (17)–(19), we have

$$\begin{aligned}
L u_{\xi\xi} &\geq \tilde{u}^{ik}\tilde{u}^{jl}D_\xi\tilde{u}_{ij}D_\xi\tilde{u}_{kl} + \tilde{u}^{ij}D_{p_k p_l}A_{ij}D_\xi u_k D_\xi u_l \\
&\quad - C\left[(1 + \tilde{u}_{ii})\mathcal{T} + (\tilde{u}_{ii})^2\right] - C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}} \\
&\geq \tilde{u}^{ik}\tilde{u}^{jl}D_\xi\tilde{u}_{ij}D_\xi\tilde{u}_{kl} - C\left[(1 + \tilde{u}_{ii})\mathcal{T} + (\tilde{u}_{ii})^2\right] - C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}},
\end{aligned} \tag{57}$$

where the first inequality is established by (21) in Lemma 2.1, and the second inequality is obtained by using the A3w condition. Note that the term $\sum_{k=1}^{n} \tilde{B}_{p_k} D_{ii} u_k$ in (21) does not appear on the right-hand side of the first inequality in (57) since it can be subtracted using the definition of the linearized operator L in (23). By a direct computation, we obtain

$$
\begin{aligned}
|LA_{\xi\xi}| &\le C\big[(1 + \tilde{u}_{ii})\mathcal{T} + \tilde{u}_{ii}\big] + \left|\frac{B_{p_l}}{B} A_{\xi\xi,l}\right| \\
&\le C\big[(1 + \tilde{u}_{ii})\mathcal{T} + \tilde{u}_{ii}\big] + C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}},
\end{aligned}
\tag{58}
$$

where (20) in Lemma 2.1 is used in the second inequality. It follows from (16) and (20) in Lemma 2.1 that

$$
\begin{aligned}
\big|Lv'\big| &\le C\mathcal{T} + C\big|L(u_k)\big| \\
&\le C\mathcal{T} + CB^{-\frac{1}{n-1}}.
\end{aligned}
\tag{59}
$$

Note that there is no $1 + \tilde{u}_{ii}$ in the coefficient of $B^{-\frac{1}{n-1}}$ in (59) since the term $\tilde{B}_{p_i} D_l u_k$ is already subtracted in $L(u_k)$.

Hence, combining (57), (58), and (59), we have

$$
L\big(\tilde{u}_{\xi\xi} - v'\big) \ge \tilde{u}^{ik}\tilde{u}^{jl} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{kl} - C\big[(1 + \tilde{u}_{ii})\mathcal{T} + \tilde{u}_{ii}^2\big] - C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}}.
\tag{60}
$$

By Cauchy's inequality, we obtain

$$
\tilde{u}^{ij} D_i\big(\tilde{u}_{\xi\xi} - v'\big) D_j\big(\tilde{u}_{\xi\xi} - v'\big) \le (1 + \theta)\tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi} + C(\theta)\tilde{u}^{ij} D_i v' D_j v'
\tag{61}
$$

for any $\theta > 0$, where $C(\theta)$ is a positive constant depending on θ. Inserting (48), (60), and (61) into (56), by calculations, we get

$$
\begin{aligned}
0 \ge LH \\
\ge{}& \frac{1}{\tilde{u}_{\xi\xi} - v'}\tilde{u}^{ik}\tilde{u}^{jl} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{kl} - \frac{1 + \theta}{(\tilde{u}_{\xi\xi} - v')^2}\tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi} \\
&- C\frac{1}{\tilde{u}_{\xi\xi} - v'}\Big\{\big[(1 + \tilde{u}_{ii})\mathcal{T} + (\tilde{u}_{ii})^2\big] + C(1 + \tilde{u}_{ii})B^{-\frac{1}{n-1}}\Big\} \\
&- \frac{C(\theta)}{(\tilde{u}_{\xi\xi} - v')^2}\tilde{u}^{ij} D_i v' D_j v' - C\alpha\big(B^{-\frac{1}{n-1}} + \mathcal{T}\big) \\
&+ \alpha \sum_{i=1}^{n} \tilde{u}_{ii} + \kappa\big[\epsilon_1 \mathcal{T} - C\big(B^{-\frac{1}{2(n-1)}} + 1\big)\big].
\end{aligned}
\tag{62}
$$

Next, we shall deal with the terms on the right-hand side of (62). Without loss of generality, we assume that $\{\tilde{u}_{ij}\}$ is diagonal at x_0 with the maximum eigenvalue $\tilde{u}_{11}$. We can always assume that $\tilde{u}_{11} > 1$ and is as large as we want; otherwise we are done. Since v' is bounded, $\tilde{u}_{11}$ and $\tilde{u}_{\xi\xi}$ are comparable in the sense that, for any $\theta > 0$, there exists a further constant $C(\theta)$ such that

$$
\big|\tilde{u}_{11} - \tilde{u}_{\xi\xi} + v'\big| < \theta \tilde{u}_{11}
\tag{63}
$$

if $\tilde{u}_{11} > C(\theta)$. Therefore, using (63), we have

$$
\begin{aligned}
&\frac{1}{\tilde{u}_{\xi\xi} - v'}\big[(1 + \tilde{u}_{ii})\mathcal{T} + (\tilde{u}_{ii})^2\big]\\
&\quad \le \frac{1}{(1 - \theta)\tilde{u}_{11}}\big[(1 + \tilde{u}_{ii})\mathcal{T} + (\tilde{u}_{ii})^2\big]\\
&\quad \le C\left(\mathcal{T} + \sum_{i=1}^n \tilde{u}_{ii}\right)
\end{aligned}
\tag{64}
$$

for some constant C if $\theta \in (0, 1/2)$.

We shall treat the first two terms on the right-hand side of (62). For this purpose, we will make a more detailed calculations than [15] since the equality $D_i A_{\xi\xi} = D_\xi A_{i\xi}$ does not hold in general. Because in (62) the vector ξ is not equal to e_1, the situation here is different from the formula below (24) in [21]. Next, we divide the discussions into the two cases (a) and (b) assumed in the statement of this theorem.

In case (a), $A_{ij} = f_{ij}(x, u_i)\delta_{ij}$. We define

$$
\mathcal{P} := \frac{1}{\tilde{u}_{11}}\tilde{u}^{ik}\tilde{u}^{jl}D_\xi \tilde{u}_{ij}D_\xi \tilde{u}_{kl} - \frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_i \tilde{u}_{\xi\xi}D_j \tilde{u}_{\xi\xi},
\tag{65}
$$

we shall get a lower bound of the quantity $\mathcal{P}$ in terms of $\mathcal{T}$. We define the matrix

$$
\{a_{ik}\} := \big\{\tilde{u}^{jl}D_\xi \tilde{u}_{ij}D_\xi \tilde{u}_{kl}\big\}
\tag{66}
$$

and let Λ be its maximum eigenvalue. For the first term of $\mathcal{P}$, we have

$$
\frac{1}{\tilde{u}_{11}}\tilde{u}^{ik}\tilde{u}^{jl}D_\xi \tilde{u}_{ij}D_\xi \tilde{u}_{kl} = \frac{\mathrm{trace}\{\tilde{u}^{ik}a_{kj}\}}{\tilde{u}_{11}} \ge \frac{\tilde{u}^{11}\Lambda}{\tilde{u}_{11}} = \frac{\Lambda}{\tilde{u}_{11}^2}.
\tag{67}
$$

For the second term of $\mathcal{P}$, using $\{\tilde{u}_{ij}\} := \{u_{ij} - A_{ij}\}$, we have

$$
\begin{aligned}
&\frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_i \tilde{u}_{\xi\xi}D_j \tilde{u}_{\xi\xi}\\
&\quad = \frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}(D_\xi u_{ki} - D_i A_{k\xi})(D_\xi u_{lj} - D_i A_{l\xi})\xi_k \xi_l\\
&\quad = \frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}\big[D_\xi(u_{ik} - A_{ik}) + (D_\xi A_{ki} - D_i A_{k\xi})\big]\\
&\qquad \cdot \big[D_\xi(u_{jl} - A_{jl}) + (D_\xi A_{jl} - D_j A_{l\xi})\big]\xi_k \xi_l\\
&\quad = \frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi \tilde{u}_{i\xi}D_\xi \tilde{u}_{j\xi} + \frac{1 - 2\theta}{1 - \theta}\frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi \tilde{u}_{i\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi})\\
&\qquad + \frac{1 - 2\theta}{1 - \theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi})\\
&\quad \le \frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi \tilde{u}_{i\xi}D_\xi \tilde{u}_{j\xi} + \frac{1 - 2\theta}{\theta}\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi}),
\end{aligned}
\tag{68}
$$

where the last inequality is valid by using Cauchy's inequality. We now calculate the terms $D_i A_{\xi\xi}$ and $D_\xi A_{i\xi}$. For general $A(x, Du)$, we have

$$D_i A_{\xi\xi} = D_{x_i} A_{\xi\xi} + (D_{p_k} A_{\xi\xi}) u_{ki}, \quad \forall i = 1, \dots, n. \tag{69}$$

For $A_{ij} = f_{ij}(x, u_i)\delta_{ij}$, we have

$$D_\xi A_{i\xi} = (D_{x_k} f_{ii})\xi_i \xi_k + (D_{p_i} f_{ii})\xi_i u_{i\xi}, \quad \forall i = 1, \dots, n. \tag{70}$$

Substituting (69) and (70) into the last term of (68), we have

$$
\begin{aligned}
\frac{1-2\theta}{\theta} & \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi}) \\
&= \frac{1-2\theta}{\theta} \frac{1}{\tilde{u}_{11}^2} \tilde{u}^{ij}(D_\xi A_{i\xi} D_\xi A_{j\xi} - D_\xi A_{i\xi} D_j A_{\xi\xi} - D_i A_{\xi\xi} D_\xi A_{j\xi} + D_i A_{\xi\xi} D_j A_{\xi\xi}) \\
&\leq \frac{1-2\theta}{\theta} \frac{C}{\tilde{u}_{11}^2}(\mathcal{T} + \tilde{u}_{11}) \\
&\leq C(\mathcal{T} + 1),
\end{aligned}
\tag{71}
$$

where $\tilde{u}^{ij}\tilde{u}_{jk} = \delta_{ik}$ and $\{\tilde{u}_{ij}\} := \{u_{ij} - A_{ij}\}$ are used in the first inequality, and $\tilde{u}_{11} \geq \frac{1-2\theta}{\theta}$ is assumed in the last inequality. Combining (65), (67), (68), and (71), we now get the lower bound of $\mathcal{P}$ in terms of $\mathcal{T}$,

$$\mathcal{P} \geq \frac{1}{\tilde{u}_{11}^2}(\Lambda - a_{kl}\xi_k\xi_l) - C(\mathcal{T} + 1) \geq -C(\mathcal{T} + 1), \tag{72}$$

where the definitions of the matrix $\{a_{ik}\}$ and its maximum eigenvalue Λ are used successively. On the other hand, it follows from $D_i H = 0$ in (54) that

$$
\begin{aligned}
\tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi} &\leq 2\tilde{u}^{ii}\big[|D_i v'|^2 + (\tilde{u}_{\xi\xi} - v')^2(\alpha D_k u D_{ik} u + \kappa D_i \Phi)^2\big] \\
&\leq 2\tilde{u}^{ii}|D_i v'|^2 + C(\tilde{u}_{\xi\xi} - v')^2\left(\alpha^2 \sum_{i=1}^n \tilde{u}_{ii} + \kappa^2 \mathcal{T}\right).
\end{aligned}
\tag{73}
$$

Combining (63), (72), and (73), we obtain

$$
\begin{aligned}
\frac{1}{\tilde{u}_{\xi\xi} - v'} & \tilde{u}^{ik}\tilde{u}^{jl} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{kl} - \frac{1+\theta}{(\tilde{u}_{\xi\xi} - v')^2} \tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi} \\
&\geq \frac{1}{1-\theta}\left(\frac{1}{\tilde{u}_{11}} \tilde{u}^{ik}\tilde{u}^{jl} D_\xi \tilde{u}_{ij} D_\xi \tilde{u}_{kl} - \frac{1+\theta}{(1-\theta)\tilde{u}_{11}^2} \tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi}\right) \\
&\geq \frac{1}{1-\theta}\left(\mathcal{P} - \frac{3\theta}{(1-\theta)\tilde{u}_{11}^2} \tilde{u}^{ij} D_i \tilde{u}_{\xi\xi} D_j \tilde{u}_{\xi\xi}\right) \\
&\geq -\frac{C(\mathcal{T}+1)}{1-\theta} - \frac{3\theta}{[(1-\theta)\tilde{u}_{11}]^2}\left[2\tilde{u}^{ii}|D_i v'|^2 + C(\tilde{u}_{\xi\xi} - v')^2\left(\alpha^2 \sum_{i=1}^n \tilde{u}_{ii} + \kappa^2 \mathcal{T}\right)\right] \\
&\geq -C(\mathcal{T}+1) - C\theta\alpha^2 \sum_{i=1}^n \tilde{u}_{ii} - C\theta\kappa^2 \mathcal{T}
\end{aligned}
\tag{74}
$$

for $\theta \in (0, 1/2)$, where C becomes a further constant in the last inequality. We can assume $\tilde{u}_{11} \geq 1$, otherwise we have already obtained the desired estimate. Similar to (37), we also have

$$B^{-\frac{1}{n-1}} \leq \frac{\mathcal{T}}{n-1}. \tag{75}$$

For the last term in (62), using Cauchy's inequality, we have

$$\epsilon_1 \mathcal{T} - C\left(B^{-\frac{1}{2(n-1)}} + 1\right)$$
$$\geq \epsilon_1 \mathcal{T} - \epsilon B^{-\frac{1}{n-1}} - C(\epsilon) \tag{76}$$
$$\geq \frac{\epsilon_1}{2}\mathcal{T} - C(\epsilon_1),$$

where we take $\epsilon = \frac{n-1}{2}\epsilon_1$ and use (75) in the second inequality.

Inserting (63), (64), (74), (75), and (76) into (62), we obtain, for $\tilde{u}_{11} \geq \max\{C(\theta), 1\}$,

$$\alpha \sum_{i=1}^{n} \tilde{u}_{ii} + \frac{\kappa \epsilon_1}{2}\mathcal{T}$$
$$\leq C\left\{\left[1 + \kappa C(\epsilon_1)\right] + \left(1 + \alpha^2 \theta\right)\sum_{i=1}^{n}\tilde{u}_{ii} + \left(1 + \alpha + \kappa^2 \theta\right)\mathcal{T}\right\}. \tag{77}$$

By choosing $\kappa \gg \alpha \gg 1$ and fixing a small positive $\theta = 1/\kappa^2$, we can get from (77) that

$$\sum_{i=1}^{n} \tilde{u}_{ii}(x_0) \leq C, \tag{78}$$

which implies a corresponding estimate for $|D^2 u(x_0)|$ in case (a).

In case (b), $|D_{p_k}A_{ij}| < \delta$ for all $i, j, k = 1, \ldots, n$, and sufficiently small δ. We define

$$\mathcal{P}' := \frac{1}{\tilde{u}_{11}}\tilde{u}^{ik}\tilde{u}^{jl}D_\xi\tilde{u}_{ij}D_\xi\tilde{u}_{kl} - \frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}D_j\tilde{u}_{\xi\xi}. \tag{79}$$

Similarly to (68), we can get

$$\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}D_j\tilde{u}_{\xi\xi}$$
$$= \frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi\tilde{u}_{i\xi}D_\xi\tilde{u}_{j\xi} + \frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi\tilde{u}_{i\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}) \tag{80}$$
$$+ \frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi}).$$

For the middle term on the right-hand side of (80), rather than using Cauchy's inequality, it can be dealt with by using $D_\xi\tilde{u}_{i\xi} = D_i u_{\xi\xi} - D_\xi A_{i\xi}$, (69), (70), (54), and $|D_{p_k}A_{ij}| < \delta$, namely

$$\frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_\xi\tilde{u}_{i\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi})$$
$$= \frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}\left[D_i\tilde{u}_{\xi\xi} - (D_\xi A_{i\xi} - D_i A_{\xi\xi})\right](D_\xi A_{j\xi} - D_j A_{\xi\xi})$$

$$\leq \frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}) \tag{81}$$

$$= \frac{2}{\tilde{u}_{11}^2}\tilde{u}^{ij}\big[D_i\nu' - (\tilde{u}_{\xi\xi} - \nu')(\alpha D_k u D_{ik}u + \kappa D_i\Phi)\big](D_\xi A_{j\xi} - D_j A_{\xi\xi})$$

$$\leq C\big[1 + \delta(\alpha + \kappa\mathcal{T})\big],$$

provided $\tilde{u}_{11} \geq 1$. We can analyze the last term on the right-hand side of (80) for general matrix A to get

$$\frac{1}{\tilde{u}_{11}^2}\tilde{u}^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi}) \leq C(\mathcal{T} + 1), \tag{82}$$

provided $\tilde{u}_{11} \geq 1$. Combining (79), (80), (81), and (82), we obtain

$$\mathcal{P}' \geq \frac{1}{\tilde{u}_{11}^2}(\Lambda - a_{kl}\xi_k\xi_l) - C\big[1 + \mathcal{T} + \delta(\alpha + \kappa\mathcal{T})\big] \tag{83}$$

$$\geq -C\big[1 + \mathcal{T} + \delta(\alpha + \kappa\mathcal{T})\big].$$

Combining (63), (83), and (73), we obtain

$$\frac{1}{\tilde{u}_{\xi\xi} - \nu'}\tilde{u}^{ik}\tilde{u}^{jl}D_\xi\tilde{u}_{ij}D_\xi\tilde{u}_{kl} - \frac{1+\theta}{(\tilde{u}_{\xi\xi} - \nu')^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}D_j\tilde{u}_{\xi\xi}$$

$$\geq \frac{1}{1-\theta}\left(\frac{1}{\tilde{u}_{11}}\tilde{u}^{ik}\tilde{u}^{jl}D_\xi\tilde{u}_{ij}D_\xi\tilde{u}_{kl} - \frac{1+\theta}{(1-\theta)\tilde{u}_{11}^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}D_j\tilde{u}_{\xi\xi}\right) \tag{84}$$

$$\geq \frac{1}{1-\theta}\left(\mathcal{P}' - \frac{2\theta}{(1-\theta)\tilde{u}_{11}^2}\tilde{u}^{ij}D_i\tilde{u}_{\xi\xi}D_j\tilde{u}_{\xi\xi}\right)$$

$$\geq -C\big[1 + \mathcal{T} + \delta(\alpha + \kappa\mathcal{T})\big] - C\theta\alpha^2\sum_{i=1}^n\tilde{u}_{ii} - C\theta\kappa^2\mathcal{T}$$

for $\theta \in (0, 1/2)$. Using the estimate in (83) and deducing as in case (a), for $\tilde{u}_{11} \geq \max\{C(\theta), 1\}$, we obtain in place of (77)

$$\alpha\sum_{i=1}^n\tilde{u}_{ii} + \frac{\kappa\epsilon_1}{2}\mathcal{T}$$

$$\leq C\left\{\big[1 + \kappa C(\epsilon_1) + \delta\alpha\big] + (1 + \alpha^2\theta)\sum_{i=1}^n\tilde{u}_{ii} + (1 + \alpha + \kappa^2\theta + \kappa\delta)\mathcal{T}\right\}. \tag{85}$$

By choosing $\alpha = 2C + 1$ and $\kappa = \frac{2[C(\alpha+3)+1]}{\epsilon_1}$ successively and fixing the positive constants $\theta = 1/\kappa^2$ and $\delta = 1/\kappa$, we can get from (85) that (78) holds again, which implies a corresponding estimate for $|D^2 u(x_0)|$ in case (b).

Next, we consider the case $x_0 \in \partial\Omega$, namely the function G in (50) attains its maximum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$ and a unit vector ξ. The estimation of the rest of the Hessian $D^2 u$ splits into two stages according to a different direction of ξ.

Case (i). ξ tangential. Since $\nu'(x_0, \xi) = 0$, at x_0 we obtain

$$
\begin{aligned}
0 \geq D_\nu G \\
= D_\nu \left\{ \left[\tilde{u}_{\xi\xi} - \nu'(x, \xi) \right] e^{\frac{\alpha}{2}|Du|^2 + \kappa \Phi} \right\} \\
= e^{\frac{\alpha}{2}|Du|^2 + \kappa \Phi} \left\{ (\tilde{u}_{\xi\xi} - \nu'(x, \xi)) D_\nu \left(\frac{\alpha}{2}|Du|^2 + \kappa \Phi \right) + D_\nu (\tilde{u}_{\xi\xi} - \nu'(x, \xi)) \right\} \\
= e^{\frac{\alpha}{2}|Du|^2 + \kappa \Phi} \left\{ \left[\alpha D_k u D_\nu (D_k u) + \kappa D_\nu \Phi \right] \tilde{u}_{\xi\xi} + D_\nu u_{\xi\xi} - D_\nu (A_{\xi\xi} + \nu') \right\} \\
= e^{\frac{\alpha}{2}|Du|^2 + \kappa \Phi} \left\{ \left[\kappa D_\nu \Phi + \alpha D_k u (\varphi_k - D_i u D_k \nu_i) \right] \tilde{u}_{\xi\xi} \right. \\
\left. + D_\nu u_{\xi\xi} - D_\nu (A_{\xi\xi} + \nu') \right\} \\
\geq e^{\frac{\alpha}{2}|Du|^2 + \kappa \Phi} \left\{ (\kappa c_0 - \alpha M) \tilde{u}_{\xi\xi} + D_\nu u_{\xi\xi} - D_\nu (A_{\xi\xi} + \nu') \right\},
\end{aligned}
\tag{86}
$$

where $c_0 = Ka \inf(e^{K(\underline{u}-u)})$, $M = \max_{x \in \partial\Omega} |D_k u(\varphi_k - D_i u D_k \nu_i)|$. The above inequality gives a relationship between $\tilde{u}_{\xi\xi}(x_0)$ and $D_\nu u_{\xi\xi}(x_0)$ at x_0, namely

$$
D_\nu u_{\xi\xi} \leq -(\kappa c_0 - \alpha M) \tilde{u}_{\xi\xi} + D_\nu (A_{\xi\xi} + \nu').
\tag{87}
$$

On the other hand, by tangential differentiating the boundary condition twice, we obtain

$$
(D_k u) \delta_i \delta_j \nu_k + (\delta_i D_k u) \delta_j \nu_k + (\delta_j D_k u) \delta_i \nu_k + \nu_k \delta_i \delta_j D_k u = \delta_i \delta_j \varphi \quad \text{on } \partial\Omega.
\tag{88}
$$

Hence, for the tangential direction ξ at x_0, we have

$$
\begin{aligned}
D_\nu u_{\xi\xi} \geq -2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j + (\delta_i \nu_j) \xi_i \xi_j D_{\nu\nu} u - C \\
\geq -2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j - C,
\end{aligned}
\tag{89}
$$

where the double normal derivative estimate (46) on $\partial\Omega$ is used in the second inequality. Thus, at x_0 we have

$$
\begin{aligned}
(\kappa c_0 - \alpha M) \tilde{u}_{\xi\xi} \leq 2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j + D_\nu (A_{\xi\xi} + \nu') + C \\
\leq 2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j + C|DD_\nu u| + C \\
\leq 2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j + C,
\end{aligned}
\tag{90}
$$

where the mixed tangential normal derivative estimate (28) and the double normal derivative estimate (46) are used to obtain the last inequality, and the constant C changes from line to line. Without loss of generality, we can assume the normal at x_0 to be $\nu = (0, \ldots, 1)$, and we can assume $\{\tilde{u}_{ij}(x_0)\}_{i,j<n}$ is diagonal with maximum eigenvalue $\tilde{u}_{11}(x_0) > 1$. Then, at x_0 we obtain

$$
(\kappa c_0 - \alpha M) \tilde{u}_{\xi\xi} \leq C(\tilde{u}_{11} + 1).
\tag{91}
$$

Since $G(x_0, e_1) \leq G(x_0, \xi)$, we have

$$
\tilde{u}_{11}(x_0) \leq \tilde{u}_{\xi\xi}(x_0) + \nu'(x_0, e_1) - \nu'(x_0, \xi).
\tag{92}
$$

Combining (91) and (92) and choosing $\kappa \geq \frac{2}{c_0}(\alpha M + C)$, we obtain

$$u_{\xi\xi}(x_0) \leq C. \tag{93}$$

Case (ii). ξ non-tangential. We write ξ as

$$\xi = \alpha\tau + \beta\nu, \tag{94}$$

where $\alpha = \xi \cdot \tau$, $|\tau| = 1$, $\tau \cdot \nu = 0$, $\beta = \xi \cdot \nu \neq 0$, and

$$\alpha^2 + \beta^2 = 1. \tag{95}$$

Therefore, at x_0 we have

$$\begin{aligned}
D_{\xi\xi}\tilde{u} &= \alpha^2 \tilde{u}_{\tau\tau} + \beta^2 \tilde{u}_{\nu\nu} + 2\alpha\beta\tilde{u}_{\tau\nu} \\
&= \alpha^2 \tilde{u}_{\tau\tau} + \beta^2 \tilde{u}_{\nu\nu} + \nu'(x,\xi),
\end{aligned} \tag{96}$$

where the definition of ν' in (51) is used. Since $G(x_0, \tau) \leq G(x_0, \xi)$, we obtain

$$\begin{aligned}
G(x_0, \xi) &= \alpha^2 G(x_0, \tau) + \beta^2 G(x_0, \nu) \\
&\leq \alpha^2 G(x_0, \xi) + \beta^2 G(x_0, \nu).
\end{aligned} \tag{97}$$

Using (95) in (97), we get

$$G(x_0, \xi) \leq G(x_0, \nu), \tag{98}$$

which implies

$$D_{\xi\xi}u(x_0) \leq C + D_{\nu\nu}u(x_0) \leq C, \tag{99}$$

where the double normal derivative estimate (46) on $\partial\Omega$ is used again.

Therefore, we can conclude from the two cases (i) and (ii) that if G attains its global maximum at $x_0 \in \partial\Omega$ and a unit vector ξ, then $D_{\xi\xi}u(x_0)$ is bounded from above. Taking (78) into account, we can derive estimate (11). Hence, Theorem 1.1 is proved. $\square$

4 C^0, C^1 estimates and the existence

In this section, we derive the lower order *a priori* derivative estimates (i.e., the C^0 and C^1 estimates) for Monge–Ampère type equations (1) with the Neumann boundary value condition (2). Using these estimates together with the second derivative estimate in Theorem 1.1, we give the proof of Theorem 1.2.

Since $B_z > 0$, the comparison principle for the classical solutions is valid. Under the assumptions that $\underline{u}$ satisfies (4)–(5) and $\bar{u}$ satisfies (1) and $D_\nu \bar{u} \leq \varphi(x)$ on $\partial\Omega$, by comparison principle, we have

$$\underline{u} \leq u \leq \bar{u} \tag{100}$$

in $\bar{\Omega}$, which gives the C^0 estimate of the solution u.

The C^1 estimate is established in [15, 22] in a general form. For convenience, we repeat it here.

Theorem 4.1 *Let $u \in C^2(\Omega) \cap C^{1,1}(\bar{\Omega})$ satisfy*

$$D^2 u \geq -\mu_0\big(1 + |Du|^2\big)I \tag{101}$$

in a C^2 domain $\Omega \subset \mathbb{R}^n$, with $D_\nu u \geq -\delta$ on $\partial\Omega$, where μ_0 and δ are nonnegative constants. Then we have the estimate

$$|Du| \leq C, \tag{102}$$

where C depends on μ_0, δ, Ω, and $\sup|u|$.

From the ellipticity and the quadratic lower bound for A in (13), it is easy to get inequality (101). Setting $\delta = \max\{-\min_{\partial\Omega}\varphi, 0\}$, u satisfying (2) automatically satisfies $D_\nu u \geq -\delta$ on $\partial\Omega$. Hence, the gradient estimate (102) holds for the solution u of problem (1)–(2).

With the preparations of *a priori* estimates (100), (102), and (11), we are able to prove the existence result in Theorem 1.2.

Proof of Theorem 1.2　We shall prove the existence of a viscosity solution u of the Neumann problem of the degenerate Monge–Ampère equation (1)–(2) by approximations.

We consider the following approximating problem:

$$\begin{aligned}
\det\big[D^2 u - A(\cdot, Du)\big] &= B_\epsilon > 0, \quad \text{in } \Omega, \\
D_\nu u &= \varphi(x), \quad \text{on } \partial\Omega,
\end{aligned} \tag{103}$$

where

$$B_\epsilon(\cdot, u, Du) := B(\cdot, u, Du) + \epsilon$$

for small positive constant ϵ. If B satisfies conditions (6)–(8), it is obvious that B_ϵ still satisfies the same conditions.

By Remark 1, there exists a unique elliptic solution $u_\epsilon \in C^{3,\alpha}$ of problem (103), where $\alpha \in (0, 1)$. Since $\bar{u}$ is still the supersolution of problem (103), and for sufficiently small $\epsilon > 0$, $\underline{u}$ is still the subsolution of problem (103) as well, we get the uniform estimate

$$\underline{u} \leq u_\epsilon \leq \bar{u} \tag{104}$$

for sufficiently small $\epsilon > 0$. Notice that the gradient estimate in Theorem 4.1 is independent of the lower bound of the right-hand side term of the equation. Therefore, we have

$$|Du_\epsilon| \leq C, \tag{105}$$

where the constant C is independent of ϵ. By Theorem 1.1, we obtain the uniform second derivative estimate

$$\sup_\Omega\big|D^2 u_\epsilon\big| \leq C. \tag{106}$$

From the remarks in the paragraph below Theorem 1.1 (or from the proof of Theorem 1.1), we know that the constant C in (106) is independent of ϵ. With the uniform estimates (104), (105), and (106), we can pass the limit to u_ϵ. Now letting $\epsilon \to 0$ (passing to a subsequence if necessary), we get a viscosity solution u of the degenerate Monge–Ampère equation (1) with the Neumann boundary condition (2), which is degenerate elliptic and satisfies the estimate

$$\|u\|_{C^{1,1}(\bar{\Omega})} \leq C. \tag{107}$$

Therefore, the proof of Theorem (1.2) is completed.   □

Remark 7 In Theorem 1.2, we proved the existence of a viscosity solution $u \in C^{1,1}(\bar{\Omega})$ by smooth approximations. Such uniqueness of viscosity solutions in our $A_z = 0$, $B_z > 0$, and $\varphi_z = 0$ case is not covered in Sect. 4 of [23]. It would be interesting to investigate the uniqueness of the viscosity solutions of the degenerate Monge–Ampère equation (1) satisfying the Neumann boundary condition (2).

Remark 8 In the current paper, we show that "the existence of a subsolution implies the existence of a solution", which is the new point of the existence result to the Neumann problem (1)–(2) of degenerate Monge–Ampère type equations. This idea is also new even for nondegenerate Monge–Ampère type equations. For the nondegenerate case in [15], for a more general problem with A and φ depending also on u,

$$\det\left[D^2 u - A(\cdot, u, Du)\right] = B(\cdot, u, Du) \quad \text{in } \Omega,$$
$$D_\nu u = \varphi(\cdot, u) \quad \text{on } \partial\Omega, \tag{108}$$

the result is that "the existence of a supersolution implies the existence of a solution". If we consider such a general problem (108) in the degenerate elliptic setting, the supersolution u is only degenerate elliptic and is not enough to construct a barrier function. It needs new ideas to solve the general Neumann problem (108) in the degenerate case.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1] School of Science, Nanjing University of Science and Technology, 210094 Nanjing, P.R. China. [2] School of Mathematics and Shing-Tung Yau Center of Southeast University, Southeast University, 211189 Nanjing, P.R. China.

Publisher's Note

References

1. Ma, X.N., Trudinger, N.S., Wang, X.-J.: Regularity of potential functions of the optimal transportation problem. Arch. Ration. Mech. Anal. **177**, 151–183 (2005)
2. Loeper, G.: On the regularity of solutions of optimal transportation problems. Acta Math. **202**, 241–283 (2009)
3. Andriyanova, E.: On the Dirichlet problem for degenerate elliptic Monge–Ampère type equations. Calc. Var. Partial Differ. Equ. **58**, 181 (2019)
4. Bao, J.G.: The Dirichlet problem for the degenerate elliptic Monge–Ampère equation. J. Math. Anal. Appl. **238**, 168–178 (1999)
5. Błocki, Z.: Interior regularity of the degenerate Monge–Ampère equations. Bull. Aust. Math. Soc. **68**, 81–92 (2003)
6. Guan, P.F.: C^2 a priori estimates for degenerate Monge–Ampère equations. Duke Math. J. **86**, 323–346 (1997)
7. Guan, P.F., Trudinger, N.S., Wang, X.-J.: On the Dirichlet problem for degenerate Monge–Ampère equations. Acta Math. **182**, 87–104 (1999)
8. Shi, J.H., Jiang, F.D.: Pogorelov estimates for the Monge–Ampère equations. Proc. Am. Math. Soc. **147**, 2561–2571 (2019)
9. Jiang, F.D., Trudinger, N.S.: On Pogorelov estimates in optimal transportation and geometric optics. Bull. Math. Sci. **4**, 407–431 (2014)
10. Jiang, F.D., Trudinger, N.S.: On the second boundary value problem for Monge–Ampère type equations and geometric optics. Arch. Ration. Mech. Anal. **229**, 547–567 (2018)
11. Trudinger, N.S.: Recent developments in elliptic partial differential equations of Monge–Ampère type. In: Proceedings of the International Congress of Mathematicians, vol. 3, pp. 291–302. ICM, Madrid (2006)
12. Huang, Y., Jiang, F.D., Liu, J.K.: Boundary $C^{2,\alpha}$ Estimates for Monge–Ampère Type Equations. Birkhäuser, Boston (2001)
13. Jiang, F.D., Trudinger, N.S., Yang, X.P.: On the Dirichlet problem for Monge–Ampère type equations. Calc. Var. Partial Differ. Equ. **49**, 1223–1236 (2014)
14. Lions, P.L., Trudinger, N.S., Urbas, J.I.E.: The Neumann problem for equations of Monge–Ampère type. Commun. Pure Appl. Math. **39**, 539–563 (1986)
15. Jiang, F.D., Trudinger, N.S., Xiang, N.: On the Neumann problem for Monge–Ampère type equations. Can. J. Math. **68**, 1334–1361 (2016)
16. Trudinger, N.S., Urbas, J.: On the second derivative estimates for equations of Monge–Ampère type. Bull. Aust. Math. Soc. **30**, 321–334 (1984)
17. Caffarelli, L.A., Nirenberg, L., Spruck, J.: The Dirichlet problem for the degenerate Monge–Ampère equation. Rev. Mat. Iberoam. **2**, 19–27 (1986)
18. Li, Q.-R., Wang, X.-J.: Regularity of the homogeneous Monge–Ampère equation. Discrete Contin. Dyn. Syst. **35**, 6069–6084 (2015)
19. Jiang, F.D., Shi, J.H., Yang, X.-P.: Interior $C^{1,1}$ regularity of solutions to a class of doubly degenerate Monge–Ampère type equations. Preprint (2020)
20. Jiang, F.D., Trudinger, N.S.: Oblique boundary value problems for augmented Hessian equations II. Nonlinear Anal. **154**, 148–173 (2017)
21. Liu, J.K., Trudinger, N.S.: On Pogorelov estimates for Monge–Ampère type equations. Discrete Contin. Dyn. Syst. **28**, 1121–1135 (2010)
22. Jiang, F.D., Xiang, N., Xu, J.J.: Gradient estimates for Neumann boundary value problem of Monge–Ampère type equations. Commun. Contemp. Math. **19**, 1650041 (2017)
23. Jiang, F.D., Trudinger, N.S.: Oblique boundary value problems for augmented Hessian equations III. Commun. Partial Differ. Equ. **44**, 708–748 (2019)