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Structural stability for the Boussinesq equations interfacing with Darcy equations in a bounded domain

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Abstract

A priori bounds were derived for the flow in a bounded domain for the viscous-porous interfacing fluids. We assumed that the viscous fluid was slow in Ω_1 , which was governed by the Boussinesq equations. For a porous medium in Ω_2 , we supposed that the flow satisfied the Darcy equations. With the aid of these a priori bounds we were able to demonstrate the result of the continuous dependence type for the Boussinesq coefficient λ . Following the method of a first-order differential inequality, we can further obtain the result that the solution depends continuously on the interface boundary coefficient α . These results showed that the structural stability is valid for the interfacing problem.

MSC: 35B40; 35Q30; 76D05

Keywords: Boussinesq equations; Continuous dependence; Boussinesq coefficient; Interfacing problem; A priori bounds

1 Introduction

Recently, people have become interested in obtaining stability results of solutions for physical problems of partial differential equations with changes in coefficients. Sometimes the equations themselves are changed. This stability was called the structural stability in order to distinguish it from the traditional stability on initial data and boundary data. These problems were widely studied in many papers by many authors. For the problems of continuum mechanics, it is important for the authors to establish the structural stability of the model. This importance is discussed by Hirsch and Smale [1] in the form of a differential equation. This stability estimation is basic. We want to know whether a slight change in the coefficients in the equations or boundary data, or even the equation itself, will lead to drastic changes in the solution. For a review of the nature of the structural stability, refer to the books written by Ames and Straughan [2] and Straughan [3].

There are many papers studying the structural stability on the coefficients in fluids equations in porous media. Representative is the work of Ames et al. [4, 5], Franchi and Straughan [6], Hoang and Ibragimov [7], Lin and Payne [8–10], Liu [11, 12], Liu et al.

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[13–15], Scott [16], Scott and Straughan [17], Payne et al. [18–21] and some related papers [22–24]. The previous publications of structural stability usually study one fluid in a bounded domain. Usually, there exists more than one fluid in a domain. These fluids have some interactions. It is desirable to see what effect they can have on each other. So the study of two interfacing fluids may be interesting and meaningful. In [19], the authors studied the structural stability for a flow interfacing with a porous solid. They proved that the solution depends continuously on the coefficient of the interface boundary condition.

In this paper, we want to study the continuous dependence type results on the interface boundary coefficient and the Boussinesq coefficient for the solution of the Boussinesq–Darcy problem in R^3 . The Boussinesq equations interface with the Darcy equations through the mutual boundary. Thus, we suggest an appropriate part of the plane $z = x_1 = 0$ is the mutual boundary for a porous fluid in a bounded region Ω_2 in and a nonlinear viscous fluid in Ω_1 in R^3 . We denote the interface by L . The remaining part of $\partial\Omega_1$ is denoted by Γ_1 , and the remaining part of $\partial\Omega_2$ is denoted by Γ_2 . We also denote $\partial\Omega_1 = \Gamma_1 \cup L$ and $\partial\Omega_2 = \Gamma_2 \cup L$.

Let (u_i, T, p) and (v_i, θ, q) denote the velocity, temperature and pressure in Ω_1 and Ω_2 , respectively. Then the Boussinesq flow equations are (see [25–27])

$$\begin{aligned} \frac{\partial u_i}{\partial t} - \mu \Delta u_i + \lambda u_j u_{i,j} - g_i T + p_{,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \\ \frac{\partial T}{\partial t} + u_i T_{,i} &= k_1 \Delta T, \quad \text{in } \Omega_1 \times [0, \tau], \\ u_{i,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \end{aligned} \tag{1}$$

where g_i is the gravity force function; λ is the Boussinesq coefficient. The coefficients μ and k_1 are kinematic viscosity and thermal conductivity, respectively. From [28, 29]), we can see that the Boussinesq equations are useful in studying fluid and geophysical fluid dynamics.

The Darcy equations can be written as (see Nield and Bejan [30])

$$\begin{aligned} v_i - g_i \theta + \frac{\partial q}{\partial x_i} &= 0, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial \theta}{\partial t} + v_i \frac{\partial \theta}{\partial x_i} &= k_2 \Delta \theta, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial v_j}{\partial x_i} &= 0, \quad \text{in } \Omega_2 \times [0, \tau], \end{aligned} \tag{2}$$

where Ω_1 and Ω_2 are all bounded domains. They are all simply connected and star-shaped. The boundaries $\partial\Omega_1$ and $\partial\Omega_2$ are their boundaries, respectively. τ is a positive constant which satisfies $0 < \tau < \infty$. The following boundary conditions are satisfied:

$$\begin{aligned} u_i &= 0; \quad T = G(x, t), \quad \text{on } \Gamma_1 \times [0, \tau], \\ v_i n_i &= 0, \quad \theta = \tilde{G}(x, t), \quad \text{on } \Gamma_2 \times [0, \tau], \end{aligned} \tag{3}$$

for prescribed functions $G(x, t)$ and $\tilde{G}(x, t)$ and $n_i^{(1)}, n_i^{(2)}$ denote the unit outward normals of Ω_1, Ω_2 , respectively. Obviously, $n_3^{(1)} = -n_3^{(2)} = -1$. The initial conditions are written as

$$\begin{aligned} u_i(x, 0) &= f_i(x), & T(x, 0) &= T_0(x), & \text{in } \Omega_1, \\ \theta(x, 0) &= \theta_0(x), & & & \text{in } \Omega_2, \end{aligned} \tag{4}$$

for prescribed functions f_i, T_0 and θ_0 . The interface L conditions are

$$\begin{aligned} u_3 = v_3 &\leq 0, & T &= \theta, & k_1 T_{,3} &= k_2 \theta_{,3}, \\ q &= p - 2\mu u_{3,3}, & u_{\beta,3} + u_{3,\beta} &= \frac{\alpha}{\sqrt{k_1}} u_\beta, \end{aligned} \tag{5}$$

where α is a positive coefficient and the value of α can be defined by experiment. It is determined by the given fluid and porous solid. The boundary conditions (5) were given by Nield and Bejan in [30]. In [31], Jones deduced the last condition in (5).

In this paper, we want to obtain the continuous dependence on the Boussinesq coefficient λ and the interface boundary coefficient α for the Boussinesq–Darcy interfacing problems in a bounded domain. However, there are only a few papers studying this interfacing problem in a bounded domain (see Payne and Straughan [19] and Liu et al. [13]). For the unbounded domain, refer to Liu et al. [32]. However, compared with the above literature, in this paper, there is a nonlinear term $u_i u_{i,j}$. In particular, the bound of $\int_\Omega u_{i,j} u_{i,j} dx$ is needed in this paper. But the methods proposed in [13, 19, 32] cannot be used directly. Second, some well-known Sobolev inequalities cannot be held for the interfacing problem. Our biggest innovation is to overcome these difficulties. We are sure that we can obtain some new and interesting results. We will derive some useful a priori bounds by using different inequalities. With the aid of these a priori bounds, we derive the continuous dependence on the Boussinesq coefficient and the interface boundary coefficient.

In the following discussions, we use the comma to denote partial differentiation. We also use $u_{i,k}$ to denote the partial differentiation with respect to the direction x_k . This is to say $u_{i,k} = \frac{\partial u_i}{\partial x_k}$. We also use the usual summation convention with repeated Latin subscripts summed from 1 to 3, and the Greek subscripts summed from 1 to 2. Therefore, $u_{i,i} = \sum_{i=1}^3 \left(\frac{\partial u_i}{\partial x_i}\right)^2, u_{\beta,\beta} = \sum_{\beta=1}^2 \left(\frac{\partial u_\beta}{\partial x_\beta}\right)^2$.

2 A priori bounds

In this section, we want to drive bounds for various norms of u_i in terms of known data which will be used in the next sections.

Lemma 2.1 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$. Then the temperatures satisfy*

$$\sup_{[0,\tau]} \|T\|_\infty, \sup_{[0,\tau]} \|\theta\|_\infty \leq N_M, \tag{6}$$

where $N_M = \max\{\|T_0\|_\infty, \sup_{[0,\tau]} \|G\|_\infty, \|\theta_0\|_\infty, \sup_{[0,\tau]} \|\tilde{G}\|_\infty\}$.

Proof First, we let T_{LM} denotes the maximum of the temperature on the interface L . Payne, Rodrigues and Straughan [33] have derived

$$\sup_{[0,\tau]} \|T\|_\infty \leq \max\left\{\|T_0\|_\infty, \sup_{[0,\tau]} G_\infty, T_{LM}\right\}$$

and

$$\sup_{[0,\tau]} \|\theta\|_\infty \leq \max \left\{ \|\theta_0\|_\infty, \sup_{[0,\tau]} \tilde{G}_\infty, T_{LM} \right\}.$$

However, in the area $\Omega_1 \cup \Omega_2 \times [0, \tau]$, the maximum of the temperature cannot be reached on the interface L . Therefore, we have the result (6). □

Lemma 2.2 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$ and Ω_1, Ω_2 are bounded regions. Then*

$$\int_{\Omega_1} |\mathbf{u}|^2 dx \leq e^\tau \int_{\Omega_1} |\mathbf{f}|^2 dx + g^2 N_M^2 (|\Omega_1| + |\Omega_2|) (e^\tau - 1) \doteq A_1. \tag{7}$$

Proof Multiplying (1)₁ by u_i , integrating over Ω_1 and using (6), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |\mathbf{u}|^2 dx \\ &= \mu \int_{\Omega_1} (u_{i,j} + u_{j,i}) u_i dx - \lambda \int_{\Omega_1} u_j u_{i,j} u_i dx + \int_{\Omega_1} g_i T u_i dx - \int_{\Omega_1} p_{,i} u_i dx \\ &= -\mu \int_{\Omega_1} (u_{i,j} + u_{j,i}) u_{i,j} dx + \mu \int_L (u_{\beta,3} + u_{3,\beta}) u_\beta n_3^{(1)} dA - \frac{1}{2} \lambda \oint_{\partial\Omega_1} u_3 u_i u_i n_3^{(1)} dA \\ & \quad + \frac{1}{2} \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} g^2 N_M^2 |\Omega_1| - \int_L (p - 2\mu u_{3,3}) u_i n_i^{(1)} dA \\ & \leq \frac{1}{2} \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} g^2 N_M^2 |\Omega_1| + \int_L q v_i n_i^{(2)} dA. \end{aligned}$$

By the divergence theorem and (2) and the conditions in the interface, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |\mathbf{u}|^2 dx & \leq -\mu \int_{\Omega_1} (u_{i,j} + u_{j,i}) u_{i,j} dx \\ & \quad + \frac{1}{2} \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} g^2 N_M^2 |\Omega_1| + \int_{\Omega_2} v_i (g_i \theta - v_i) dx \\ & \leq \frac{1}{2} \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} g^2 N_M^2 |\Omega_1| + \frac{1}{2} g^2 N_M^2 |\Omega_2|, \end{aligned}$$

or

$$\frac{d}{dt} \int_{\Omega_1} |\mathbf{u}|^2 dx \leq \int_{\Omega_1} |\mathbf{u}|^2 dx + g^2 N_M^2 |\Omega_1| + g^2 N_M^2 |\Omega_2|. \tag{8}$$

From (8) it follows that

$$\frac{d}{dt} \left(e^{-t} \int_{\Omega_1} |\mathbf{u}|^2 dx \right) \leq (g^2 N_M^2 |\Omega_1| + g^2 N_M^2 |\Omega_2|) e^{-t}.$$

Upon integration, we can arrive at Lemma 2.2. □

Now we define

$$F_1(t) = \int_{\Omega_1} |\mathbf{u}_t|^2 dx, \quad F_2(t) = \int_0^t \int_{\Omega_2} |\mathbf{v}_t|^2 dx d\eta,$$

$$F_3(t) = \int_{\Omega_1} u_{ij}(u_{ij} + u_{j,i}) dx. \tag{9}$$

Lemma 2.3 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$ and Ω_1, Ω_2 are bounded regions, then*

$$F_3(t) \leq \frac{1}{2\mu} F_1(t) + a_1, \tag{10}$$

where $a_1 = \frac{1}{\mu} A_1 + \frac{1}{2\mu} g^2 N_M^2 |\Omega_1| + \frac{1}{4\mu} g^2 N_M^2 |\Omega_2|$.

Proof Using the divergence theorem, we have

$$\begin{aligned} & \mu \int_{\Omega_1} u_{ij}(u_{ij} + u_{j,i}) dx \\ &= \mu \int_L u_\beta(u_{\beta,3} + u_{3,\beta})n_3^{(1)} dA + 2\mu \int_{\Omega_1} u_3 u_{3,3} n_3^{(1)} dA \\ & \quad - \int_{\Omega_1} u_i [u_{i,t} + \lambda u_j u_{ij} - g_i T + p_{,i}] dx \\ & \leq \mu \int_L u_\beta(u_{\beta,3} + u_{3,\beta})n_3^{(1)} dA - \int_L (p - 2\mu u_{3,3}) u_3 n_3^{(1)} dA \\ & \quad + \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} \int_{\Omega_1} |\mathbf{u}_t|^2 dx - \frac{1}{2} \lambda \int_{\partial\Omega_1} u_3 u_i u_i n_3^{(1)} dA + \frac{1}{2} g^2 N_M^2 |\Omega_1| \\ & \leq \frac{\mu\alpha}{\sqrt{k_1}} \int_L u_\beta u_\beta n_3^{(1)} dA - \int_L q v_3 n_3^{(2)} dA \\ & \quad + \int_{\Omega_1} |\mathbf{u}|^2 dx + \frac{1}{2} \int_{\Omega_1} |\mathbf{u}_t|^2 dx + \frac{1}{2} g^2 N_M^2 |\Omega_1|. \end{aligned} \tag{11}$$

In the light of the condition on L , we compute

$$\int_L q v_3 n_3^{(2)} dA = \int_{\Omega_2} q_i v_i dA = \int_{\Omega_2} (g_i \theta - v_i) v_i dA \leq \frac{1}{4} g^2 N_M^2 |\Omega_2|. \tag{12}$$

Combining (11) and (12), we have Lemma 2.3. □

Lemma 2.4 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$ and Ω_1, Ω_2 are bounded regions. Then*

$$\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \leq A_4(t), \tag{13}$$

where $A_4(t)$ is a positive function which will be defined later.

Proof We firstly compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} F_1(t) &= \mu \int_{\Omega_1} u_{i,t}(u_{ij} + u_{j,i})_{,jt} dx - \int_{\Omega_1} u_{i,t} p_{i,t} dx - \lambda \int_{\Omega_1} u_{i,t} u_{ij} u_{j,t} dx \\ & \quad - \lambda \int_{\Omega_1} u_{i,t} u_{i,jt} u_j dx + \int_{\Omega_1} u_{i,t} g_i T_{,t} dx \\ &= -\mu \int_{\Omega_1} u_{i,jt}(u_{ij,t} + u_{j,it}) dx - \int_L u_{3,t}(p_{,t} - 2\mu u_{3,3,t}) n_3^{(1)} dA \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha\mu}{\sqrt{k_1}} \int_L u_{\beta,t} u_{\beta,t} n_3^{(1)} dA \\
 & - \lambda \int_{\Omega_1} u_{i,t} u_{i,j} u_{j,t} dx - \lambda \int_{\Omega_1} u_{i,t} u_{i,jt} u_j dx + \int_{\Omega_1} u_{i,t} g_i T_{,t} dx \\
 = & -\mu \int_{\Omega_1} u_{i,jt} (u_{i,jt} + u_{j,it}) dx + \int_L u_{3,t} q_{,t} n_3^{(2)} dA - \lambda \int_{\Omega_1} u_{i,t} u_{i,j} u_{j,t} dx \\
 & - \lambda \int_{\Omega_1} u_{i,t} u_{i,jt} u_j dx + \int_{\Omega_1} u_{i,t} g_i T_{,t} dx \\
 = & -\mu \int_{\Omega_1} u_{i,jt} (u_{i,jt} + u_{j,it}) dx + \int_{\Omega_2} v_{i,t} (g_i \theta_{,t} - v_{i,t}) dx - \lambda \int_{\Omega_1} u_{i,t} u_{i,j} u_{j,t} dx \\
 & - \frac{1}{2} \lambda \int_{\Omega_1} u_{i,t} u_{i,t} u_3 n_3^{(1)} dx + \int_{\Omega_1} u_{i,t} g_i T_{,t} dx \\
 \leq & -\mu \int_{\Omega_1} u_{i,jt} (u_{i,jt} + u_{j,it}) dx - \frac{1}{2} \int_{\Omega_2} |\mathbf{v}_t|^2 dx - \lambda \int_{\Omega_1} u_{i,t} u_{i,j} u_{j,t} dx \\
 & + \frac{1}{2} \int_{\Omega_1} |\mathbf{u}_t|^2 dx + \frac{1}{2} g^2 \left(\int_{\Omega_2} \theta_{,t}^2 dx + \int_{\Omega_1} T_{,t}^2 dx \right). \tag{14}
 \end{aligned}$$

We find that the result given in Appendix B of Lin and Payne [10] for $\|\mathbf{u}\|_4^2$

$$\left(\int_{\Omega_1} |\mathbf{u}|^4 dx \right)^{\frac{1}{2}} \leq k \left[\left(\int_{\Omega_1} |\mathbf{u}|^2 dx \right) + \left(\int_{\Omega_1} |\mathbf{u}|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{3}{4}} \right], \quad k > 0. \tag{15}$$

So, we have for an arbitrary constant $\varepsilon_1 > 0$

$$\begin{aligned}
 & - \int_{\Omega_1} u_{i,t} u_{i,j} u_{j,t} dx \\
 & \leq \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}_t|^4 dx \right)^{\frac{1}{2}} \\
 & \leq k \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left[\left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right) + \left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{u}_t|^2 dx \right)^{\frac{3}{4}} \right] \\
 & \leq k \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right) \\
 & \quad + k \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{u}_t|^2 dx \right)^{\frac{3}{4}} \\
 & \leq k \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right) \\
 & \quad + \frac{1}{4} k^4 \varepsilon_1^{-3} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^2 \left(\int_{\Omega_1} |\mathbf{u}_t|^2 dx \right) + \frac{3}{4} \varepsilon_1 \left(\int_{\Omega_1} |\nabla \mathbf{u}_t|^2 dx \right). \tag{16}
 \end{aligned}$$

We notice that one has obtained the following results [13]:

$$\gamma \int_{\Omega_1} |\mathbf{u}|^2 dx \leq \int_{\Omega_1} u_{i,j} (u_{i,j} + u_{j,i}) dx \tag{17}$$

and

$$\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \leq C \int_{\Omega_1} u_{i,j}(u_{i,j} + u_{j,i}) dx, \tag{18}$$

where γ and C are positive constants which have been defined in [13]. Following the methods in [13], we can derive a similar result,

$$\int_{\Omega_1} |\nabla \mathbf{u}_t|^2 dx \leq C \int_{\Omega_1} u_{i,j,t}(u_{i,j,t} + u_{j,i,t}) dx. \tag{19}$$

Combining (14), (16)–(19) and using (7), we obtain

$$\frac{d}{dt} [F_1(t) + F_2(t)] \leq F_1(t) + 2k\sqrt{C}F_3^{\frac{1}{2}}(t)F_1(t) + \frac{1}{2}k^4C^2\varepsilon_1^{-3}F_3^2(t)F_1(t) \tag{20}$$

$$+ g^2 \left(\int_{\Omega_2} \theta_{,t}^2 dx + \int_{\Omega_1} T_{,t}^2 dx \right), \tag{21}$$

where we have chosen $\varepsilon_1 = \frac{4\mu}{3k_5}$.

We now use (1)₂ and (2)₂ with the boundary conditions (3) and (5) and the divergence theorem to obtain

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega_1} T_t^2 dx + \int_{\Omega_2} \theta_t^2 dx \right) \\ &= 2 \int_{\Omega_1} T_t [-u_{i,t}T_{,i} - u_iT_{,it} + k_1\Delta T] dx + 2 \int_{\Omega_2} \theta_t [-v_{i,t}\theta_{,i} - v_i\theta_{,it} + k_2\Delta\theta] dx \\ &= -2k_1 \int_{\Omega_1} |\nabla T_t|^2 dx - 2k_2 \int_{\Omega_2} |\nabla\theta_t|^2 dx + 2 \int_{\Omega_1} TT_{,it}u_{i,t} dx + 2 \int_{\Omega_2} \theta\theta_{,it}v_{i,t} dx \\ &\leq \frac{N_M^2}{2} \left(\frac{1}{k_1}F_1(t) + \frac{1}{k_2} \int_{\Omega_2} |\mathbf{v}_t|^2 dx \right). \end{aligned} \tag{22}$$

By integration of (24) we thus obtain

$$\begin{aligned} & \int_{\Omega_1} T_t^2 dx + \int_{\Omega_2} \theta_t^2 dx \\ &\leq \frac{N_M^2}{2} \left(\frac{1}{k_1} \int_0^t F_1(\eta) d\eta + \frac{1}{k_2}F_2(t) \right) + \int_{\Omega_1} (T_{0,t})^2 dx + \int_{\Omega_2} (\theta_{0,t})^2 dx. \end{aligned} \tag{23}$$

Inserting (25) into (20) and setting

$$\begin{aligned} a_1 &= \frac{A_1k^2}{\varepsilon_2} + 1 + \frac{1}{32\varepsilon_2^4}A_1^4k^8\varepsilon_3^{-3}, & a_2 &= 2k\sqrt{C}, & a_3 &= \frac{1}{2}k^4C^2\varepsilon_1^{-3}, \\ a_4 &= \frac{k^2}{\varepsilon_2} \sqrt[4]{A_1C^3}, & a_5 &= \frac{1}{32\varepsilon_2^4}k^8A_1\varepsilon_4^{-3}C^3, & a_6 &= \frac{N_M^2g^2}{2k_1}, & a_7 &= \frac{N_M^2g^2}{2k_2}, \\ a_8 &= \int_{\Omega_1} (T_{0,t})^2 dx + \int_{\Omega_2} (\theta_{0,t})^2 dx, \end{aligned} \tag{24}$$

we have

$$\begin{aligned} \frac{d}{dt}[F_1(t) + F_2(t)] &\leq a_1F_1(t) + a_2F_3^{\frac{1}{2}}(t)F_1(t) + a_3F_3^2(t)F_1(t) + a_4F_1(t)F_3^{\frac{3}{4}}(t) \\ &\quad + a_5F_1(t)F_3^3(t) + a_6 \int_0^t F_1(\eta) d\eta + a_7F_2(t) + a_8. \end{aligned} \tag{25}$$

Then, using Lemma 2.3 and the Hölder inequality in (27), we get

$$\begin{aligned} \frac{d}{dt}[F_1(t) + F_2(t)] &\leq b_1F_1^4(t) + b_2F_1^3(t) + b_3F_1^2(t) + b_4F_1(t) \\ &\quad + b_5 + a_6 \int_0^t F_1(\eta) d\eta + a_7F_2(t), \end{aligned} \tag{26}$$

for some computable positive constants b_i ($i = 1, 2, 3, 4, 5$). Now, we define

$$F(t) = F_1(t) + F_2(t) + M_0 \int_0^t F_1(\eta) d\eta, \quad M_0 > 0. \tag{27}$$

We have from (32)

$$\frac{d}{dt}F(t) \leq b_1F^4(t) + b_2F^3(t) + b_3F^2(t) + b_6F(t) + b_5, \tag{28}$$

where $b_6 = \frac{1}{b_4+M_0} \max\{1, \frac{a_7}{b_4+M_0}, \frac{a_6}{(b_4+M_0)M_0}\}$. Obviously, we have from (34)

$$\frac{d}{dt}F(t) \leq b_1(F(t) + b_7)^4, \tag{29}$$

where

$$b_7 = \max\left\{ \frac{b_2}{4b_1}, \sqrt{\frac{b_3}{6b_1}}, \sqrt[3]{\frac{b_6}{4b_1}}, \sqrt[4]{\frac{b_5}{b_1}} \right\}. \tag{30}$$

Therefore, we can get the result

$$\int_{\Omega_1} |\mathbf{u}_t|^2 dx + \int_0^t \int_{\Omega_2} |\mathbf{v}_t|^2 dx d\eta + M_0 \int_0^t \int_{\Omega_1} |\mathbf{u}_t|^2 dx d\eta \leq A_2(t), \tag{31}$$

where

$$A_2(t) = \sqrt[3]{\frac{1}{(F(0) + b_7)^{-3} - 3b_1t}}, \quad F(0) = \int_{\Omega_1} |\mathbf{f}_t|^2 dx. \tag{32}$$

In view of (9), Lemma 3 and (18), we also have

$$\int_{\Omega_1} u_{i,j}(u_{i,j} + u_{j,i}) dx \leq A_3(t) \tag{33}$$

and

$$\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \leq A_4(t), \tag{34}$$

where

$$A_3(t) = \frac{1}{2\mu}A_2(t) + a_9, \quad A_4(t) = CA_3(t). \tag{35}$$

Combining (13), (15) and (34), we may get the following lemma. □

Lemma 2.5 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$ and Ω_1, Ω_2 are bounded regions. Then*

$$\left(\int_{\Omega_1} |u|^4 dx \right)^{\frac{1}{2}} \leq k[A_1 + A_1^{\frac{1}{4}}A_4^{\frac{3}{4}}(t)] \doteq A_5(t). \tag{36}$$

3 Continuous dependence on λ

In this section, we want to establish the continuous dependence on g_i . Let (u_i, T, p) and (v_i, θ, q) be solutions of (1)–(5) with $\lambda = \lambda^{(1)}$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) be solutions of (1)–(5) with $\lambda = \lambda^{(2)}$, respectively.

We define

$$w_i = u_i - u_i^*, \quad S = T - T^*, \quad \pi = p - p^*, \quad \tilde{\lambda} = \lambda^{(1)} - \lambda^{(2)}, \tag{37}$$

and

$$w_i^m = v_i - v_i^*, \quad S^m = \theta - \theta^*, \quad \pi^m = q - q^*. \tag{38}$$

Then (w_i, T, π) satisfy the following equations:

$$\begin{aligned} \frac{\partial w_i}{\partial t} - \mu \Delta w_i + \tilde{\lambda} u_{i,j} u_j + \lambda^{(2)} u_j w_{i,j} + \lambda^{(2)} u_{i,j}^* w_j - g_i S + \pi_{,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \\ \frac{\partial S}{\partial t} + w_i T_{,i} + u_i^* S_{,i} &= k_1 \Delta S, \quad \text{in } \Omega_1 \times [0, \tau], \\ w_{i,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \end{aligned} \tag{39}$$

and (w_i^m, S^m, π^m) satisfy the equations

$$\begin{aligned} w_i^m - g_i S^m + \pi_{,i}^m &= 0, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial S^m}{\partial t} + w_i^m \theta_{,i} + v_i^* S_{,i}^m &= k_2 \Delta S^m, \quad \text{in } \Omega_2 \times [0, \tau], \\ w_{i,i}^m &= 0, \quad \text{in } \Omega_2 \times [0, \tau]. \end{aligned} \tag{40}$$

The boundary conditions are

$$\begin{aligned} w_i &= 0; \quad S = 0, \quad \text{on } \Gamma_1 \times [0, \tau], \\ w_i^m n_i &= 0, \quad S^m = 0, \quad \text{on } \Gamma_2 \times [0, \tau]. \end{aligned} \tag{41}$$

The initial conditions can be written as

$$w_i(x, 0) = 0, \quad S(x, 0) = 0, \quad \text{in } \Omega_1, \quad S^m(x, 0) = 0, \quad \text{in } \Omega_2. \tag{42}$$

The interface L conditions are

$$\begin{aligned}
 w_3 &= w_3^m, & S &= S^m, & k_1 S_{3,3} &= k_2 S_{3,3}^m, \\
 \pi^m &= \pi - 2\mu w_{3,3}, & w_{\beta,3} + w_{3,\beta} &= \frac{\alpha}{\sqrt{k_1}} w_\beta.
 \end{aligned}
 \tag{43}$$

We first give some useful lemmas.

Lemma 3.1 *Let (u_i, T, p) and (v_i, θ, q) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to $\lambda^{(1)}$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to $\lambda^{(2)}$. Then for any $t > 0$ the differences of velocities satisfy*

$$\begin{aligned}
 &\frac{d}{dt} \int_{\Omega_1} |\mathbf{w}|^2 dx + \int_{\Omega_2} |\mathbf{w}^m|^2 dx \\
 &\leq c_1(t) \int_{\Omega_1} |\mathbf{w}|^2 dx + 2g^2 \left[\int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \right] + 2(\tilde{\lambda})^2 k A_4(t) A_5(t),
 \end{aligned}$$

where $c_1(t)$ is a positive function which depends on t .

Proof We begin with the identity

$$\int_{\Omega_1} \left[\frac{\partial w_i}{\partial t} - \mu \Delta w_i + \tilde{\lambda} u_{i,j} u_j + \lambda^{(2)} u_j w_{i,j} + \lambda^{(2)} u_{i,j}^* w_j - g_i S + \pi_{,i} \right] w_i dx = 0.
 \tag{44}$$

From (44) it follows that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |\mathbf{w}|^2 dx &= \mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx - \tilde{\lambda} \int_{\Omega_1} u_{i,j} u_j w_i dx - \lambda^{(2)} \int_{\Omega_1} u_j w_{i,j} w_i dx \\
 &\quad - \lambda^{(2)} \int_{\Omega_1} u_{i,j}^* w_j w_i dx - \int_{\Omega_1} \pi_{,i} w_i dx - \int_{\Omega_1} g_i S w_i dx \\
 &\doteq I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
 \end{aligned}
 \tag{45}$$

We now deal with I_1 and I_5 . Using the divergence theorem, we have

$$\begin{aligned}
 I_1 + I_5 &= -\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + \mu \int_{\Omega_1} (w_{\beta,3} + w_{3,\beta}) w_\beta n_3^{(1)} dA \\
 &\quad - \int_{\Omega_1} (\pi - 2\mu u_{3,3}) w_i n_i^{(1)} dx \\
 &= -\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + \frac{\alpha \mu}{\sqrt{k_1}} \int_L w_\beta w_\beta n_3^{(1)} dA + \int_L \pi^m w_i^m n_i^{(2)} dA \\
 &\leq -\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + \int_{\Omega_2} (-w_i^m + \tilde{g}_i \theta + g_i^{(2)} S^m) w_i^m dx \\
 &\leq -\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx - \frac{1}{2} \int_{\Omega_2} |\mathbf{w}^m|^2 dx + g^2 \int_{\Omega_2} (S^m)^2 dx.
 \end{aligned}
 \tag{46}$$

Using the Hölder inequality, (15), Lemma 2.4, Lemma 2.5 and the Young inequality with $\delta_1 > 0$, we have

$$\begin{aligned}
 I_2 &\leq |\tilde{\lambda}| \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}|^4 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\mathbf{w}|^4 dx \right)^{\frac{1}{4}} \\
 &\leq |\tilde{\lambda}| \sqrt{kA_4(t)A_5(t)} \left[\left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right) + \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{3}{4}} \right]^{\frac{1}{2}} \\
 &\leq (\tilde{\lambda})^2 kA_4(t)A_5(t) + \left[1 + \frac{1}{4}\delta_1^{-3} \right] \int_{\Omega_1} |\mathbf{w}|^2 dx + \frac{3}{4}\delta_1 \int_{\Omega_1} |\nabla \mathbf{w}|^2 dx. \tag{47}
 \end{aligned}$$

Following a similar procedure to deriving I_3 , we obtain

$$\begin{aligned}
 I_3 &\leq \lambda^{(2)} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}|^4 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\mathbf{w}|^4 dx \right)^{\frac{1}{4}} \\
 &\leq \lambda^{(2)} \sqrt{kA_5(t)} \left\{ \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right) \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right) \right. \\
 &\quad \left. + \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{7}{4}} \right\}^{\frac{1}{2}} \\
 &\leq \lambda^{(2)} \sqrt{kA_5(t)} \left\{ \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{1}{2}} \right. \\
 &\quad \left. + \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right)^{\frac{1}{8}} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{7}{8}} \right\} \\
 &\leq \left[\frac{(\lambda^{(2)})^2}{2\delta_2} kA_5(t) + \frac{(\lambda^{(2)})^8}{8\delta_3^7} (kA_5(t))^4 \right] \int_{\Omega_1} |\mathbf{w}|^2 dx \\
 &\quad + \left(\frac{1}{2}\delta_2 + \frac{7}{8}\delta_3 \right) \int_{\Omega_1} |\nabla \mathbf{w}|^2 dx, \tag{48}
 \end{aligned}$$

where δ_2, δ_3 are positive constants to be determined later. Similarly, we have

$$\begin{aligned}
 I_4 &\leq \lambda^{(2)} \left(\int_{\Omega_1} |\nabla \mathbf{u}^*|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{w}|^4 dx \right)^{\frac{1}{2}} \\
 &\leq \lambda^{(2)} \sqrt{A_4(t)} \left[\left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right) + \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right)^{\frac{1}{4}} \left(\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \right)^{\frac{3}{4}} \right] \\
 &\leq \left[\lambda^{(2)} \sqrt{A_4(t)} + \frac{1}{4\delta_4} (\lambda^{(2)})^4 A_4^2(t) \right] \left(\int_{\Omega_1} |\mathbf{w}|^2 dx \right) + \frac{3}{4}\delta_4 \int_{\Omega_1} |\nabla \mathbf{w}|^2 dx. \tag{49}
 \end{aligned}$$

We note that I_6 can be bounded,

$$I_6 \leq g^2 \int_{\Omega_1} S^2 dx + \frac{1}{4} \int_{\Omega_1} |\mathbf{w}|^2 dx. \tag{50}$$

Similar to (19), we also have

$$\int_{\Omega_1} |\nabla \mathbf{w}|^2 dx \leq k_5 \int_{\Omega_1} w_{ij}(u_{ij} + w_{j,i}) dx. \tag{51}$$

We define the functions $F_1(t)$ and $F_2(t)$ by

$$F_1(t) = \int_{\Omega_1} |\mathbf{w}|^2 dx, \quad F_2(t) = \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta. \tag{52}$$

Inserting (45)–(51) into (44) and using (52), we have

$$\begin{aligned} \frac{d}{dt}[F_1(t) + F_2(t)] \leq & -2 \left[\mu - \frac{3}{4}(\delta_1 + \delta_4)k_5 - \left(\frac{1}{2}\delta_2 + \frac{7}{8}\delta_3 \right)k_5 \right] \int_{\Omega_1} (w_{ij} + w_{j,i})w_{ij} dx \\ & + 2 \left[\frac{1}{4}\delta_1^{-3} + \frac{(\lambda^{(2)})^2}{2\delta_2}kA_5(t) + \frac{(\lambda^{(2)})^8}{8\delta_3^7}(kA_5(t))^4 + \lambda^{(2)}\sqrt{A_4(t)} \right] \\ & + \frac{1}{4\delta_4}(\lambda^{(2)})^4 A_4^2(t) + \frac{5}{4} \int_{\Omega_1} |\mathbf{w}|^2 dx + 2g^2 \int_{\Omega_1} S^2 dx \\ & + 2g^2 \int_{\Omega_2} (S^m)^2 dx + 2(\tilde{\lambda})^2 kA_4(t)A_5(t). \end{aligned} \tag{53}$$

We choose $\delta_1, \delta_2, \delta_3$ and δ_4 small enough such that

$$\frac{3}{4}(\delta_1 + \delta_4)k_5 + \left(\frac{1}{2}\delta_2 + \frac{7}{8}\delta_3 \right)k_5 = \mu.$$

Letting

$$c_1(t) = 2 \left[\frac{1}{4}\delta_1^{-3} + \frac{(\lambda^{(2)})^2}{2\delta_2}kA_5(t) + \frac{(\lambda^{(2)})^8}{8\delta_3^7}(kA_5(t))^4 + \lambda^{(2)}\sqrt{A_4(t)} + \frac{1}{4\delta_4}(\lambda^{(2)})^4 A_4^2(t) + \frac{5}{4} \right],$$

we can get Lemma 3.1. □

Lemma 3.2 *Let (u_i, T, p) and (v_i, θ, q) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to $\lambda^{(1)}$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to $\lambda^{(2)}$. Then for any $t > 0$ we have*

$$\int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \leq \frac{1}{2k_1} N_M^2 \int_0^t F_1(\eta) d\eta + \frac{1}{2k_2} N_M^2 F_2(t).$$

Proof We multiply (39)₂ and (40)₂ by S and S^m , respectively, and integrate by parts to find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \right] \\ & = -k_1 \int_{\Omega_1} |\nabla S|^2 dx - k_2 \int_{\Omega_2} |\nabla S^m|^2 dx \\ & \quad + \int_{\Omega_1} w_i T S_{,i} dx + \int_{\Omega_2} w_i^m \theta S_{,i}^m dx \\ & \leq \frac{1}{4k_1} N_M^2 \int_{\Omega_1} |\mathbf{w}|^2 dx + \frac{1}{4k_2} N_M^2 \int_{\Omega_2} |\mathbf{w}^m|^2 dx. \end{aligned} \tag{54}$$

Integrating (54) from 0 to t one may deduce

$$\begin{aligned} & \int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \\ & \leq \frac{1}{2k_1} N_M^2 \int_0^t \int_{\Omega_1} |\mathbf{w}|^2 dx d\eta + \frac{1}{2k_2} N_M^2 \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta. \end{aligned} \tag{55}$$

Combining (52) and (55), we may obtain Lemma 3.2. □

Now, we use Lemmas 3.1 and 3.2 to obtain

$$\begin{aligned} \frac{d}{dt} [F_1(t) + F_2(t)] & \leq c_1(t)F_1(t) + \frac{g^2 N_M^2}{k_1} \int_0^t F_1(\eta) d\eta \\ & \quad + \frac{g^2 N_M^2}{k_2} F_2(t) + 2(\tilde{\lambda})^2 k A_4(t) A_5(t). \end{aligned} \tag{56}$$

Setting

$$F_3(t) = F_1(t) + F_2(t) + \frac{k_2}{k_1} \int_0^t F_1(\eta) d\eta, \tag{57}$$

we obtain

$$\frac{d}{dt} F_3(t) \leq c_2(t)F_3(t) + 2(\tilde{\lambda})^2 k A_4(t) A_5(t), \tag{58}$$

where

$$c_2(t) = \max \left\{ c_1(t) + \frac{k_2}{k_1}, \frac{g^2 N_M^2}{k_2} \right\}. \tag{59}$$

Thus after integration we may derive from (58) the estimate

$$F_3(t) \leq 2(\tilde{\lambda})^2 k \int_0^t A_4(\eta) A_5(\eta) e^{\int_s^t c_2(\eta) d\eta} ds. \tag{60}$$

Combining (57), Lemma 3.2 and (60), we have the following theorem.

Theorem 3.1 *Let (u_i, T, p) and (v_i, θ, q) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to $\lambda^{(1)}$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to $\lambda^{(2)}$. Then for any $t > 0$ we have*

$$(u_i, T, p) \rightarrow (u_i^*, T^*, p^*), \quad (v_i, \theta, q) \rightarrow (v_i^*, \theta^*, q^*), \tag{61}$$

as $\lambda^{(1)} \rightarrow \lambda^{(2)}$. The differences of velocities satisfy

$$\begin{aligned} & \int_{\Omega_1} |\mathbf{w}|^2 dx + \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta + \frac{k_2}{k_1} \int_0^t \int_{\Omega_1} |\mathbf{w}|^2 dx d\eta \\ & \leq 2(\tilde{\lambda})^2 k \int_0^t A_4(\eta) A_5(\eta) e^{\int_s^t c_2(\eta) d\eta} ds, \end{aligned} \tag{62}$$

where $\mathbf{w}, \mathbf{w}^m, S, S^m, \tilde{g}$ have been defined in (37) and (38).

Furthermore, there are two positive $c_2, c_3(t)$, such that

$$\int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \leq (\tilde{\lambda})^2 k \frac{N_M^2}{k_2} \int_0^t A_4(\eta) A_5(\eta) e^{\int_s^t c_2(\eta) d\eta} ds. \tag{63}$$

Inequalities (62) and (63) demonstrate the continuous dependence on λ in the indicated measure.

4 Continuous dependence on the interface coefficient

In this section, we want to establish the continuous dependence on the interface coefficient α . Let (u_i, T, p) and (v_i, θ, q) be solutions of (1)–(5) with $\alpha = \alpha_1$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) be solutions of (1)–(5) with $\alpha = \alpha_2$, respectively.

We define

$$w_i = u_i - u_i^*, \quad S = T - T^*, \quad \pi = p - p^*, \quad \sigma = \alpha_1 - \alpha_2, \tag{64}$$

and

$$w_i^m = v_i - v_i^*, \quad S^m = \theta - \theta^*, \quad \pi^m = q - q^*. \tag{65}$$

Then (w_i, S, π) satisfy the following equation:

$$\begin{aligned} \frac{\partial w_i}{\partial t} - \mu \Delta w_i + w_j u_{ij} + u_j^* w_{i,j} - g_i S + \pi_{,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \\ \frac{\partial S}{\partial t} + w_i T_{,i} + u_i^* S_{,i} &= k_1 \Delta S, \quad \text{in } \Omega_1 \times [0, \tau], \\ w_{i,i} &= 0, \quad \text{in } \Omega_1 \times [0, \tau], \end{aligned} \tag{66}$$

and (w_i^m, S^m, π^m) satisfy equations

$$\begin{aligned} w_i^m - g_i S^m + \pi_{,i}^m &= 0, \quad \text{in } \Omega_2 \times [0, \tau], \\ \frac{\partial S^m}{\partial t} + w_i^m \theta_{,i} + v_i^* S_{,i}^m &= k_2 \Delta S^m, \quad \text{in } \Omega_2 \times [0, \tau], \\ w_{i,i}^m &= 0, \quad \text{in } \Omega_2 \times [0, \tau]. \end{aligned} \tag{67}$$

The boundary conditions are

$$\begin{aligned} w_i = 0; \quad S = 0, \quad \text{on } \Gamma_1 \times [0, \tau], \\ w_i^m n_i = 0, \quad S^m = 0, \quad \text{on } \Gamma_2 \times [0, \tau]. \end{aligned} \tag{68}$$

The initial conditions can be written as

$$w_i(x, 0) = 0, \quad S(x, 0) = 0, \quad \text{in } \Omega_1, \quad S^m(x, 0) = 0, \quad \text{in } \Omega_2. \tag{69}$$

The interface L conditions are

$$w_3 = w_3^m, \quad S = S^m, \quad k_1 S_{,3} = k_2 S_{,3}^m,$$

$$\pi^m = \pi - 2\mu w_{3,3}, \quad w_{\beta,3} + w_{3,\beta} = \frac{\sigma}{\sqrt{k_1}} u_\beta + \frac{\alpha_2}{\sqrt{k_1}} w_\beta. \tag{70}$$

We give some useful lemmas.

Lemma 4.1 *If $T_0, \theta_0, G, \tilde{G} \in L^\infty$ and Ω_1, Ω_2 are bounded regions, then*

$$\int_L u_\beta u_\beta \, dA \leq A_6(t),$$

where $A_6(t)$ is a positive function which depends on t .

Proof We use Eqs. (1), (2) to derive

$$\begin{aligned} \int_{\Omega_1} u_i u_{i,t} \, dx &= \mu \int_{\Omega_1} (u_{i,j} + u_{j,i})_{,j} u_i \, dx - \int_{\Omega_1} u_j u_{i,j} u_i \, dx + \int_{\Omega_1} g_i T u_i \, dx - \int_{\Omega_1} p_{,i} u_i \, dx \\ &= -\mu \int_{\Omega_1} (u_{i,j} + u_{j,i}) u_{i,j} \, dx + \mu \int_L (u_{\beta,3} + u_{3,\beta}) u_\beta n_3^{(1)} \, dA \\ &\quad - \int_L (p - 2\mu u_{3,3}) u_i n_i^{(1)} \, dA - \int_{\Omega_1} u_j u_{i,j} u_i \, dx + \int_{\Omega_1} g_i T u_i \, dx. \end{aligned} \tag{71}$$

Using the interface conditions, we obtain from (71)

$$\begin{aligned} \frac{\mu\alpha_1}{\sqrt{k_1}} \int_L u_\beta u_\beta \, dA &= -\mu \int_{\Omega_1} (u_{i,j} + u_{j,i}) u_{i,j} \, dx - \int_{\Omega_1} u_i u_{i,t} \, dx \\ &\quad + \int_L q v_i n_i^{(2)} \, dA - \int_{\Omega_1} u_j u_{i,j} u_i \, dx + \int_{\Omega_1} g_i T u_i \, dx \\ &= - \int_{\Omega_1} u_i u_{i,t} \, dx + \int_{\Omega_2} v_i (g_i \theta - v_i) \, dx \\ &\quad - \int_{\Omega_1} u_j u_{i,j} u_i \, dx + \int_{\Omega_1} g_i T u_i \, dx. \end{aligned} \tag{72}$$

By using the Hölder inequality, the AG mean inequality, (31), (34), Lemma 2.2 and Lemma 2.5, we have from (72)

$$\begin{aligned} &\frac{\mu\alpha_1}{\sqrt{k_1}} \int_L u_\beta u_\beta \, dA \\ &\leq \left(\int_{\Omega_1} |\mathbf{u}|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}_t|^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{4} g^2 N_M^2 |\Omega_2| \\ &\quad + \left(\int_{\Omega_1} |\mathbf{u}|^4 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\nabla \mathbf{u}|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\Omega_1} g_i g_i T^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_1} |\mathbf{u}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \sqrt{A_1 A_2(t)} + \frac{1}{4} g^2 N_M^2 |\Omega_2| + A_5(t) \sqrt{A_4(t)} + \sqrt{g^2 N_M^2 |\Omega_1| A_1}. \end{aligned} \tag{73}$$

Therefore

$$\int_L u_\beta u_\beta \, dA \leq A_6(t), \tag{74}$$

where

$$A_6(t) = \frac{\sqrt{k_1}}{\mu\alpha_1} \left\{ \sqrt{A_1 A_2(t)} + \frac{1}{4} g^2 N_M^2 |\Omega_2| + A_5(t) \sqrt{A_4(t)} + \sqrt{g^2 N_M^2 |\Omega_1| A_1} \right\}. \tag{75}$$

□

Lemma 4.2 *Let (u_i, T, p) and (v_i, θ, q) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to $\lambda^{(1)}$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to $\lambda^{(2)}$. Then for any $t > 0$ we have*

$$\begin{aligned} & \int_{\Omega_1} |\mathbf{w}|^2 dx + \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta + \frac{k_2}{k_1} \int_0^t \int_{\Omega_1} |\mathbf{w}|^2 dx d\eta \\ & \leq \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_0^t e^{\int_s^t c_4(\eta) d\eta} \int_L u_\beta u_\beta dA ds. \end{aligned} \tag{76}$$

Proof We begin with the identity

$$\int_{\Omega_1} \left[\frac{\partial w_i}{\partial t} - \mu \Delta w_i + \lambda w_j u_{i,j} + \lambda u_j^* w_{i,j} - g_i S + \pi_{,i} \right] w_i dx = 0. \tag{77}$$

From (77) it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_1} |\mathbf{w}|^2 dx &= 2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i})_j w_i dx - 2 \int_{\Omega_1} \pi_{,i} w_i dx - 2\lambda \int_{\Omega_1} w_j u_{i,j} w_i dx \\ &\quad - 2\lambda \int_{\Omega_1} u_j^* w_{i,j} w_i dx + 2 \int_{\Omega_1} g_i S w_i dx. \end{aligned} \tag{78}$$

Integrating by parts as in Sect. 3 now leads to

$$\begin{aligned} & 2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i})_j w_i dx - 2 \int_{\Omega_1} \pi_{,i} w_i dx \\ &= -2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + 2\mu \int_{\Omega_1} (w_{\beta,3} + w_{3,\beta}) w_\beta n_3^{(1)} dA \\ &\quad - 2 \int_{\Omega_1} (\pi - 2\mu u_{3,3}) w_i n_i^{(1)} dx \\ &= -2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + \frac{2\alpha_1 \mu}{\sqrt{k_1}} \int_L w_\beta w_\beta n_3^{(1)} dA \\ &\quad + \frac{2\sigma \mu}{\sqrt{k_1}} \int_L u_\beta w_\beta n_3^{(1)} dA + 2 \int_L \pi^m w_i^m n_i^{(2)} dA \\ &\leq -2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx + 2 \int_{\Omega_2} (-w_i^m + g_i S^m) w_i^m dx \\ &\quad + \frac{2\alpha_1 \mu}{\sqrt{k_1}} \int_L w_\beta w_\beta n_3^{(1)} dA + \frac{2\sigma \mu}{\sqrt{k_1}} \int_L u_\beta w_\beta n_3^{(1)} dA \\ &\leq -2\mu \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx - \int_{\Omega_2} |\mathbf{w}^m|^2 dx + 2g^2 \int_{\Omega_2} (S^m)^2 dx \\ &\quad + \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_L u_\beta u_\beta dA. \end{aligned} \tag{79}$$

Inserting (79), (48), (49), and (50) into (78) and using (51) and (52), we have

$$\begin{aligned} & \frac{d}{dt}[F_1(t) + F_2(t)] \\ & \leq -2 \left[\mu - \frac{3}{4} \delta_4 k_5 - \left(\frac{1}{2} \delta_2 + \frac{7}{8} \delta_3 \right) k_5 \right] \int_{\Omega_1} (w_{i,j} + w_{j,i}) w_{i,j} dx \\ & \quad + 2 \left[\frac{\lambda^2}{2\delta_2} k A_5(t) + \frac{\lambda^8}{8\delta_3^7} (k A_5(t))^4 + \lambda \sqrt{A_4(t)} + \frac{1}{4\delta_4} \lambda^4 A_4^2(t) + \frac{1}{4} \right] \int_{\Omega_1} |\mathbf{w}|^2 dx \\ & \quad + 2g^2 \int_{\Omega_1} S^2 dx + 2g^2 \int_{\Omega_2} (S^m)^2 dx + \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_L u_\beta u_\beta dA. \end{aligned} \tag{80}$$

Choosing $\delta_2, \delta_3, \delta_4$ such that

$$\frac{3}{4} \delta_4 k_5 + \left(\frac{1}{2} \delta_2 + \frac{7}{8} \delta_3 \right) k_5 = \mu,$$

and using Lemma 7 in (80), we have

$$\begin{aligned} \frac{d}{dt}[F_1(t) + F_2(t)] & \leq c_3(t) \int_{\Omega_1} |\mathbf{w}|^2 dx + \frac{g^2}{k_1} N_M^2 \int_0^t \int_{\Omega_1} |\mathbf{w}|^2 dx d\eta \\ & \quad + \frac{g^2}{k_2} N_M^2 \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta + \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_L u_\beta u_\beta dA, \end{aligned} \tag{81}$$

where

$$c_3(t) = 2 \left[\frac{\lambda^2}{2\delta_2} k A_5(t) + \frac{\lambda^8}{8\delta_3^7} (k A_5(t))^4 + \lambda \sqrt{A_4(t)} + \frac{1}{4\delta_4} \lambda^4 A_4^2(t) + \frac{1}{4} \right]. \tag{82}$$

In view of (52), we have from (81)

$$\begin{aligned} \frac{d}{dt}[F_1(t) + F_2(t)] & \leq c_3(t) F_1(t) + \frac{g^2}{k_1} N_M^2 \int_0^t F_1(\eta) d\eta \\ & \quad + \frac{g^2}{k_2} N_M^2 F_2(t) + \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_L u_\beta u_\beta dA. \end{aligned} \tag{83}$$

Defining $F_3(t)$ as in (59), we have from (83)

$$\frac{d}{dt} F_3(t) \leq c_4(t) F_3(t) + \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_L u_\beta u_\beta dA, \tag{84}$$

where

$$c_4(t) = \max \left\{ c_3(t) + \frac{k_2}{k_1}, \frac{g^2 N_M^2}{k_2} \right\}. \tag{85}$$

Thus after integration we may derive Lemma 4.2. □

Combining Lemma 8 and Lemma 9, we have the following theorem.

Theorem 4.1 *Let (u_i, T, p) and (v_i, θ, q) be the classical solutions to the initial-boundary value problem (1)–(5) corresponding to $\alpha = \alpha_1$, and (u_i^*, T^*, p^*) and (v_i^*, θ^*, q^*) also be the classical solutions to the initial-boundary value problem (1)–(5) but corresponding to $\alpha = \alpha_2$. Then for any $t > 0$ we have*

$$(u_i, T, p) \rightarrow (u_i^*, T^*, p^*), \quad (v_i, \theta, q) \rightarrow (v_i^*, \theta^*, q^*), \tag{86}$$

as $\alpha_1 \rightarrow \alpha_2$. The differences of velocities satisfy

$$\begin{aligned} & \int_{\Omega_1} |\mathbf{w}|^2 dx + \int_0^t \int_{\Omega_2} |\mathbf{w}^m|^2 dx d\eta + \frac{k_2}{k_1} \int_0^t \int_{\Omega_1} |\mathbf{w}|^2 dx d\eta \\ & \leq \frac{\sigma^2 \mu}{2\sqrt{k_1}} \int_0^t e^{\int_s^t c_4(\eta) d\eta} A_6(s) ds, \end{aligned} \tag{87}$$

where $\mathbf{w}, \mathbf{w}^m, S, S^m, \sigma$ have been defined in (64) and (65), and c_5, c_6 are positive constants which will be defined later.

Moreover, the differences of the temperatures satisfy

$$\int_{\Omega_1} S^2 dx + \int_{\Omega_2} (S^m)^2 dx \leq \frac{\sigma^2 N_M^2 \mu}{4k_2 \sqrt{k_1}} \int_0^t A_6(s) e^{\int_s^t c_4(\eta) d\eta} ds. \tag{88}$$

Inequalities (87) and (88) are a priori bounds demonstrating the continuous dependence of the solution on the interface coefficient α .

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Authors’ contributions

YL proposed the main idea of this paper and wrote the whole paper. ZS prepared the manuscript initially. LC performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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