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The first initial-boundary value problem of parabolic Monge–Ampère equations outside a bowl-shaped domain

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Abstract

In this paper, we study the parabolic Monge–Ampère equations $-u_t \det(D^2u) = g$ outside a bowl-shaped domain with g being the perturbation of $g_0(|x|)$ at infinity. Under the weaker conditions compared with the problem outside a cylinder, we obtain the existence and uniqueness of viscosity solutions with asymptotic behavior for the first initial-boundary value problem by using the Perron method.

Keywords: Parabolic Monge–Ampère equations; Initial-boundary value problem; Bowl-shaped domain; Perron method; Asymptotic behavior

1 Introduction

Monge–Ampère equation is a class of fully nonlinear partial differential equations. The Dirichlet problem of elliptic Monge–Ampère equations on exterior domains is closely related to a celebrated result of Jörgens ($n = 2$ [1]), Calabi ($n \leq 5$ [2]), and Pogorelov ($n \geq 2$ [3]). It asserts that any classical convex solution of elliptic Monge–Ampère equation

$$\det D^2u = 1 \quad \text{in } \mathbb{R}^n$$

must be a quadratic polynomial. A simpler and more analytical proof was given by Cheng and Yau [4]. Caffarelli [5] proved that this result holds true for viscosity solutions. Then the result was extended to the Dirichlet problem of elliptic Monge–Ampère equation on exterior domains by Caffarelli and Li in [6] where the existence and uniqueness of the viscosity solutions were proved by the Perron method. Other results for elliptic Monge–Ampère equations on exterior domains can be referred to [7–11] and the references therein. The blow-up solutions to the Monge–Ampère equation and convex solutions of the Monge–Ampère systems can be referred to [12, 13].

The Jörgens–Calabi–Pogorelov theorem for parabolic Monge–Ampère equation

$$-u_t \det D^2u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0] \tag{1.1}$$

was established by Gutiérrez and Huang [14]. It is stated that if $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ is a parabolically convex solution of (1.1) such that, for some positive constants d_1, d_2,

$-d_1 \le u_t(x,t) \le -d_2$, $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$, then u must be the form $u(x,t) = Ct + P(x)$ with $C < 0$ and P being a convex quadratic polynomial. Then the Jörgens–Calabi–Pogorelov parabolic theorem was generalized to the equation $u_t = \rho(\log\det D^2u)$ with $\rho = \rho(z) \in C^2(\mathbb{R})$ by Xiong and Bao [15], the equation $u_t - \log\det D^2u = f$ by Wang and Bao [16], and the equation $-u_t \det D^2u = f$ by Zhang, Bao, and Wang [17]. In [18], the author, using the Perron method, studied the first initial-boundary value problem for parabolic Monge–Ampère equation outside a cylinder

$$-u_t \det D^2u = g \quad \text{in } \left(\mathbb{R}^n \backslash \overline{\Omega}\right) \times (0, \tilde{T}], \tag{1.2}$$

$$u = \phi(x,t) \quad \text{on } \partial\Omega \times [0, \tilde{T}], \tag{1.3}$$

$$u = \psi(x) \quad \text{in } \left(\mathbb{R}^n \backslash \Omega\right) \times \{t = 0\}, \tag{1.4}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $u_t = \partial u/\partial t$, D^2u is the Hessian matrix of u with respect to the spatial variables x, $\tilde{T} > 0$ and Ω is a smooth, bounded, and strictly convex open subset in $\mathbb{R}^n$, $g = g(x,t) = 1 + O(|x|^{-\alpha})$, $|x| \to \infty$ with $\alpha > 2$, $\phi(x,t)$ and $\psi(x)$ are given continuous functions satisfying the compatibility condition. The existence and uniqueness of viscosity solutions with asymptotic behavior at infinity to (1.2)–(1.4) were obtained. The first initial-boundary value problems of parabolic Monge–Ampère equations $u_t = \rho(\log\det D^2u)$ and $u_t - \log\det D^2u = f$ on exterior domains were also studied in [19–21]. Recently, the author and Bao [22] obtained the existence of entire solutions of the Cauchy problem for parabolic Monge–Ampère equations $-u_t \det D^2u = g$ with $g = g_0(|x|) + O(|x|^{-\alpha})$ at infinity.

This kind of first initial-boundary value problem (1.2)–(1.4) on exterior domains is motivated by the interior problem of parabolic Monge–Ampère equations [23, 24]

$$\begin{cases} -u_t \det D^2u = g(x,t) & \text{in } \Omega \times (0, \tilde{T}], \\ u = \phi(x,t) & \text{on } (\partial\Omega \times [0, \tilde{T}]) \cup (\Omega \times \{t = 0\}). \end{cases}$$

In this paper, we study the parabolic Monge–Ampère equations $-u_t \det D^2u = g(x,t)$ with $g = g_0(|x|) + O(|x|^{-\alpha})$ (see the following details for g_0 and α) outside a bowl-shaped domain.

Let $D \subset \mathbb{R}^{n+1}$ be a bounded domain and $t \in \mathbb{R}$, define

$$D(t) = \left\{x : (x,t) \in D\right\}.$$

Set $t_0 = \inf\{t : D(t) \neq \emptyset\}$. The parabolic boundary of D is defined by

$$\partial_p D = \left(\overline{D(t_0)} \times \{t_0\}\right) \cup \bigcup_{t\in\mathbb{R}} \left(\partial D(t) \times \{t\}\right),$$

where $\overline{D}$ denotes the closure of D and $\partial D(t)$ denotes the boundary of $D(t)$. The side boundary of D is defined by $SD = \bigcup_{t\in\mathbb{R}}(\partial D(t) \times \{t\})$. The set $D \subset \mathbb{R}^{n+1}$ is called a bowl-shaped domain if for each t, $D(t)$ is convex and for $t_1 \le t_2$, $D(t_1) \subset D(t_2)$. One can also refer to [14].

Let D be a bowl-shaped domain and $T = \sup\{t : D(t) \neq \emptyset\}$, $\mathbb{R}^{n+1}_T = \mathbb{R}^n \times (t_0, T]$. Then $SD = \partial D(t) \times [t_0, T]$. In the following, we shall abuse the notations SD and $\partial D(t) \times [t_0, T]$.

We shall consider the first initial-boundary value problem of parabolic Monge–Ampère equations

$$-u_t \det D^2 u = g(x,t) \quad \text{in } \mathbb{R}^{n+1}_T \backslash \overline{D}, \tag{1.5}$$

$$u = \phi(x,t) \quad \text{on } \partial D(t) \times [t_0, T], \tag{1.6}$$

$$u = \psi(x) \quad \text{in } \big(\mathbb{R}^n \backslash D(t_0)\big) \times \{t = t_0\}. \tag{1.7}$$

Let $\tilde{D} \subset \mathbb{R}^{n+1}$, if for $(x,t) \in \tilde{D}$ a function u is $2k$th continuous differentiable with spatial variables $x \in \mathbb{R}^n$ and kth continuous differentiable with time variable t, we say that $u \in C^{2k,k}(\tilde{D})$. Let $USC(\tilde{D})$ and $LSC(\tilde{D})$ be the sets of upper and lower semicontinuous real-valued functions on $\tilde{D}$, respectively. We say that a function $u \in USC(\tilde{D})$ (or $LSC(\tilde{D})$) is parabolically convex if u is convex in x and nonincreasing in t. The following definition of viscosity solutions is referred to [25].

Definition 1.1 Suppose that $u \in USC(\mathbb{R}^{n+1}_T \backslash \overline{D})$ ($LSC(\mathbb{R}^{n+1}_T \backslash \overline{D})$) is locally parabolically convex. We say that u is a viscosity subsolution (supersolution) of (1.5) if for any function $\varphi \in C^{2,1}(\mathcal{N}_r(\bar{x},\bar{t}))$ (with some $\mathcal{N}_r(\bar{x},\bar{t}) := \{(x,t) : |x - \bar{x}| < r, \bar{t} - r^2 < t \le \bar{t}\} \subset \mathbb{R}^{n+1}_T \backslash \overline{D}$, whenever

$$u(x,t) - \varphi(x,t) \le (\ge)\, u(\bar{x},\bar{t}) - \varphi(\bar{x},\bar{t}) \quad \text{for any } (x,t) \in \mathcal{N}_r(\bar{x},\bar{t}),$$

we must have

$$-\varphi_t(\bar{x},\bar{t}) \det D^2 \varphi(\bar{x},\bar{t}) \ge (\le)\, f(\bar{x},\bar{t}).$$

For the supersolution, we also need that $D^2\varphi(\bar{x},\bar{t}) > 0$ in the matrix sense.

$u \in C^0(\mathbb{R}^{n+1}_T \backslash \overline{D})$ is a viscosity solution of (1.5) if it is both a viscosity subsolution and supersolution of (1.5).

Definition 1.2 We say that $u \in USC(\overline{\mathbb{R}^{n+1}_T \backslash D})$ ($LSC(\overline{\mathbb{R}^{n+1}_T \backslash D})$) is a viscosity subsolution (supersolution) of problem (1.5)–(1.7) if u is a viscosity subsolution (supersolution) of (1.5), $u \le (\ge)\, \phi(x,t)$ on $\partial D(t) \times [t_0, T]$, and $u \le (\ge)\, \psi(x)$ for $(x,t) \in (\mathbb{R}^n \backslash D(t_0)) \times \{t = t_0\}$.

Then $u \in C^0(\overline{\mathbb{R}^{n+1}_T \backslash D})$ is a viscosity solution of (1.5)–(1.7) if it is a viscosity solution of (1.5), $u = \phi(x,t)$ on $\partial D(t) \times [t_0, T]$, and $u = \psi(x)$ for $(x,t) \in (\mathbb{R}^n \backslash D(t_0)) \times \{t = t_0\}$.

We assume that g and ψ satisfy the following assumptions:

(G) $g \in C^0(\mathbb{R}^n \times [t_0, T])$ is a positive function satisfying

$$0 < \inf_{\mathbb{R}^n \times [t_0,T]} g \le \sup_{\mathbb{R}^n \times [t_0,T]} g < \infty,$$

and for the constant $\alpha > 0$,

$$g(x,t) = g_0\big(|x|\big) + O\big(|x|^{-\alpha}\big), \quad \text{uniformly for } t, |x| \to \infty,$$

where $g_0 \in C^0([0,+\infty))$ is positive,

$$g_0(r) = O\big(r^{\beta}\big), \quad r \to +\infty,$$

and β is a constant, $\beta \geq -\alpha$,

$$\frac{-n(\min\{\alpha, n\} - 2)}{n-1} < \beta < \infty. \tag{1.8}$$

(Ψ) Assume that there exists a constant $\gamma > 0$ such that $\psi \in C^0(\mathbb{R}^n \backslash D(t_0))$ satisfies in the viscosity sense

$$\begin{cases} \det D^2\psi = \frac{g(x,t_0)}{\gamma}, D^2\psi > 0 & \text{in } \mathbb{R}^n \backslash \overline{D(t_0)}, \\ \psi = \phi(x, t_0) & \text{on } \partial D(t_0), \end{cases}$$

and for some $b \in \mathbb{R}^n$ and some constant c, $\psi(x)$ satisfies

$$\limsup_{|x|\to\infty} |x|^{\min\{\alpha,n\}-2+\beta-\frac{\beta}{n}} \left|\psi(x) - \left(u_0\left(|x|\right) + b \cdot x + c\right)\right| < \infty, \quad \text{if } \alpha \neq n, \tag{1.9}$$

$$\limsup_{|x|\to\infty} |x|^{n-2+\beta-\frac{\beta}{n}} \left(\ln|x|\right)^{-1} \left|\psi(x) - \left(u_0\left(|x|\right) + b \cdot x + c\right)\right| < \infty, \quad \text{if } \alpha = n, \tag{1.10}$$

where

$$u_0\left(|x|\right) = \left(\frac{n}{\gamma}\right)^{\frac{1}{n}} \int_0^{|x|} \left(\int_0^s z^{n-1} g_0(z)\, dz\right)^{\frac{1}{n}} ds \tag{1.11}$$

is the solution of elliptic Monge–Ampère equations

$$\det D^2 u_0 = \frac{g_0(|x|)}{\gamma}$$

with $u_0(0) = 0$, $u_0'(0) = 0$.

Our main result is as follows.

Theorem 1.1 *Let D be a bowl-shaped domain in $\mathbb{R}^{n+1}$, $n \geq 3$, and SD be smooth and strictly convex. Assume that g and ψ satisfy (G) and (Ψ) respectively and $\phi \in C^{2,1}(\overline{D})$, ϕ is decreasing in t. Then, for the $b \in \mathbb{R}^n$ and the constant c in (1.9) and (1.10), there exists a unique viscosity solution $u \in C^0(\overline{\mathbb{R}^{n+1}_T \backslash D})$ of (1.5), (1.6), and (1.7) satisfying, for $t \in [t_0, T]$,*

$$\limsup_{|x|\to\infty} |x|^{\min\{\alpha,n\}-2+\beta-\frac{\beta}{n}} \left|u(x,t) - \left(-\gamma(t-t_0) + u_0\left(|x|\right) + b \cdot x + c\right)\right| < \infty, \quad \textit{if } \alpha \neq n, \tag{1.12}$$

and

$$\limsup_{|x|\to\infty} |x|^{n+\beta-2-\frac{\beta}{n}} \left(\ln|x|\right)^{-1} \left|u(x,t) - \left(-\gamma(t-t_0) + u_0\left(|x|\right) + b \cdot x + c\right)\right| < \infty, \quad \textit{if } \alpha = n. \tag{1.13}$$

So we extend the previous results [18–20] from $g \equiv 1$ or $g = 1 + O(|x|^{-\alpha})$ to $g = g_0(|x|) + O(|x|^{-\alpha})$. Moreover, in the Dirichlet problem of elliptic Monge–Ampère equations on exterior domains, an important lemma (Lemma 5.1 [6]) is used to construct the viscosity subsolutions with asymptotic behavior. Similarly, for the parabolic Monge–Ampère equations, a viscosity subsolution with asymptotic behavior is needed to be constructed by an important lemma (Lemma 2.1 [19]) on a cylinder $Q = \Omega \times (0, \tilde{T}] \subset \mathbb{R}^{n+1}$. To construct the viscosity subsolutions of parabolic Monge–Ampère equations applying the lemma, we added the strong condition $\phi_{x_i,t}(x,t) = 0$ for any $x \in \partial\Omega$, $0 \le t \le \tilde{T}$ [18–20], which is not natural. In this paper, we establish a lemma on a bowl-shaped domain and then we use this lemma to construct the viscosity subsolutions without the strong condition $\phi_{x_i,t}(x,t) = 0$.

This paper is arranged as follows. In Sect. 2, we give the important lemma on a bowl-shaped domain with which the viscosity subsolution is constructed. Theorem 1.1 is proved in Sect. 3.

2 An important lemma

Lemma 2.1 *Let D be a bowl-shaped domain in $\mathbb{R}^{n+1}$. Suppose that SD is smooth and strictly convex and $\Phi(x,t) \in C^{2,1}(\overline{D})$. Then there exists some constant C_0, depending only on n, Φ, D, such that, for any $(\xi,\lambda) \in SD$, $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, there exists $\bar{x}(\xi,\lambda,t) \in \mathbb{R}^{n+1}$ satisfying*

$$\left|\bar{x}(\xi,\lambda,t)\right| \le C_0$$

and

$$v_{\xi,\lambda}(x,t) < \Phi(x,t) \quad \text{on } SD\backslash\left\{(\xi,\lambda)\right\}, \tag{2.1}$$

where, for $(x,t) \in \mathbb{R}^n \times [t_0,T]$,

$$v_{\xi,\lambda}(x,t) = \Phi(\xi,t) + \frac{c_*}{2}\left(\left|(x,t) - \bar{x}(\xi,\lambda,t)\right|^2 - \left|(\xi,\lambda) - \bar{x}(\xi,\lambda,t)\right|^2\right),$$

and c_* is any bounded positive constant.

In addition, for some positive constant c_0 and some bounded domain $D_1 \subset \mathbb{R}^n \times [t_0,T]$, we have

$$\frac{\partial v_{\xi,\lambda}}{\partial t} < -c_0 \quad \text{in } D_1. \tag{2.2}$$

Proof Let $(\xi,\lambda) \in SD$, and Φ locally has the expansion

$$\begin{aligned}\Phi(x,t) &= \Phi(\xi,t) + (x-\xi)\cdot D_x\Phi(\xi,t) + \frac{1}{2}(x-\xi)' D^2\Phi(\xi,t)(x-\xi)\\ &\ge \Phi(\xi,t) + (x-\xi)\cdot D_x\Phi(\xi,t) - \overline{C}|x-\xi|^2,\end{aligned}$$

where $D_x\Phi$ is the gradient of Φ in x, $D^2\Phi$ is the Hessian matrix of Φ in x, $(\xi,t) \in \overline{D}$, and $\overline{C} = \frac{1}{2}\max_{\overline{D}}|D^2\Phi|$.

Let

$$\bar{x}(\xi,\lambda,t) = -\frac{1}{c_*}\left(\Phi_{x_1}(\xi,t),\dots,\Phi_{x_n}(\xi,t),0\right) + \hat{c}\nu(\xi,\lambda) + (\xi,\lambda),$$

where $\nu(\xi,\lambda)$ is the unit internal normal vector of SD at (ξ,λ) and $\hat{c}$ is sufficiently large but bounded positive constant to be determined. Then

$$\begin{aligned} v_{\xi,\lambda}(x,t) &= \Phi(\xi,t) + \frac{c_*}{2}\left(\left|(x,t)\right|^2 - \left|(\xi,\lambda)\right|^2\right) - c_*(x-\xi,t-\lambda)\cdot\bar{x}(\xi,\lambda,t) \\ &= \Phi(\xi,t) + \frac{c_*}{2}|x-\xi|^2 + \frac{c_*}{2}(t-\lambda)^2 \\ &\quad + (x-\xi)\cdot D_x\Phi(\xi,t) - c_*\hat{c}(x-\xi,t-\lambda)\cdot\nu(\xi,\lambda). \end{aligned} \tag{2.3}$$

So

$$\begin{aligned} &(v_{\xi,\lambda} - \Phi)(x,t) \\ &\quad \le \overline{C}|x-\xi|^2 + \frac{c_*}{2}|x-\xi|^2 + \frac{c_*}{2}(t-\lambda)^2 - c_*\hat{c}(x-\xi,t-\lambda)\cdot\nu(\xi,\lambda). \end{aligned}$$

By a translation, without loss of generality, we can assume that $\xi = 0$, $\lambda = 0$. Then

$$\begin{aligned} v_{\xi,\lambda}(x,t) &:= \tilde{v}(x,t) \\ &= \Phi(0,t) + \frac{c_*}{2}|x|^2 + \frac{c_*}{2}t^2 + x\cdot D_x\Phi(0,t) - c_*\hat{c}(x,t)\cdot\nu(0,0) \end{aligned}$$

and

$$(\tilde{v} - \Phi)(x,t) \le \overline{C}|x|^2 + \frac{c_*}{2}|x|^2 + \frac{c_*}{2}t^2 - c_*\hat{c}(x,t)\cdot\nu(0,0).$$

We again rotate the coordinates to have $\nu(0,0)$ as one of the axes. That is, let M be an orthogonal matrix such that $Me_{n+1} = \nu(0,0)$. Set $M(y,\gamma) = (x,t)$, then

$$(\tilde{v} - \Phi)(x,t) := (\overline{v} - \overline{\Phi})(y,\gamma) \le C_1|y|^2 + C_2\gamma^2 - c_*\hat{c}\gamma \tag{2.4}$$

and

$$\tilde{v}(x,t) := \overline{v}(y,\gamma) \le C_3 - c_*\hat{c}\gamma, \tag{2.5}$$

where C_1, C_2 are bounded and depend on $\overline{C}$, c_* and M, C_3 is bounded and depends on c_*, $\|\Phi\|_{C^{2,1}(\overline{D})}$, M and diam$D$. Since SD is strictly convex, then SD can be locally represented by

$$\gamma = \rho(y) = O\left(|y|^2\right). \tag{2.6}$$

Thus, by (2.4),

$$(\tilde{v} - \Phi)(x,t) = (\overline{v} - \overline{\Phi})(y,\gamma) \le C_1|y|^2 + C_2\rho^2(y) - \hat{c}c_*\rho(y).$$

Again by the fact that SD is strictly convex, there exists a constant $\delta > 0$ depending only on D such that

$$\rho(y) \ge \delta|y|^2, \quad \forall |y| < \delta. \tag{2.7}$$

So

$$
\begin{aligned}
(\tilde{v} - \Phi)(x,t) &= (\overline{v} - \overline{\Phi})(y,\gamma) \\
&\leq C_1|y|^2 + C_2\rho^2(y) - \hat{c}c_*\delta|y|^2, \quad \forall |y| < \delta.
\end{aligned}
$$

Clearly, by (2.6), for sufficiently large but bounded constant $\hat{c}$,

$$
(\tilde{v} - \Phi)(x,t) < 0, \quad \forall 0 < |y| < \delta, \gamma = \rho(y),
$$

where $\hat{c}$ depends only on δ, $\|\Phi\|_{C^{2,1}(\overline{D})}$, c_*, and M.

On the other hand, by (2.7), we have

$$
\gamma \geq \delta^3, \quad \forall (y,\gamma) \in SD \backslash \bigl\{ \bigl(y, \rho(y)\bigr) : |y| < \delta \bigr\}.
$$

Then, for any $(y,\gamma) \in SD\backslash\{(y,\rho(y)) : |y| < \delta\}$, by (2.5),

$$
\tilde{v}(x,t) = \overline{v}(y,\gamma) \leq C_3 - \hat{c}\delta^3 c_*.
$$

Choosing $\hat{c}$ large enough (depending only on $c_*, \delta, \mathrm{diam}D, \|\Phi\|_{C^{2,1}(\overline{D})}, M$) but still bounded, we get

$$
\tilde{v}(x,t) - \Phi(x,t) < 0, \quad \forall (y,\gamma) \in SD \backslash \bigl\{ \bigl(y,\rho(y)\bigr) : |y| < \delta \bigr\}.
$$

From (2.3), we know that

$$
\frac{\partial v_{\xi,\lambda}}{\partial t} = \Phi_t(\xi,t) + c_*(t-\lambda) + (x-\xi)\cdot D_{x,t}\Phi(\xi,t) - c_*\hat{c}v_{n+1}(\xi,\lambda),
$$

where

$$
D_{x,t}\Phi(\xi,t) = \left(\frac{\partial^2\Phi}{\partial x_1 \partial t}, \dots, \frac{\partial^2\Phi}{\partial x_n \partial t} \right)'.
$$

Therefore, for some positive constant c_0 and some bounded domain $D_1 \subset \mathbb{R}^n \times [t_0, T]$, similar to the above arguments, by translation and rotation of the coordinates, we can choose $\hat{c}$ sufficiently large but bounded such that (2.2) holds. The lemma is proved. □

Remark 2.1 By (2.2), it is easy to see that even if $\Phi_{x_i,t}(x,t) \not\equiv 0$, $(x,t) \in SD$, we still have $-(v_{\xi,\lambda})_t \det D^2 v_{\xi,\lambda} \geq g(x,t)$ in some bounded domain D_1. Then we avoid the bad condition $\Phi_{x_i,t}(x,t) = 0$ for any $(x,t) \in SD$ in [18–20].

3 Proof of Theorem 1.1

For the reader's convenience, we first give the following lemmas whose proof can be found in [18, 26].

Lemma 3.1 ([18]) *Let $\Omega \subset \Omega_1$ be two open strictly convex subsets with smooth boundaries in $\mathbb{R}^n$ and $Q = \Omega \times (t_0, T]$, $Q_1 = \Omega_1 \times (t_0, T]$. Suppose that $v \in C^0(\overline{Q})$ and $u \in C^0(\overline{Q_1})$ are* *parabolically convex and satisfy respectively*

$$-v_t \det D^2 v \geq f \quad \textit{in } Q$$

and

$$-u_t \det D^2 u \geq f \quad \textit{in } Q_1.$$

Furthermore,

$$u \leq v \quad \textit{in } Q, \qquad u = v \quad \textit{on } \partial\Omega \times [t_0, T].$$

Let

$$w(x,t) = \begin{cases} v(x,t), & (x,t) \in Q, \\ u(x,t), & (x,t) \in Q_1. \end{cases}$$

Then $w \in C^0(Q_1)$ is parabolically convex and satisfies, in the viscosity sense,

$$-w_t \det D^2 w \geq f \quad \textit{on } Q_1.$$

Lemma 3.2 ([26]) *Let Ω_1 be an open strictly convex subsets with smooth boundary in $\mathbb{R}^n$, $Q_1 = \Omega_1 \times (t_0, T]$, and $f \in C^0(Q_1)$ be nonnegative. Suppose that $\mathbb{S}_0$ is a nonempty family of subsolutions to the equation*

$$-u_t \det\big(D^2 u\big) = f \quad \textit{in } Q_1, \tag{3.1}$$

and

$$u(x,t) = \sup\big\{\omega(x,t) | \omega \in \mathbb{S}_0\big\}, \quad (x,t) \in Q_1,$$

then u is a viscosity subsolution of (3.1).

Proof of Theorem 1.1 Through an affine transformation in the x-space and by subtracting a linear function to u, we may assume that $b = 0$. The proof is divided into six steps.

Step 1. Construct a viscosity subsolution of (1.5), (1.6), (1.7).

Let $R > 0$, $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$. Without loss of generality, we may assume that $B_2(0) \subset\subset D(t_0)$. Let $R_1 = \operatorname{diam}(D(T))$, then $D(T) \subset\subset B_{R_1}(0)$, choose $R_2 > 2R_1$. Then $B_2(0) \subset\subset D(t_0) \subset\subset D(T) \subset\subset B_{R_1}(0) \subset\subset B_{R_2}(0)$. By Lemma 2.1, for any $(\xi, \lambda) \in \partial D(t) \times [t_0, T]$, there exists $\overline{x}(\xi, \lambda, t) \in \mathbb{R}^{n+1}$, $|\overline{x}(\xi, \lambda, t)| < \infty$ such that

$$v_{\xi,\lambda}(x,t) < \phi(x,t), \quad (x,t) \in \big(\partial D(t) \times [t_0, T]\big) \setminus \big\{(\xi, \lambda)\big\},$$

where

$$\begin{aligned} &v_{\xi,\lambda}(x,t) \\ &\quad = \phi(\xi,t) + \frac{c_*}{2}\Big[\big|(x,t) - \bar{x}(\xi,\lambda,t)\big|^2 - \big|(\xi,\lambda) - \bar{x}(\xi,\lambda,t)\big|^2\Big], \quad (x,t) \in \mathbb{R}^n \times [t_0, T]. \end{aligned}$$

Then, by (2.2), we can choose c_* large enough but bounded such that

$$-(v_{\xi,\lambda})_t \det D^2 v_{\xi,\lambda} \geq \max_{(x,t)\in\overline{B_{R_2}(0)}\times[t_0,T]} g \geq g(x,t), \quad (x,t) \in B_{R_2}(0) \times (t_0,T],$$

$$\det D^2 v_{\xi,\lambda}(x,t_0) \geq g(x,t_0)/\gamma, \quad x \in B_{R_2}(0).$$

Set

$$v(x,t) = \sup_{(\xi,\lambda)\in\partial D(t)\times[t_0,T]} v_{\xi,\lambda}(x,t), \quad (x,t) \in \mathbb{R}^n \times [t_0,T].$$

Then, by (2.1),

$$v(x,t) = \phi(x,t), \quad (x,t) \in \partial D(t) \times [t_0,T], \tag{3.2}$$

and by [27] and Lemma 3.2,

$$-v_t \det D^2 v \geq g(x,t), \quad (x,t) \in B_{R_2}(0) \times (t_0,T], \tag{3.3}$$

$$\det D^2 v(x,t_0) \geq g(x,t_0)/\gamma, \quad x \in B_{R_2}(0). \tag{3.4}$$

So

$$\det D^2 v(x,t_0) \geq \det D^2 \psi(x), \quad x \in B_{R_2}(0) \backslash \overline{D(t_0)}.$$

Choose two positive continuous functions $\bar{g}(|x|)$, $\underline{g}(|x|)$ such that

$$\gamma \bar{g}\big(|x|\big) \geq g(x,t) \geq \gamma \underline{g}\big(|x|\big)$$

and

$$\gamma \underline{g}\big(|x|\big) = g_0\big(|x|\big) - c_1|x|^{-\alpha}, \quad |x| \to \infty,$$

$$\gamma \bar{g}\big(|x|\big) = g_0\big(|x|\big) + c_2|x|^{-\alpha}, \quad |x| \to \infty,$$

where c_1 and c_2 are positive constants. For $a > 0$, we define functions

$$\begin{aligned} u_1(x,t) = {}& -\gamma(t-t_0) + \inf_{B_{R_1}\times[t_0,T]} v \\ & + \int_{2R_1}^{|x|} \left(\int_1^s n z^{n-1} \bar{g}(z)\,dz + a \right)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [t_0,T], \\ u_2(x,t) = {}& -\gamma(t-t_0) + \sup_{B_{R_1}\times[t_0,T]} v \\ & + \int_2^{|x|} \left(\int_1^s n z^{n-1} \underline{g}(z)\,dz + a \right)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [t_0,T]. \end{aligned}$$

Then u_1, u_2 are parabolically convex. Moreover,

$$-(u_1)_t \det D^2 u_1 = \gamma \bar{g}\big(|x|\big) \geq g(x,t), \quad (x,t) \in \big(\mathbb{R}^n \backslash \{0\}\big) \times (t_0,T], \tag{3.5}$$

$$-(u_2)_t \det D^2 u_2 = \gamma \underline{g}\big(|x|\big) \le g(x,t), \quad (x,t) \in \big(\mathbb{R}^n \backslash \{0\}\big) \times (t_0, T], \tag{3.6}$$

$$\det D^2 u_1(x,t) = \overline{g}\big(|x|\big) \ge g(x,t)/\gamma, \quad x \in \mathbb{R}^n \backslash \{0\}, t_0 \le t \le T, \tag{3.7}$$

$$\det D^2 u_2(x,t) = \underline{g}\big(|x|\big) \le g(x,t)/\gamma, \quad x \in \mathbb{R}^n \backslash \{0\}, t_0 \le t \le T. \tag{3.8}$$

For $|x| \le R_1$, $t_0 \le t \le T$,

$$\begin{aligned} u_1(x,t) &= -\gamma(t-t_0) + \inf_{B_{R_1} \times [t_0,T]} v + \int_{2R_1}^{R_1} \left(\int_1^s n z^{n-1} \overline{g}(z)\,dz + a \right)^{\frac{1}{n}} ds \\ &\le \inf_{B_{R_1} \times [t_0,T]} v \le v(x,t). \end{aligned} \tag{3.9}$$

We can choose $a_0 > 0$ such that, for $a \ge a_0$, the following three inequalities hold simultaneously:

$$\begin{aligned} u_1(x,t) &= -\gamma(t-t_0) + \inf_{B_{R_1} \times [t_0,T]} v + \int_{2R_1}^{R_2} \left(\int_1^s n z^{n-1} \overline{g}(z)\,dz + a \right)^{\frac{1}{n}} ds \\ &\ge v(x,t), \quad \text{for } |x| = R_2, t_0 \le t \le T, \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_2(x,t) &= -\gamma(t-t_0) + \sup_{B_{R_1} \times [t_0,T]} v + \int_{2}^{R_2} \left(\int_1^s n z^{n-1} \underline{f}(z)\,dz + a \right)^{\frac{1}{n}} ds \\ &\ge v(x,t), \quad \text{for } |x| = R_2, t_0 \le t \le T, \end{aligned} \tag{3.11}$$

$$u_2(x,t) \ge \phi(x,t), \quad (x,t) \in \partial D(t) \times [t_0, T]. \tag{3.12}$$

In addition, for $(x,t) \in \mathbb{R}^n \times [t_0, T]$, we have

$$\begin{aligned} u_1(x,t) &= -\gamma(t-t_0) + u_0\big(|x|\big) + v_1(a) \\ &\quad - \int_{|x|}^{\infty} \left[\left(\int_1^s n z^{n-1} \overline{g}(z)\,dz + a \right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz \right)^{\frac{1}{n}} \right] ds, \end{aligned}$$

where the function $u_0(|x|)$ is (1.11), and

$$\begin{aligned} v_1(a) &= \int_{2R_1}^{\infty} \left[\left(\int_1^s n z^{n-1} \overline{g}(z)\,dz + a \right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz \right)^{\frac{1}{n}} \right] ds \\ &\quad - u_0(2R_1) + \inf_{B_{R_1} \times [t_0,T]} v. \end{aligned}$$

Then $v_1(a)$ is strictly increasing in $(0, +\infty)$ and

$$\lim_{a \to +\infty} v_1(a) = +\infty.$$

Furthermore, by (1.8), we have $\beta + n > 0$. Since $\bar{g}(z) = \frac{g_0(z)}{\gamma} + \frac{c_2}{\gamma} z^{-\alpha}$, $g_0(z) = O(z^\beta)$, $z \to \infty$, we know that, as $s \to +\infty$,

$$\begin{aligned}&\left(\int_1^s nz^{n-1}\bar{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}} \\ &\quad = O\bigl(s^{1-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}\bigr), \quad \text{if } \alpha \neq n,\end{aligned}$$

and

$$\begin{aligned}&\left(\int_1^s nz^{n-1}\bar{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}} \\ &\quad = O\bigl(s^{1-\beta+\frac{\beta}{n}-n} \ln s\bigr), \quad \text{if } \alpha = n.\end{aligned}$$

As a result, as $|x| \to \infty$,

$$\begin{aligned}&\int_{|x|}^\infty \left[\left(\int_1^s nz^{n-1}\bar{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}}\right] ds \\ &\quad = \int_{|x|}^\infty O\bigl(s^{1-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}\bigr)\,ds \\ &\quad = O\bigl(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}\bigr), \quad \text{if } \alpha \neq n,\end{aligned}$$

and

$$\begin{aligned}&\int_{|x|}^\infty \left[\left(\int_1^s nz^{n-1}\bar{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}}\right] ds \\ &\quad = O\bigl(|x|^{2-\beta+\frac{\beta}{n}-n} \ln|x|\bigr), \quad \text{if } \alpha = n,\end{aligned}$$

where $2 - \beta + \beta/n - \min\{\alpha, n\} < 0$ by (1.8). Thus, as $|x| \to \infty$,

$$u_1(x,t) = -\gamma(t-t_0) + u_0\bigl(|x|\bigr) + \nu_1(a) + O\bigl(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}\bigr), \quad \text{if } \alpha \neq n,$$

and

$$u_1(x,t) = -\gamma(t-t_0) + u_0\bigl(|x|\bigr) + \nu_1(a) + O\bigl(|x|^{2-\beta+\frac{\beta}{n}-n} \ln|x|\bigr), \quad \text{if } \alpha = n.$$

Similarly, we can obtain that

$$\begin{aligned}u_2(x,t) &= -\gamma(t-t_0) + u_0\bigl(|x|\bigr) + \nu_2(a) \\ &\quad - \int_{|x|}^\infty \left[\left(\int_1^s nz^{n-1}\underline{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}}\right] ds,\end{aligned}$$

where

$$\begin{aligned}\nu_2(a) &= \int_2^\infty \left[\left(\int_1^s nz^{n-1}\underline{g}(z)\,dz + a\right)^{\frac{1}{n}} - \left(\int_0^s \frac{n}{\gamma} z^{n-1} g_0(z)\,dz\right)^{\frac{1}{n}}\right] ds \\ &\quad - u_0(2) + \sup_{B_{R_1}\times[t_0,T]} \nu.\end{aligned}$$

It is clear that $\nu_2(a)$ is also strictly increasing in $(0,+\infty)$ and

$$\lim_{a\to+\infty}\nu_2(a)=+\infty.$$

Then, as $|x|\to\infty$, we have

$$\begin{cases} u_2(x,t)=-\gamma(t-t_0)+u_0(|x|)+\nu_2(a)+O(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}), & \text{if } \alpha\neq n,\\ u_2(x,t)=-\gamma(t-t_0)+u_0(|x|)+\nu_2(a)+O(|x|^{2-\beta+\frac{\beta}{n}-n}\ln|x|), & \text{if } \alpha=n.\end{cases}$$

For the sufficiently large constant c in (1.9) and (1.10), there exist $a_1(c)$ and $a_2(c)$ such that $\nu_1(a_1(c))=\nu_2(a_2(c))=c$. Thus, as $|x|\to\infty$, we have

$$\begin{cases} u_1(x,t)=-\gamma(t-t_0)+u_0(|x|)+c+O(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}), & \text{if } \alpha\neq n,\\ u_1(x,t)=-\gamma(t-t_0)+u_0(|x|)+c+O(|x|^{2-\beta+\frac{\beta}{n}-n}\ln|x|), & \text{if } \alpha=n,\end{cases}$$

and

$$\begin{cases} u_2(x,t)=-\gamma(t-t_0)+u_0(|x|)+c+O(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}), & \text{if } \alpha\neq n,\\ u_2(x,t)=-\gamma(t-t_0)+u_0(|x|)+c+O(|x|^{2-\beta+\frac{\beta}{n}-n}\ln|x|), & \text{if } \alpha=n.\end{cases} \tag{3.13}$$

So

$$\lim_{|x|\to\infty}\big(u_1(x,t)-u_2(x,t)\big)=0,\quad t_0\le t\le T. \tag{3.14}$$

By (3.7), (3.8), (3.14) and the comparison principle, we obtain

$$u_1(x,t_0)\le u_2(x,t_0),\quad x\in\mathbb{R}^n\backslash B_2(0). \tag{3.15}$$

By (3.5), (3.6), (3.14), (3.15) and the comparison principle, we obtain

$$u_1(x,t)\le u_2(x,t),\quad (x,t)\in\big(\mathbb{R}^n\backslash B_2(0)\big)\times[t_0,T]. \tag{3.16}$$

For $a\ge a_0$, define

$$\underline{u}_a(x,t)=\begin{cases}\max\{\nu(x,t),u_1(x,t)\}, & |x|\le R_2, t_0\le t\le T,\\ u_1(x,t), & |x|\ge R_2, t_0\le t\le T.\end{cases}$$

By (3.10), we know that $\underline{u}_a\in C^0(\mathbb{R}^n\times[t_0,T])$. By Lemma 3.1, $\underline{u}_a$ satisfies in the viscosity sense

$$-(\underline{u}_a)_t\det D^2\underline{u}_a\ge g(x,t),\quad (x,t)\in\big(\mathbb{R}^n\backslash\{0\}\big)\times(t_0,T]$$

and

$$\det D^2\underline{u}_a(x,t_0)\ge g(x,t_0)/\gamma=\det D^2\psi(x),\quad x\in\mathbb{R}^n\backslash\overline{D(t_0)}.$$

As $|x| \to \infty$,

$$\begin{cases} \underline{u}_a(x,t) = -\gamma(t-t_0) + u_0(|x|) + c + O(|x|^{2-\beta+\frac{\beta}{n}-\min\{\alpha,n\}}), & \text{if } \alpha \neq n, \\ \underline{u}_a(x,t) = -\gamma(t-t_0) + u_0(|x|) + c + O(|x|^{2-\beta+\frac{\beta}{n}-n}\ln|x|), & \text{if } \alpha = n. \end{cases} \tag{3.17}$$

So

$$\lim_{|x|\to\infty} \big(\underline{u}_a(x,t_0) - \psi(x)\big) = 0.$$

In addition, we have, by (3.9) and (3.2),

$$\underline{u}_a(x,t_0) = v(x,t_0) = \phi(x,t_0) = \psi(x), \quad x \in \partial D(t_0).$$

Thus, from the comparison principle, we know that

$$\underline{u}_a(x,t_0) \le \psi(x), \quad x \in \mathbb{R}^n \backslash D(t_0).$$

Moreover, thanks to (3.9) and (3.2),

$$\underline{u}_a(x,t) = v(x,t) = \phi(x,t), \quad (x,t) \in \partial D(t) \times [t_0,T]. \tag{3.18}$$

Then $\underline{u}_a$ is a viscosity subsolution of (1.5), (1.6), and (1.7).

By (3.4), (3.8), (3.11), (3.12) and the comparison principle,

$$v(x,t_0) \le u_2(x,t_0), \quad x \in \overline{B_{R_2} \backslash D(t_0)}. \tag{3.19}$$

Then, by (3.3), (3.6), (3.11), (3.12), (3.19) and the comparison principle,

$$v(x,t) \le u_2(x,t), \quad (x,t) \in \overline{\big(B_{R_2} \times [t_0,T]\big) \backslash D}.$$

So, combining with (3.16), we have

$$\underline{u}_a(x,t) \le u_2(x,t), \quad (x,t) \in \overline{\mathbb{R}^{n+1}_T \backslash D}.$$

Step 2. Define the Perron solution of (1.5), (1.6), and (1.7).

Let $\mathcal{S}$ denote the set of locally parabolically convex functions $\omega \in C^0(\overline{\mathbb{R}^{n+1}_T \backslash D})$ which are viscosity subsolutions of (1.5), (1.6), and (1.7) satisfying

$$\omega(x,t) \le u_2(x,t).$$

Then $\underline{u}_a \in \mathcal{S}$. So $\mathcal{S} \neq \emptyset$. Define

$$u(x,t) = \sup\big\{\omega(x,t) : \omega \in \mathcal{S}\big\}, \quad (x,t) \in \overline{\mathbb{R}^{n+1}_T \backslash D}.$$

Step 3. We prove that u has the asymptotic behavior at infinity.

On the one hand, by the definition of u, we get

$$u(x,t) \le u_2(x,t).$$

Secondly, since $\underline{u}_a \in \mathcal{S}$, we get

$$u(x,t) \ge \underline{u}_a(x,t).$$

By (3.13) and (3.17), we have

$$\limsup_{|x|\to\infty} |x|^{\min\{\alpha,n\}+\beta-\frac{\beta}{n}-2}\big|u(x,t)-\big(-\gamma(t-t_0)+u_0\big(|x|\big)+c\big)\big| < \infty, \quad \text{if } \alpha \neq n,$$

and

$$\limsup_{|x|\to\infty} |x|^{n+\beta-\frac{\beta}{n}-2}\big(\ln|x|\big)^{-1}\big|u(x,t)-\big(-\gamma(t-t_0)+u_0\big(|x|\big)+c\big)\big| < \infty, \quad \text{if } \alpha = n.$$

Step 4. We prove that $u(x,t) = \phi(x,t)$, $(x,t) \in \partial D(t) \times [t_0,T]$, and $u(x,t_0) = \psi(x)$, $x \in \mathbb{R}^n \setminus D(t_0)$.

We first prove that $u(x,t_0) = \psi(x)$, $x \in \mathbb{R}^n \setminus D(t_0)$. Since $\phi \in C^{2,1}(\overline{D})$, there exist some positive constants $q_2 \ge q_1$ such that $-q_2 \le \phi_t(x,t) \le -q_1$ on $\overline{D}$. Choose positive constants p_1, p_2,

$$p_1 \le \min\left\{1, \frac{\gamma}{q_1}\right\}, \qquad p_2 \ge \max\left\{1, \frac{\gamma}{q_2}\right\}$$

such that

$$p_1 q_1 g(x,t_0)/\gamma \le g(x,t), \qquad p_2 q_2 g(x,t_0)/\gamma \ge g(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_T \setminus \overline{D}.$$

Let

$$\underline{U}(x,t) = -p_2 q_2 (t-t_0) + \psi(x), \quad (x,t) \in \overline{\mathbb{R}^{n+1}_T \setminus D},$$
$$\overline{U}(x,t) = -p_1 q_1 (t-t_0) + \psi(x), \quad (x,t) \in \overline{\mathbb{R}^{n+1}_T \setminus D}.$$

Then, in the viscosity sense,

$$-\underline{U}_t \det D^2\underline{U} = p_2 q_2 \det D^2\psi = p_2 q_2 g(x,t_0)/\gamma \ge g(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_T \setminus \overline{D},$$
$$-\overline{U}_t \det D^2\overline{U} = p_1 q_1 \det D^2\psi = p_1 q_1 g(x,t_0)/\gamma \le g(x,t), \quad (x,t) \in \mathbb{R}^{n+1}_T \setminus \overline{D}.$$

In addition, on $\partial D(t) \times [t_0,T]$,

$$\begin{aligned}
\underline{U}(x,t) &= -p_2 q_2 (t-t_0) + \psi(x) \\
&= -p_2 q_2 (t-t_0) + \phi(x,t_0) \\
&\le -q_2 (t-t_0) + \phi(x,t_0) \\
&\le \phi(x,t),
\end{aligned}$$

$$\begin{aligned}\overline{U}(x,t) &= -p_1q_1(t-t_0) + \psi(x)\\ &= -p_1q_1(t-t_0) + \phi(x,t_0)\\ &\geq -q_1(t-t_0) + \phi(x,t_0)\\ &\geq \phi(x,t).\end{aligned}$$

As $|x| \to \infty$,

$$\lim_{|x|\to\infty} \bigl(\underline{U}(x,t) - u(x,t)\bigr) \leq 0$$

and

$$\lim_{|x|\to\infty} \bigl(\overline{U}(x,t) - u(x,t)\bigr) \geq 0.$$

Obviously, for $x \in \mathbb{R}^n \backslash \overline{D(t_0)}$,

$$\underline{U}(x,t_0) = \overline{U}(x,t_0) = \psi(x).$$

Then $\underline{U}(x,t)$ and $\overline{U}(x,t)$ are viscosity subsolution and supersolution of (1.5), (1.6), and (1.7) respectively. So, $\underline{U} \in \mathcal{S}$. Moreover, for any $\omega \in \mathcal{S}$, we obtain $\omega(x,t) \leq \overline{U}(x,t)$. Thus

$$\underline{U}(x,t) \leq u(x,t) \leq \overline{U}(x,t), \quad (x,t) \in \overline{\mathbb{R}_T^{n+1} \backslash D}.$$

Therefore, $u(x,t_0) = \psi(x)$, $x \in \mathbb{R}^n \backslash D(t_0)$.

Now we prove that $u(x,t) = \phi(x,t)$, $(x,t) \in \partial D(t) \times [t_0,T]$. For any $\overline{\xi} \in \partial D(t)$, $t_0 \leq \overline{\gamma} \leq T$, on the one hand, since $\underline{u}_a \in \mathcal{S}$, then by (3.18),

$$\liminf_{(x,t)\to(\bar{\xi},\bar{\gamma})} u(x,t) \geq \lim_{(x,t)\to(\bar{\xi},\bar{\gamma})} \underline{u}_a(x,t) = \phi(\bar{\xi},\bar{\gamma}).$$

On the other hand, we have

$$\limsup_{(x,t)\to(\bar{\xi},\bar{\gamma})} u(x,t) \leq \phi(\bar{\xi},\bar{\gamma}).$$

Indeed, for every $\omega \in \mathcal{S}$, we have

$$\begin{cases} -\omega_t + \Delta\omega \geq 0, & (x,t) \in (B_{R_1} \times (t_0,T]) \backslash \overline{D},\\ \omega \leq \phi, & (x,t) \in \partial D(t) \times [t_0,T],\\ \omega \leq \overline{U}, & (x,t) \in ((B_{R_1} \backslash D(t_0)) \times \{t = t_0\}) \cup (\partial B_{R_1} \times [t_0,T]).\end{cases}$$

Let v^+ satisfy

$$\begin{cases} -v_t^+ + \Delta v^+ = 0, & (x,t) \in (B_{R_1} \times (t_0,T]) \backslash \overline{D},\\ v^+ = \phi, & (x,t) \in \partial D(t) \times [t_0,T],\\ v^+ = \overline{U}, & (x,t) \in ((B_{R_1} \backslash D(t_0)) \times \{t = t_0\}) \cup (\partial B_{R_1} \times [t_0,T]).\end{cases}$$

By the comparison principle, $\omega \le v^+$, $(x,t) \in \overline{(B_{R_1} \times [t_0,T])\backslash D}$. So $u \le v^+$, $(x,t) \in \overline{(B_{R_1} \times [t_0,T])\backslash D}$ and

$$\limsup_{(x,t)\to(\bar{\xi},\bar{\gamma})} u(x,t) \le \lim_{(x,t)\to(\bar{\xi},\bar{\gamma})} v^+(x,t) = \phi(\bar{\xi},\bar{\gamma}).$$

Step 5. We prove that u is a viscosity solution of (1.5).

As the proof of Theorem 1.4 in [21], we can prove that u is a viscosity solution of (1.5).

Step 6. We prove the uniqueness.

Suppose that u and v all satisfy (1.5), (1.6), (1.7), and (1.12) or (1.13). Then

$$\lim_{x\to\infty} \big(u(x,t) - v(x,t)\big) = 0.$$

By the comparison principle, $u \equiv v$, $(x,t) \in \overline{\mathbb{R}^{n+1}_T \backslash D}$.

Theorem 1.1 is proved. □

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