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Analysis of reaction–diffusion systems where a parameter influences both the reaction terms as well as the boundary

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Abstract

We study positive solutions to steady-state reaction–diffusion models of the form

$$\begin{cases} -\Delta u = \lambda f(v); & \Omega, \\ -\Delta v = \lambda g(u); & \Omega, \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; & \partial\Omega, \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} v = 0; & \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ ($N > 1$) with smooth boundary $\partial\Omega$, or $\Omega = (0,1)$, $\frac{\partial z}{\partial \eta}$ is the outward normal derivative of z. We assume that f and g are continuous increasing functions such that $f(0) = 0 = g(0)$ and $\lim_{s\to\infty} \frac{f(Mg(s))}{s} = 0$ for all $M > 0$. In particular, we extend the results for the single equation case discussed in (Fonseka et al. in J. Math. Anal. Appl. 476(2):480-494, 2019) to the above system.

MSC: 35J15; 35J25; 35J60

1 Introduction

In [7] the authors analyzed and established several results for positive solutions for reaction–diffusion models of the form

$$\begin{cases} -\Delta u = \lambda f(u); & \Omega, \\ \frac{\partial u}{\partial \eta} + \mu(\lambda) u = 0; & \partial\Omega, \end{cases}$$

where $f \in C^2([0,\infty))$, and $\mu \in C([0,\infty))$ is strictly increasing such that $\mu(0) \geq 0$. In recent history, there has been a lot of interests in models where the parameter influences the equation and boundary conditions (see [5–7], and [8]). In this paper, we are interested in extending the results in [7] to systems of the form

$$\begin{cases} -\Delta u = \lambda f(v); & \Omega, \\ -\Delta v = \lambda g(u); & \Omega, \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; & \partial\Omega, \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} v = 0; & \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ ($N > 1$) with smooth boundary $\partial\Omega$, or $\Omega = (0,1)$, $\frac{\partial z}{\partial \eta}$ is the outward normal derivative of z, and f, g satisfy the following conditions:

(H_1) $f, g \in C[0,\infty)$, increasing, $f(0) = 0 = g(0)$, and $\lim_{s\to\infty} \frac{f(Mg(s))}{s} = 0$ for all $M > 0$ (combined sublinear effect at infinity). Further, there exists $a > 0$ such that $f, g \in C^1[0,a)$ and $f'(0), g'(0) > 0$.

Without loss of generality, we assume that $f'(0) \geq g'(0)$ throughout the paper. We first recall recent results for the eigenvalue problem

$$\begin{cases} -\Delta v = Ev; & \Omega, \\ \frac{\partial v}{\partial \eta} + \gamma\sqrt{E} v = 0; & \partial\Omega, \end{cases} \tag{1.2}$$

where $\gamma > 0$. Namely, let $E_1(\gamma)$ be its principal eigenvalue (see [8]), and let v be the corresponding normalized positive eigenfunction of (1.2). Now consider the eigenvalue problem

$$\begin{cases} -\Delta \phi = \bar{E} g'(0)\phi; & \Omega, \\ \frac{\partial \phi}{\partial \eta} + \sqrt{\bar{E}}\phi = 0; & \partial\Omega. \end{cases} \tag{1.3}$$

Noting that the substitution $E = \overline{E} g'(0)$ reduces (1.3) to (1.2), we easily see that the principal eigenvalue of (1.3) is $\frac{E_1(\gamma)}{g'(0)}$ with $\gamma = \frac{1}{\sqrt{g'(0)}}$. Define

$$A_1 := \frac{E_1(\gamma)}{g'(0)}. \tag{1.4}$$

Next, for a given $\tilde{\lambda}$, we recall the for the eigenvalue problem

$$\begin{cases} -\Delta v = (\sigma + \tilde{\lambda})v; & \Omega, \\ \frac{\partial v}{\partial \eta} + \gamma\sqrt{\tilde{\lambda}} v = 0; & \partial\Omega. \end{cases} \tag{1.5}$$

Denoting by $\sigma_{\tilde{\lambda}}(\tilde{\lambda}, \gamma)$ its principal eigenvalue and by $\phi_{\tilde{\lambda}} > 0$ the corresponding eigenfunction of (1.5) such that $\|\phi_{\tilde{\lambda}}\| = 1$, the following results were established in [8]:

$$\begin{cases} \sigma_{\tilde{\lambda}} > 0; & \tilde{\lambda} < E_1(\gamma), \\ \sigma_{\tilde{\lambda}} = 0; & \tilde{\lambda} = E_1(\gamma), \\ \sigma_{\tilde{\lambda}} < 0; & \tilde{\lambda} > E_1(\gamma). \end{cases} \tag{1.6}$$

Hence, by the substitution $\lambda g'(0) = \tilde{\lambda}$, denoting by σ_λ the principal eigenvalue of

$$\begin{cases} -\Delta\phi = (\sigma + \lambda g'(0))\phi; & \Omega, \\ \frac{\partial\phi}{\partial\eta} + \sqrt{\lambda}\phi = 0; & \partial\Omega, \end{cases} \tag{1.7}$$

we deduce the following:

$$\begin{cases} \sigma_\lambda > 0; & \lambda < A_1, \\ \sigma_\lambda = 0; & \lambda = A_1, \\ \sigma_\lambda < 0; & \lambda > A_1. \end{cases} \tag{1.8}$$

Next, for $0 < a < b$, define

$$Q_1(a) := \min\left\{\frac{a}{f(a)}, \frac{a}{g(a)}\right\} \tag{1.9}$$

and

$$Q_2(b) := \max\left\{\frac{b}{f(b)}, \frac{b}{g(b)}\right\}. \tag{1.10}$$

Further, let

$$C_1 = \inf_{\epsilon} \frac{N}{\epsilon^N} \frac{R^{N-1}}{R-\epsilon}, \tag{1.11}$$

where R is the radius of the largest inscribed ball in Ω. Let w be the unique solution of

$$\begin{cases} -\Delta w = 1; & \Omega, \\ \frac{\partial w}{\partial\eta} + w = 0; & \partial\Omega. \end{cases} \tag{1.12}$$

Then we first establish the following:

Theorem 1.1 *Let (H_1) hold. Then (1.1) has a positive solution (u_λ, v_λ) for $\lambda > A_1$ such that $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Further, if there exists $0 < a < b$ such that $\frac{Q_1}{Q_2} > C_1\|w\|_\infty$ and $Q_1 > \max\{A_1, 1\}\|w\|_\infty$, then (1.1) has at least three positive solutions for*

$$\max\{A_1, C_1Q_2, 1\} < \lambda < \frac{Q_1}{\|w\|_\infty} \quad \text{(see Fig. 1)}.$$

Next we establish the following:

Theorem 1.2 *If there exists $r > 0$ such that $f, g \in C^2([0, r)), f(0) = 0 = g(0), f'(0) = g'(0) > 0$, and $f''(s), g''(s) < 0$ for all $s \in [0, r)$, then (1.1) has a positive solution (u_λ, v_λ) for $\lambda > A_1$ and $\lambda \approx A_1$ such that $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to 0$ as $\lambda \to A_1^+$* (see Fig. 2).

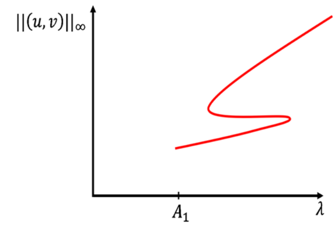

Figure 1 Bifurcation diagram for positive solutions when the hypothesis of Theorem 1.1 holds

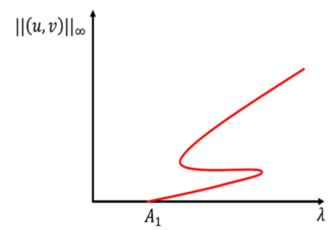

Figure 2 Bifurcation diagram for positive solutions when the hypotheses of Theorems 1.1 and 1.2 hold

Remark 1.1 Note that if both $\frac{s}{f(s)}$ and $\frac{s}{g(s)}$ are strictly increasing for $s > 0$, then the multiplicity of positive solutions for $\lambda > A_1$ is not possible (see Appendix).

We present some preliminaries in Sect. 2. We provide proofs of Theorems 1.1 and 1.2 in Sect. 3. In Sect. 4, we discuss an example. We state and prove Lemma 5.1 in Appendix to justify Remark 1.1. Our existence and multiplicity results are established via the method of sub- and supersolutions. We conclude this Introduction by citing two additional related references [3] and [9].

2 Preliminaries

In this section, we introduce definitions of a subsolution and a supersolution of (1.1), and state sub-supersolution theorems which are used to prove the existence and multiplicity for positive solutions. By a subsolution of (1.1) we mean $(\psi_1, \psi_2) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ that satisfies

$$\begin{cases} -\Delta \psi_1 \leq \lambda f(\psi_2); & \Omega, \\ -\Delta \psi_2 \leq \lambda g(\psi_1); & \Omega, \\ \frac{\partial \psi_1}{\partial \eta} + \sqrt{\lambda}\psi_1 \leq 0; & \partial\Omega, \\ \frac{\partial \psi_2}{\partial \eta} + \sqrt{\lambda}\psi_2 \leq 0; & \partial\Omega, \end{cases}$$

and by a supersolution of (1.1) we mean $(Z_1, Z_2) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ that satisfies

$$\begin{cases} -\Delta Z_1 \geq \lambda f(Z_2); & \Omega, \\ -\Delta Z_2 \geq \lambda g(Z_1); & \Omega, \\ \frac{\partial Z_1}{\partial \eta} + \sqrt{\lambda} Z_1 \geq 0; & \partial\Omega, \\ \frac{\partial Z_2}{\partial \eta} + \sqrt{\lambda} Z_2 \geq 0; & \partial\Omega. \end{cases}$$

Now we state two results (see [2, 10], and [11]) we will use later.

Lemma 2.1 *Let (ψ_1,ψ_2) and (Z_1,Z_2) be a subsolution and a supersolution of (1.1), respectively, such that $(\psi_1,\psi_2) \le (Z_1,Z_2)$. Then (1.1) has a solution $(u,v) \in C^2(\Omega)\cap C^1(\bar{\Omega})$ such that $(u,v) \in [(\psi_1,\psi_2),(Z_1,Z_2)]$.*

Lemma 2.2 *Let f and g be nonnegative and increasing, and suppose there exist a subsolution $(\psi_1,\overline{\psi}_1)$, a strict supersolution $(\phi_2,\overline{\phi}_2)$, a strict subsolution $(\psi_2,\overline{\psi}_2)$, and a supersolution $(\phi_1,\overline{\phi}_1)$ for (1.1) such that $(\psi_1,\overline{\psi}_1) \le (\psi_2,\overline{\psi}_2) \le (\phi_1,\overline{\phi}_1)$, $(\psi_1,\overline{\psi}_1) \le (\phi_2,\overline{\phi}_2) \le (\phi_1,\overline{\phi}_1)$, and $(\psi_2,\overline{\psi}_2) \not\le (\phi_2,\overline{\phi}_2)$. Then (1.1) has at least three positive solutions (u_i,v_i), $i = 1,2,3$, such that $(u_1,v_1) \in [(\psi_1,\overline{\psi}_1),(\phi_2,\overline{\phi}_2)]$, $(u_2,v_2) \in [(\psi_2,\overline{\psi}_2),(\phi_1,\overline{\phi}_1)]$, and $(u_3,v_3) \in [(\psi_1,\overline{\psi}_1),(\phi_1,\overline{\phi}_1)]\backslash([(\psi_1,\overline{\psi}_1),(\phi_2,\overline{\phi}_2)] \cup [(\psi_2,\overline{\psi}_2),(\phi_1,\overline{\phi}_1)])$.*

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 First, we show the existence of positive solutions for $\lambda > A_1$. Let $(\psi_1,\psi_2) = (m\phi_\lambda, m\phi_\lambda)$, where ϕ_λ is the normalized positive eigenfunction of (1.7). Define $H(s) = (\sigma_\lambda + \lambda g'(0))s - \lambda f(s)$. Then $H(0) = 0, H'(0) = \sigma_\lambda + \lambda(g'(0) - f'(0)) < 0$ (by (1.8)). This implies that $H(s) \le 0$ for $s \approx 0$. Thus $(\sigma_\lambda + \lambda g'(0))m\phi_\lambda - \lambda f(m\phi_\lambda) \le 0$ for $m \approx 0$. Hence $-\Delta\psi_1 \le \lambda f(\psi_2)$ for $m \approx 0$. Analyzing $\tilde{H}(s) = (\sigma_\lambda + \lambda g'(0))s - \lambda g(s)$, by a similar argument we obtain $-\Delta\psi_2 \le \lambda g(\psi_1)$. Further, on the boundary, we have $\frac{\partial\psi_1}{\partial\eta} + \sqrt{\lambda}\psi_1 = 0$ and $\frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}\psi_2 = 0$. Thus (ψ_1,ψ_2) is a subsolution of (1.1) for $m \approx 0$.

Now let e_λ be a positive solution of

$$\begin{cases} -\Delta e = 1; & \Omega, \\ \frac{\partial e}{\partial\eta} + \sqrt{\lambda}e = 0; & \partial\Omega. \end{cases} \tag{3.1}$$

We consider three different cases. If both f and g are bounded, then let $(Z_1,Z_2) = (\lambda M_\lambda \frac{e_\lambda}{\|e_\lambda\|_\infty}, \lambda M_\lambda \frac{e_\lambda}{\|e_\lambda\|_\infty})$ and choose M_λ large such that $\frac{M_\lambda \lambda}{\|e_\lambda\|_\infty} \ge \lambda f(\frac{M_\lambda e_\lambda}{\|e_\lambda\|_\infty})$. This implies $-\Delta Z_1 - \lambda f(Z_2) \ge 0$ for $M_\lambda \gg 1$, and by a similar argument we see that $-\Delta Z_2 - \lambda g(Z_1) \ge 0$ for $M_\lambda \gg 1$. Also, on the boundary, we have $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = 0$ and $\frac{\partial Z_2}{\partial\eta} + \sqrt{\lambda}Z_2 = 0$. Hence (Z_1,Z_2) is a supersolution for $M_\lambda \gg 1$. If $g(x) \to \infty$ as $x \to \infty$, let $(Z_1,Z_2) = (M_\lambda e_\lambda, \lambda g(M_\lambda\|e_\lambda\|_\infty)e_\lambda)$. Then by choosing M_λ large we obtain

$$\frac{1}{\lambda\|e_\lambda\|_\infty} \ge \frac{f(\lambda\|e_\lambda\|_\infty g(M_\lambda\|e_\lambda\|_\infty))}{M_\lambda\|e_\lambda\|_\infty},$$

which implies that $M_\lambda - \lambda f(\lambda g(M_\lambda\|e_\lambda\|_\infty)e_\lambda) \ge 0$. Hence $-\Delta Z_1 - \lambda f(Z_2) \ge 0$. We also have $\lambda g(M_\lambda\|e_\lambda\|_\infty) - \lambda g(M_\lambda e_\lambda) \ge 0$. This implies that $-\Delta Z_2 - \lambda g(Z_1) \ge 0$. Further, on the boundary, we have $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = \frac{\partial Z_2}{\partial\eta} + \sqrt{\lambda}Z_2 = 0$. Hence (Z_1,Z_2) is a supersolution of (1.1) for $M_\lambda \gg 1$. Finally, we consider the case where $f(x) \to \infty$ as $x \to \infty$ and $g(x)$ is bounded. In this case, let $(Z_1,Z_2) = (\lambda f(M_\lambda\|e_\lambda\|_\infty)e_\lambda, M_\lambda e_\lambda)$. Then $\lambda f(M_\lambda\|e_\lambda\|_\infty) - \lambda f(M_\lambda e_\lambda) \ge 0$, which implies that $-\Delta Z_1 - \lambda f(Z_2) \ge 0$. Also, we have $M_\lambda \ge \lambda g(\lambda f(M_\lambda\|e_\lambda\|_\infty)e_\lambda)$ for $M_\lambda \gg 1$. This implies that $-\Delta Z_2 - \lambda g(Z_1) \ge 0$. Further, on the boundary, we have $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = \frac{\partial Z_2}{\partial\eta} + \sqrt{\lambda}Z_2 = 0$. Hence (Z_1,Z_2) is a supersolution of (1.1) for $M_\lambda \gg 1$. Now we can choose M_λ large enough such that $(\psi_1,\psi_2) \le (Z_1,Z_2)$. Hence by Lemma (2.1), (1.1) has a positive solution for $\lambda > A_1$.

Next, we show that there exists a positive solution (u_λ,v_λ) for $\lambda \gg 1$ such that $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Define $h \in C^2([0,\infty))$ such that $h(0) < 0$, $h(s) \le f(s)$, and

$h(s) \leq g(s)$ for $s \in (0,\infty)$ and $\lim_{s\to\infty} h(s) > 0$. Then the Dirichlet boundary value problem

$$\begin{cases} -\Delta w = \lambda h(w); & \Omega, \\ w = 0; & \partial\Omega, \end{cases}$$

has a positive solution $\overline{w}_\lambda$ for $\lambda \gg 1$ such that $\|\overline{w}_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$ (see [4]).

It is easy to show that $(\overline{w}_\lambda, \overline{w}_\lambda)$ is a subsolution of (1.1) for $\lambda \gg 1$ since $h \leq f, h \leq g$, and $\frac{\partial \overline{w}}{\partial \eta} < 0; \partial\Omega$. We can also choose $M_\lambda \gg 1$ such that $(Z_1, Z_2) \geq (\overline{w}_\lambda, \overline{w}_\lambda)$, where (Z_1, Z_2) is a supersolution of (1.1) as constructed before. By Lemma 2.1, (1.1) has a positive solution $(u_\lambda, v_\lambda) \in [(\overline{w}_\lambda, \overline{w}_\lambda), (Z_1, Z_2)]$ for $\lambda \gg 1$. Clearly, $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$ since $\|\overline{w}\|_\infty \to \infty$ as $\lambda \to \infty$.

Next, we establish our multiplicity result. We first construct a positive strict supersolution $(\tilde{z}_1, \tilde{z}_2)$ for (1.1) when $1 < \lambda < \frac{Q_1}{\|w\|_\infty}$, where w is the solution of (1.12). Let $(\tilde{z}_1, \tilde{z}_2) = (a\frac{w}{\|w\|_\infty}, a\frac{w}{\|w\|_\infty})$. Then $\frac{a}{\|w\|_\infty f(a)} > \lambda$ gives us $\frac{a}{\|w\|_\infty} > \lambda f(a)$. Since f is increasing, we have $\frac{a}{\|w\|_\infty} \geq \lambda f(a\frac{w}{\|w\|_\infty})$, which implies that $-\Delta\tilde{z}_1 \geq \lambda f(\tilde{z}_2)$ in Ω. Similarly, we can show that $-\Delta\tilde{z}_2 \geq \lambda g(\tilde{z}_1)$ in Ω. On the boundary, we have $\frac{\partial\tilde{z}_1}{\partial\eta} + \sqrt{\lambda}\tilde{z}_1 = \frac{a}{\|w\|_\infty}[\frac{\partial w}{\partial\eta} + \sqrt{\lambda}w] > \frac{a}{\|w\|_\infty}[\frac{\partial w}{\partial\eta} + w] = 0$ since $\lambda > 1$. Similarly, we have $\frac{\partial\tilde{z}_2}{\partial\eta} + \sqrt{\lambda}\tilde{z}_2 > 0$. Thus $(\tilde{z}_1, \tilde{z}_2)$ is a strict supersolution of (1.1).

Now we construct a strict subsolution $(\tilde{\psi}_1, \tilde{\psi}_2)$ of (1.1) for $\lambda \geq C_1Q_2 = C_1\max\{\frac{b}{g(b)}, \frac{b}{f(b)}\}$. Note that in [1] the authors showed that the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(v); & \Omega, \\ -\Delta v = \lambda g(u); & \Omega, \\ u = 0; & \partial\Omega, \\ v = 0; & \partial\Omega, \end{cases}$$

has a strict subsolution $(\overline{u}_0, \overline{v}_0)$ for $\lambda \geq C_1\max\{\frac{b}{g(b)}, \frac{b}{f(b)}\}$ such that $\|\overline{u}_0\|_\infty \geq b$ and $\|\overline{v}_0\|_\infty \geq b$. Let $(\tilde{\psi}_1, \tilde{\psi}_2)$ be the first iteration of $(\overline{u}_0, \overline{v}_0)$, that is, the solution to the problem

$$\begin{cases} -\Delta\tilde{\psi}_1 = \lambda f(\overline{v}_0); & \Omega, \\ -\Delta\tilde{\psi}_2 = \lambda g(\overline{u}_0); & \Omega, \\ \frac{\partial\tilde{\psi}_1}{\partial\eta} + \sqrt{\lambda}\tilde{\psi}_1 = 0; & \partial\Omega, \\ \frac{\partial\tilde{\psi}_2}{\partial\eta} + \sqrt{\lambda}\tilde{\psi}_2 = 0; & \partial\Omega. \end{cases}$$

Then by comparison principle $(\tilde{\psi}_1, \tilde{\psi}_2) > (\overline{u}_0, \overline{v}_0); \bar{\Omega}$. Hence $(\tilde{\psi}_1, \tilde{\psi}_2)$ is a strict subsolution of (1.1) such that $\|\tilde{\psi}_1\|_\infty \geq b > a$ and $\|\tilde{\psi}_2\|_\infty \geq b > a$. Thus we have $(\tilde{\psi}_1, \tilde{\psi}_2) \nleq (\tilde{z}_1, \tilde{z}_2)$. Further, for $\lambda > A_1$, we can construct the subsolution (ψ_1, ψ_2) as before for $m \approx 0$, and for any $\lambda > 0$, we can construct the supersolution (Z_1, Z_2) as before for $M_\lambda \gg 1$. Also, for $m \approx 0$ and $M_\lambda \gg 1$, we obtain $(\psi_1, \psi_2) \leq (\tilde{\psi}_1, \tilde{\psi}_2) \leq (Z_1, Z_2)$ and $(\psi_1, \psi_2) \leq (\tilde{z}_1, \tilde{z}_2) \leq (Z_1, Z_2)$. Hence by Lemma (2.2), (1.1) has at least three positive solutions for $\lambda \in (\max\{A_1, C_1Q_2, 1\}, \frac{Q_1}{\|w\|_\infty})$. □

Proof of Theorem 1.2 Let $(\psi_1, \psi_2) = (m\phi_\lambda, m\phi_\lambda)$ be as in the proof of Theorem 1.1. Then (ψ_1, ψ_2) is a subsolution of (1.1) for $m \approx 0$. Since $f''(s) < 0$ and $g''(s) < 0$ for $s \approx 0$, there exists $A > 0$ such that $f''(s) \leq -A$ and $g''(s) \leq -A$ for $s \approx 0$. Let $(\phi_1, \phi_2) = (\delta_\lambda \phi_\lambda, \delta_\lambda \phi_\lambda)$, where $\delta_\lambda = -\frac{2\sigma_\lambda}{\lambda A \min_{\overline{\Omega}} \phi_\lambda}$. Note that $\delta_\lambda > 0$ and $\delta_\lambda \to 0$ as $\lambda \to A_1^+$ since $\sigma_\lambda \to 0$ and $\min_{\overline{\Omega}} \phi_\lambda \nrightarrow 0$ as $\lambda \to A_1^+$ (see [8]). Clearly, $\|\phi_1\|_\infty \to 0$ and $\|\phi_2\|_\infty \to 0$ as $\lambda \to A_1^+$. Then by Taylor's theorem we have $f(\phi_2) = f(0) + f'(0)\phi_2 + \frac{f''(\xi)}{2}\phi_2^2 = \phi_2 + \frac{f''(\xi)}{2}\phi_2^2$ for some $\xi \in [0, \phi_2]$. Then we have

$$\begin{aligned}
-\Delta\phi_1 - \lambda f(\phi_2) &= \delta_\lambda \big(\sigma_\lambda + \lambda g'(0)\big)\phi_\lambda - \lambda\left[\delta_\lambda \phi_\lambda f'(0) + \frac{f''(\xi)}{2}(\delta_\lambda \phi_\lambda)^2\right] \\
&\geq \delta_\lambda \phi_\lambda \left[\sigma_\lambda + \frac{\lambda A}{2}\delta_\lambda \min_{\bar{\Omega}} \phi_\lambda\right] = 0
\end{aligned}$$

by our choice of δ_λ. A similar argument shows that $-\Delta\phi_2 - \lambda g(\phi_1) \geq 0$. Further, on the boundary, we have $\frac{\partial \phi_1}{\partial \eta} + \sqrt{\lambda}\phi_1 = \frac{\partial \phi_2}{\partial \eta} + \sqrt{\lambda}\phi_2 = 0$. Thus (ϕ_1, ϕ_2) is a supersolution of (1.1). Choosing $m \approx 0$, we also have $(\psi_1, \psi_2) \leq (\phi_1, \phi_2)$. By Lemma 2.1 there exists a positive solution $(u_\lambda, v_\lambda) \in [(\psi_1, \psi_2), (\phi_1, \phi_2)]$ for $\lambda > A_1$ and $\lambda \approx A_1$ such that $\|u_\lambda\|_\infty \to 0$, $\|v_\lambda\|_\infty \to 0$ as $\lambda \to A_1^+$. □

4 Example

For an example to illustrate Theorems 1.1–1.2, consider $f = f_{\alpha,k}$ and $g = g_k$ as follows:

$$f = f_{\alpha,k}(s) = \begin{cases} e^{\frac{s}{s+1}} - 1; & s \leq k, \\ [e^{\frac{\alpha s}{\alpha+s}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{k}{k+1}} - 1]; & s \geq k, \end{cases}$$

$$g = g_k(s) = \begin{cases} 2(1+s)^{\frac{1}{2}} - 2; & s \leq k, \\ [\frac{1}{2}(1+s)^2 - \frac{1}{2}(1+k)^2] + [2(1+k)^{\frac{1}{2}} - 2]; & s \geq k, \end{cases}$$

where $k > 0$ is a constant, and $\alpha > 0$ is a parameter. Note that though g is superlinear at infinity, since f is bounded, $\frac{f(Mg(s))}{s} \longrightarrow 0$ as $s \longrightarrow \infty$ for all $M > 0$. Choose $a = k, b = \alpha$, and $\alpha > k$. Then we note that $Q_1(k) = \min\{\frac{k}{f(k)}, \frac{k}{g(k)}\} \longrightarrow \infty$ as $k \longrightarrow \infty$ since $\frac{k}{f(k)}, \frac{k}{g(k)} \longrightarrow \infty$ as $k \longrightarrow \infty$ and $Q_2(\alpha) = \max\{\frac{\alpha}{f(\alpha)}, \frac{\alpha}{g(\alpha)}\} \longrightarrow 0$ as $\alpha \longrightarrow \infty$ since $\frac{\alpha}{f(\alpha)}, \frac{\alpha}{g(\alpha)} \longrightarrow 0$ as $\alpha \longrightarrow \infty$. Hence we can choose $k = k_0$ such that $Q_1(k_0) > \max\{A_1, 1\}\|w\|_\infty$ and choose $\alpha \gg 1$ such that $\frac{Q_1(k_0)}{Q_2(\alpha)} \geq C_1\|w\|_\infty$. Hence $\frac{Q_1}{\|w\|_\infty} > \max\{A_1, C_1 Q_2, 1\}$ for $k = k_0$ and $\alpha \gg 1$. It is also clear that f and g satisfy (H_1). This implies that (1.1) has at least one positive solution for $\lambda > A_1$ and at least three positive solutions for $\lambda \in (\max\{A_1, C_1 Q_2, 1\}, \frac{Q_1}{\|w\|_\infty})$ for $k = k_0$ and $\alpha \gg 1$. Thus Theorem 1.1 holds in this example for $k = k_0$ and $\alpha \gg 1$. Further, since $f'(0) = g'(0)$ and $f, g \in C^2([0, k))$ and $f'' < 0$, $g'' < 0$ for all $s \in [0, k)$, Theorem 1.2 holds. Hence the bifurcation diagram for positive solutions is as shown in Fig. 2.

Appendix

Lemma 5.1 *Let (H_1) hold, and let $\frac{s}{f(s)}$ and $\frac{s}{g(s)}$ be strictly increasing for $s > 0$. Then* (1.1) *has a unique positive solution for $\lambda > A_1$.*

Proof: The existence for $\lambda > A_1$ is clear from Theorem 1.1. Let $(u_i, v_i), i = 1, 2$, be two distinct positive solutions for $\lambda > A_1$. Since $(m\phi_\lambda, m\phi_\lambda)$ is a subsolution for $m \approx 0$, there exists a minimal positive solution, and hence without loss of generality, we can assume that $(u_1, v_1) \geq (u_2, v_2)$, by which we mean that $u_1 \geq u_2, v_1 \geq v_2$. Now

$$
\begin{aligned}
L &= \int_\Omega \left\{ g(u_1)g(u_2)\left[\frac{u_1}{g(u_1)} - \frac{u_2}{g(u_2)}\right] + f(v_1)f(v_2)\left[\frac{v_1}{f(v_1)} - \frac{v_2}{f(v_2)}\right]\right\} dx \\
&= \int_\Omega \left[g(u_2)u_1 - g(u_1)u_2 + f(v_2)v_1 - f(v_1)v_2\right] dx \\
&= \int_\Omega \left\{[u_2\Delta v_1 - u_1\Delta v_2] + [v_2\Delta u_1 - v_1\Delta u_2]\right\} dx \\
&= \int_{\partial\Omega} \left\{\left[u_2\frac{\partial v_1}{\partial\eta} - u_1\frac{\partial v_2}{\partial\eta}\right] + \left[v_2\frac{\partial u_1}{\partial\eta} - v_1\frac{\partial u_2}{\partial\eta}\right]\right\} dx \\
&= \int_{\partial\Omega} [-\sqrt{\lambda}u_2v_1 + \sqrt{\lambda}u_1v_2 - \sqrt{\lambda}v_2u_1 + \sqrt{\lambda}v_1u_2]\, dx = 0.
\end{aligned}
$$

This is clearly a contradiction when $\frac{s}{f(s)}$ and $\frac{s}{g(s)}$ are strictly increasing and $(u_1, v_1), (u_2, v_2)$ are distinct. Thus (1.1) has a unique positive solution for $\lambda > A_1$.

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