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Homoclinic orbits of sub-linear Hamiltonian systems with perturbed terms

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Abstract

By using variational methods, we obtain the existence of homoclinic orbits for perturbed Hamiltonian systems with sub-linear terms. To the best of our knowledge, there is no published result focusing on the perturbed and sub-linear Hamiltonian systems.

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Keywords: Perturbed Hamiltonian systems; Sub-linear; Homoclinic orbits

1 Introduction and the main result

In this paper, we study the existence of homoclinic orbits for the following second order Hamiltonian systems with perturbed terms:

$$-\ddot{u}(t) + A(t)u(t) - \lambda u(t) = \chi(t)\nabla F\big(u(t)\big) + h(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $u \in \mathbb{R}^N$, $A(t)$ is continuous T-periodic $N \times N$ symmetric matrix valued function, $\lambda \in \mathbb{R}$, $h \in \mathbb{R}^N$, $F(u) \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\nabla F(u)$ denotes its gradient with respect to the u variable.

As usual, we say that $u(t)$ is a *homoclinic orbit* of (1.1) if $u(t)$ is a solution of (1.1) and $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ as $|t| \to \infty$.

If $\lambda = 0$ and $h(t) \equiv 0$, then the system (1.1) becomes to the second order Hamiltonian system

$$-\ddot{u}(t) + A(t)u(t) = \chi(t)\nabla F\big(u(t)\big), \quad t \in \mathbb{R}. \tag{1.2}$$

Since homoclinic orbits play a key role in the research of fluid mechanics and gas dynamics. Therefore, homoclinic orbits of Hamiltonian systems have been studied by many authors [1–12, 14–21]. If the matrix $A(t)$ is positive definite uniformly in t, the authors [7–9, 18, 19] have obtained the existence of homoclinic orbits for (1.2). *However*, we shall consider the cases of (1.1) where $A(t)$ is not uniformly positively definite for $t \in \mathbb{R}$ and the nonlinearities $\nabla F(u)$ is *sub-linear* as at infinity.

In this paper, we are interested in the strongly indefinite case for (1.1).

(L_1) $A(t)$ is T-periodic, $\lambda \notin \sigma(-\frac{d^2}{dt^2} + A(t))$, the spectrum of $-\frac{d^2}{dt^2} + A(t)$.

Let $(\cdot,\cdot)$ be the inner product in $\mathbb{R}^N$, and the associated norm is denoted by $|\cdot|$. Assume that

($X1$) $h \in L^2(\mathbb{R},\mathbb{R}^N)$, $\chi \in L^2(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R},\mathbb{R})$ and $\chi(t) \geq 0$ for all $t \in \mathbb{R}$.

($F1$) $F(u) \in C^1(\mathbb{R}^N,\mathbb{R})$, $F(u) \geq 0$ for all $u \in \mathbb{R}^N$, $F(0) = 0$ and

$$\lim_{|u|\to\infty} \frac{|\nabla F(u)|}{|u|} = 0. \tag{1.3}$$

By $F(0) = 0$ and the differential mean value theorem, we have

$$F(u) = F(u) - F(0) = \bigl(\nabla F(su), u\bigr) \quad \text{for some } s \in (0,1).$$

It follows from (1.3) that

$$\lim_{|u|\to\infty} \frac{|F(u)|}{|u|^2} = \lim_{|u|\to\infty} \frac{|(\nabla F(su),u)|}{|u|^2} \leq \lim_{|u|\to\infty} \frac{|\nabla F(su)|}{|su|}|s| = 0,$$

so we get $\lim_{|u|\to\infty} \frac{F(u)}{|u|^2} = 0$, i.e., the nonlinearity $F(u)$ shows sub-quadratic growth at infinity. Now, our main result reads as follows.

Theorem 1.1

(1) *If* $F(0) = 0$ *and* $h(t) \equiv 0$, *then* (1.1) *has at least one homoclinic orbit.*

(2) *If* (L_1), ($X1$) *and* ($F1$) *hold, and* $h(t) \not\equiv 0$, *then* (1.1) *has at least one nontrivial homoclinic orbit.*

Remark 1.1 To the best of our knowledge, there is no published result focusing on the perturbed Hamiltonian systems (1.1). The main novelties of this paper are as follows: (1) We extend the existing results of (1.2) to the more general Hamiltonian systems (1.1). (2) Most existing results of the systems (1.2) are based on the following superlinear conditions at 0:

$$\bigl|\nabla F(u)\bigr| = o\bigl(|u|\bigr) \quad \text{as } |u| \to 0, \quad \text{i.e.,} \quad \lim_{|u|\to 0} \frac{|\nabla F(u)|}{|u|} = 0. \tag{1.4}$$

However, we remove the conditions (1.4); (3) When the nonlinearity $\nabla F(u)$ is *sub-linear* as $|u| \to \infty$, as far as we know, the results on homoclinic orbits of (1.2) obtained in the literature are very scarce. *Moreover*, we obtain the existence of homoclinic orbits of the more general Hamiltonian systems (1.1) with $\nabla F(u)$ being *sub-linear* as $|u| \to \infty$.

2 Variational frameworks and proof of the main result

Let E^- be a separable closed subspace of a Hilbert space E with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $E^+ = (E^-)^\perp$. For some $R > 0$, set

$$M := \bigl\{u \in E^- : \|u\| \leq R\bigr\}. \tag{2.1}$$

Then M is a submanifold of E^- with boundary ∂M. On E we will also use a topology τ generated by the norm

$$\|u\|_\tau := \max\left(\|P_+ u\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}\bigl|\langle P_- u, e_k\rangle\bigr|\right),$$

where $P_\pm : E \to E^\pm$ is the orthogonal projection of E onto $E^\pm$ and $\{e_k\}$ is a total orthonormal sequence in E^-. Obviously,

$$u^j \xrightarrow{\tau} u \quad \text{implies that} \quad P_+u^j \to P_+u \quad \text{and} \quad P_-u^j \rightharpoonup P_-u. \tag{2.2}$$

Notations We shall denote by $\|\cdot\|_{L^q}$ and $\|\cdot\|_q$ the usual $L^q(\mathbb{R},\mathbb{R}^N)$-norm and $L^q(\mathbb{R},\mathbb{R})$-norm ($1 \le q \le \infty$), respectively. We denote by C different positive constants. Let $\Phi \in C^1(E,\mathbb{R})$. We say that Φ is τ-upper semicontinuous if $u^j \xrightarrow{\tau} u$ implies $\Phi(u) \ge \overline{\lim}_{j\to\infty} \Phi(u^j)$, and Φ' is weakly sequentially continuous if $u^j \rightharpoonup u$ implies $\Phi'(u^j) \rightharpoonup \Phi'(u)$.

Next, we shall use the following generalized saddle point theorem to prove our main result.

Lemma 2.1 ([13]) *Suppose that $\Phi \in C^1(E,\mathbb{R})$ is τ-upper semicontinuous and Φ' is weakly sequentially continuous. If*

$$b := \inf_{E^+} \Phi > \sup_{\partial M} \Phi, \qquad d := \sup_M \Phi < \infty, \tag{2.3}$$

then, for some $c \in [b,d]$, there is a $(PS)_c$ sequence $\{u^j\} \subset E$, i.e., $\Phi(u^j) \to c$ and $\Phi'(u^j) \to 0$.

Under assumption (L_1), $B := -\frac{d^2}{dt^2} + A(t) - \lambda$ is a selfadjoint operator acting on $L^2 := L^2(\mathbb{R},\mathbb{R}^N)$ with domain $\mathcal{D}(B) = H^2(\mathbb{R},\mathbb{R}^N)$ and we have the orthogonal decomposition $L^2 = L^- \oplus L^+$, $u = u^- + u^+$ such that B is negative (resp., positive) in L^- (resp., in L^+). Let $E := \mathcal{D}(|B|^{1/2})$ be equipped, respectively, with the inner product and norm

$$\langle u,v\rangle := \big(|B|^{1/2}u, |B|^{1/2}v\big)_{L^2}, \qquad \|u\| := \big\||B|^{1/2}u\big\|_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R},\mathbb{R}^N)$. Then we have the decomposition

$$E = E^- \oplus E^+$$

and

$$E^\pm = E \cap L^\pm,$$

orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $\langle\cdot,\cdot\rangle$. By (L_1), $E = H^1(\mathbb{R},\mathbb{R}^N)$ with equivalent norms. Then E is a Hilbert space and it is not difficult to show that $E \subset C^0(\mathbb{R},\mathbb{R}^N)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$ (see, e.g., [18]).

Therefore, the corresponding functional of (1.1) can be written as

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} \big[|\dot{u}|^2 + \big(A(t)u,u\big) - \lambda u^2\big]\,dt - \Psi(u)\\ &= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \Psi(u), \quad u \in E,\end{aligned}$$

where $\Psi(u) := \int_{\mathbb{R}}[\chi(t)F(u) + h(t)u]\,dt$. By assumptions (L_1), $(X1)$ and $(F1)$, it is easy to verify that $\Phi, \Psi \in C^1(E,\mathbb{R})$ and the derivatives are given by

$$\Phi'(u)v = \langle u^+, v^+\rangle - \langle u^-, v^-\rangle - \Psi'(u)v, \qquad \Psi'(u)v = \int_{\mathbb{R}} \big[\chi(t)\big(\nabla F(u), v\big) + h(t)v\big]\,dt, \tag{2.4}$$

where $u = u^- + u^+$, $v = v^- + v^+ \in E = E^- \oplus E^+$. Equation (2.4) implies that (1.1) is the corresponding Euler–Lagrange equation for Φ. Therefore, we have reduced the problem of finding homoclinic orbits of (1.1) to that of seeking critical points of the functional Φ on E.

In order to apply Lemma 2.1 to prove our result, we need the following two lemmas.

Lemma 2.2 *Under conditions of Theorem 1.1, the functional Φ is τ-upper semicontinuous and Φ' is weakly sequentially continuous.*

Proof First, we show that the functional Φ is τ-upper semicontinuous. Let $u^j \xrightarrow{\tau} u$ and $\Phi(u^j) \geq C_0$ for some constant C_0. By (2.2), we have

$$(u^j)^+ \to u^+, \qquad (u^j)^- \rightharpoonup u^- \quad \text{and} \quad u^j \rightharpoonup u \quad \text{in } E, \qquad u^j \to u \quad \text{a.e. on } \mathbb{R}^N, \tag{2.5}$$

going to a subsequence if necessary. Clearly, $(X1)$ and $(F1)$ imply $\chi(t)F(u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$, which together with (2.5) and Fatou's lemma implies

$$\varliminf_{j\to\infty} \int_{\mathbb{R}} \chi(t)F(u^j)\,dt \geq \int_{\mathbb{R}} \chi(t)F(u)\,dt. \tag{2.6}$$

By (2.5), we have $u^j \rightharpoonup u$ in $L^2(\mathbb{R},\mathbb{R}^N)$, it follows from $h \in L^2(\mathbb{R},\mathbb{R}^N)$ (see $(X1)$) that

$$\lim_{j\to\infty} \int_{\mathbb{R}} h(t)u^j\,dt = \lim_{j\to\infty} (h,u^j) = (h,u) = \int_{\mathbb{R}} h(t)u\,dt. \tag{2.7}$$

By (2.6), (2.7), $\Phi(u^j) \geq C_0$, the definition of Φ and the weak lower semicontinuity of the norm, we get

$$\begin{aligned}
-C_0 &\geq \varliminf_{j\to\infty} \bigl(-\Phi(u^j)\bigr) \\
&= \varliminf_{j\to\infty} \frac{1}{2}\bigl(\|(u^j)^-\|^2 - \|(u^j)^+\|^2\bigr) + \int_{\mathbb{R}} \bigl(\chi(t)F(u^j) + h(t)u^j\bigr)\,dt \\
&\geq \frac{1}{2}\bigl(\|u^-\|^2 - \|u^+\|^2\bigr) + \int_{\mathbb{R}} \bigl(\chi(t)F(u) + h(t)u\bigr)\,dt = -\Phi(u).
\end{aligned}$$

It implies that $\Phi(u) \geq C_0$. Therefore, Φ is τ-upper semicontinuous.

Now, we prove Φ' is weakly sequentially continuous on E. By (2.5) and the definition of Φ', we have

$$\begin{aligned}
&\lim_{j\to\infty} \Phi'(u^j)\varphi \\
&\quad= \lim_{j\to\infty} \left\{\langle (u^j)^+, \varphi\rangle - \langle (u^j)^-, \varphi\rangle - \int_{\mathbb{R}} \bigl[\chi(t)\bigl(\nabla F(u^j), \varphi\bigr) + h(t)\varphi\bigr]\,dt\right\} \\
&\quad= \langle u^+, \varphi\rangle - \langle u^-, \varphi\rangle - \lim_{j\to\infty} \int_{\mathbb{R}} \bigl[\chi(t)\bigl(\nabla F(u^j), \varphi\bigr) + h(t)\varphi\bigr]\,dt, \quad \forall \varphi \in C_0^\infty(\mathbb{R},\mathbb{R}^N).
\end{aligned}$$

It follows from $F \in C^1$, $\varphi \in C_0^\infty(\mathbb{R},\mathbb{R}^N)$ and $u^j \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ (by (2.5)) that

$$\lim_{j\to\infty} \Phi'(u^j)\varphi = \langle u^+, \varphi\rangle - \langle u^-, \varphi\rangle - \int_{\mathbb{R}} \bigl[\chi(t)\bigl(\nabla F(u), \varphi\bigr) + h(t)\varphi\bigr]\,dt = \Phi'(u)\varphi,$$

i.e., Φ' is weakly sequentially continuous on E. The proof is finished. □

Lemma 2.3 *Under conditions of Theorem 1.1, the geometric assumption (2.3) in Lemma 2.1 is true, i.e.,*

$$b := \inf_{E^+} \Phi > \sup_{\partial M} \Phi, \qquad d := \sup_{M} \Phi < \infty.$$

Proof Obviously, if $\chi(t) \equiv 0$ $(t \in \mathbb{R})$, then assumption (L_1) implies that (1.1) becomes to a linear equation and it is easy to see that it has a solution. Therefore, we may assume that $\|\chi\|_\infty \neq 0$. By the Sobolev inequality, there is a constant $C_0 > 0$ such that

$$C_0\|u\|_{L^2}^2 \leq \|u\|^2, \quad \forall u \in E. \tag{2.8}$$

Clearly, $(F1)$ implies that

$$\left|\nabla F(u)\right| \leq \frac{C_0}{3\|\chi\|_\infty}|u| + C, \qquad \left|F(u)\right| \leq \frac{C_0}{3\|\chi\|_\infty}|u|^2 + C|u|, \quad u \in \mathbb{R}^N. \tag{2.9}$$

For $u \in E^+$, by (2.8), (2.9), the definition of Φ and the Hölder inequality, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \bigl(\chi(t)F(u) + h(t)u\bigr)\,dt \\
&\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \left|\chi(t)\right| \left(\frac{C_0}{3\|\chi\|_\infty}|u|^2 + C|u|\right) dt - \int_{\mathbb{R}} \left|h(t)\right||u|\,dt \\
&\geq \frac{1}{2}\|u\|^2 - \frac{C_0}{3}\|u\|_{L^2}^2 - \bigl(C\|\chi\|_2 + \|h\|_{L^2}\bigr)\|u\|_{L^2} \\
&\geq \frac{1}{6}\|u\|^2 - \frac{1}{C_0^{1/2}}\bigl(C\|\chi\|_2 + \|h\|_{L^2}\bigr)\|u\|.
\end{aligned}$$

It follows from $\|\chi\|_2 + \|h\|_{L^2} < \infty$ (see $(X1)$) that $b := \inf_{E^+} \Phi > -\infty$

For $u \in E^-$, by (2.8) and $\chi(t)F(u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ (see $(X1)$ and $(F1)$), we have

$$\begin{aligned}
\Phi(u) &= -\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} \bigl(\chi_n F_n(u_n) + h_n u_n\bigr)\,dt \\
&\leq -\frac{1}{2}\|u\|^2 + \|h\|_{L^2}\|u\|_{L^2} \\
&\leq -\frac{1}{2}\|u\|^2 + \frac{1}{C_0^{1/2}}\|h\|_{L^2}\|u\|.
\end{aligned}$$

It follows from $\|h\|_{L^2} < \infty$ (see $(X1)$) that for R large enough we have

$$b := \inf_{E^+} \Phi > \sup_{\partial M} \Phi, \qquad d := \sup_{M} \Phi < \infty,$$

where M is defined in (2.1). The proof is finished. □

Now, Lemmas 2.2 and 2.3 imply that Lemma 2.1 holds. Next, we give a detailed proof of Theorem 1.1.

Proofs of Theorem 1.1(1) and (2) (1) Obviously, 0 is a trivial homoclinic orbit of (1.1) if $F(0) = 0$ and $h(t) \equiv 0$.

(2) By Lemma 2.1, for some $c \in [b,d]$, there is a sequence $\{u^j\} \subset E$ such that

$$\Phi(u^j) \to c, \qquad \Phi'(u^j) \to 0. \tag{2.10}$$

Let $\hat{u}^j = (u^j)^+ - (u^j)^-$, then $\|\hat{u}^j\| = \|u^j\|$. Therefore, by (2.8)–(2.10), the Hölder inequality and the fact $E = E^- \oplus E^+$ orthogonal with respect to $(\cdot,\cdot)_{L^2}$, we have

$$\begin{aligned}
C\|u^j\| &= C\|\hat{u}^j\| \ge \Phi'(u^j)\hat{u}^j \\
&= \|(u^j)^+\|^2 + \|(u^j)^-\|^2 - \int_{\mathbb{R}} \left[\chi(t)\left(\nabla F(u^j), \hat{u}^j\right) + h(t)\hat{u}^j\right] dt \\
&\ge \|u^j\|^2 - \int_{\mathbb{R}} \left[|\chi(t)|\left(\frac{C_0}{3\|\chi\|_\infty}|u^j| + C\right) + |h(t)|\right]\left(|(u^j)^+| + |(u^j)^-|\right) dt \\
&\ge \|u^j\|^2 - \left(\frac{C_0}{3}\|u^j\|_{L^2} + C\|\chi\|_2 + \|h\|_{L^2}\right)\left(\|(u^j)^+\|_{L^2} + \|(u^j)^-\|_{L^2}\right) \\
&\ge \|u^j\|^2 - 2\|u^j\|_{L^2}\left(\frac{C_0}{3}\|u^j\|_{L^2} + C\|\chi\|_2 + \|h\|_{L^2}\right) \\
&\ge \frac{1}{3}\|u^j\|^2 - \frac{2}{C_0^{1/2}}\left(C\|\chi\|_2 + \|h\|_{L^2}\right)\|u^j\|.
\end{aligned}$$

It follows from $\|\chi\|_2 + \|h\|_{L^2} < \infty$ (see $(X1)$) that $\{u^j\}$ is bounded in E.

Consequently, up to a subsequence, we may assume that $u^j \rightharpoonup u$ in E. By (2.10) and the fact that Φ' is weakly sequentially continuous (see Lemma 2.2), we have

$$0 = \lim_{j\to\infty} \Phi'(u^j)v = \Phi'(u)v, \quad \forall v \in E.$$

Therefore, u is a homoclinic orbit of (1.1). The fact $h(t) \not\equiv 0$ implies the system (1.1) has no trivial solution, i.e., 0 is not a solution of (1.1), thus u is a nontrivial homoclinic orbit of (1.1). The proof is finished. □

3 Conclusion

We obtain the existence of homoclinic orbits for a class of perturbed Hamiltonian systems with sub-linear terms. To the best of our knowledge, there is no published result focusing on the perturbed and sub-linear Hamiltonian systems.

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