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Fujita type theorem for a class of coupled quasilinear convection–diffusion equations

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Abstract
In this paper, we establish the Fujita type theorem for a homogeneous Neumann outer problem of the coupled quasilinear convection–diffusion equations and formulate the critical Fujita exponent. Besides, the influence of diffusion term, reaction term, and convection term on the global existence and the blow-up property of the problem is revealed. Finally, we discuss the large time behavior of the solution to the outer problem in the critical case and describe the asymptotic behavior of the solution.

MSC: 35B33; 35K20; 35K59

Keywords: Critical Fujita exponent; Fujita type theorem; Convection–diffusion equations

1 Introduction

In this paper, we consider the critical Fujita exponent of the following coupled quasilinear convection–diffusion equations:

$$\frac{\partial u}{\partial t} = \Delta u^m + \kappa \frac{x}{|x|^2} \cdot \nabla u^m + |x|^\lambda v^p, \quad x \in \mathbb{R}^n \backslash \overline{B}_1, t > 0, \tag{1}$$

$$\frac{\partial v}{\partial t} = \Delta v^m + \kappa \frac{x}{|x|^2} \cdot \nabla v^m + |x|^\mu u^q, \quad x \in \mathbb{R}^n \backslash \overline{B}_1, t > 0, \tag{2}$$

$$\frac{\partial u^m}{\partial \boldsymbol{\nu}}(x,t) = \frac{\partial v^m}{\partial \boldsymbol{\nu}}(x,t) = 0, \quad x \in \partial B_1, t > 0, \tag{3}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \mathbb{R}^n \backslash \overline{B}_1, \tag{4}$$

where $p, q > m > 1$, $\kappa \in \mathbb{R}$, $\lambda \geq 0$, $\mu = \frac{\lambda(q-m)+2(q-p)}{p-m} \geq 0$. In addition, B_1 denotes the unit ball in $\mathbb{R}^n$, $\boldsymbol{\nu}$ denotes the unit inner normal vector to ∂B_1, and $0 \leq u_0$, $v_0 \in C_0(\mathbb{R}^n)$ are nontrivial.

In 1966, the first result of the exponent of the quasilinear diffusion equation was introduced by Fujita [5]. Precisely, he investigated the Cauchy problem of the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n, t > 0,$$

and showed that the problem does not have any nontrivial global nonnegative solution if $1 < p < p_c = 1 + 2/n$, whereas there exist both nontrivial global (with small initial data) and nonglobal nonnegative (with large initial data) solutions when $p > p_c = 1 + 2/n$. Among many other results, it proved $p = p_c$ belonged to the blow-up case by Hayakawa [12], Kobayashi et al. [13], and Weissler [27]. This is what we know as the blow-up theorem of Fujita type, and p_c is called the critical Fujita exponent. Early results of the Fujita type theorem can be seen in the review articles of Deng [2], Levine [15], and relevant references. In recent years, there are a lot of Fujita's results, such as [2, 3, 8, 10, 11, 14, 16–21, 24–26, 30, 31] and the references therein.

Among those works, Galaktionov et al. [6, 7] considered the critical Fujita exponent of the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u^m + u^p, \quad x \in \mathbb{R}^n, t > 0 \, (p, m > 1)$$

and proved that the critical Fujita exponent is $p_c = m + 2/n$. Aguirre and Escobedo [1] demonstrated the Fujita type theorem of the following convective–diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^m + \boldsymbol{b}_0 \cdot \nabla u^q + u^p, \quad x \in \mathbb{R}^n, t > 0,$$

where $q \geq 1$, $p > 1$, $\boldsymbol{b}_0 \in \mathbb{R}^n$. They demonstrated that the critical Fujita exponent was

$$p_c = \min\left\{1 + \frac{2}{n}, 1 + \frac{2q}{n+1}\right\}.$$

Zheng and Wang [29] studied more general nonlinear convection–diffusion systems

$$\begin{aligned}
&|x|^{\lambda_1}\frac{\partial u}{\partial t} = \Delta u^m + \kappa \frac{x}{|x|^2} \cdot \nabla u^m + |x|^{\lambda_2} u^p, \quad x \in \mathbb{R}^n \backslash \overline{\Omega}, t > 0, \\
&\frac{\partial u^m}{\partial \boldsymbol{\nu}}(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\
&u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \backslash \overline{\Omega},
\end{aligned}$$

where $p > m \geq 1$, $\kappa \in \mathbb{R}$, $-2 < \lambda_1 \leq \lambda_2$, Ω is the bounded area in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $B_{R_1} \subset \Omega \subset B_{R_2}$ for some $0 < R_1 \leq R_2$, and B_R denotes the ball in $\mathbb{R}^n$ with radius R and center at the origin, and $\boldsymbol{\nu}$ is a unit outer normal vector to ∂B_1. It displayed that the critical Fujita exponent is

$$p_c = \begin{cases} m + \frac{2+\lambda_2}{n+\kappa+\lambda_1}, & \kappa > -n - \lambda_1, \\ +\infty, & \kappa \leq -n - \lambda_1. \end{cases}$$

Following from a lot of results, it shows that critical Fujita exponent of a single equation is usually a constant, while the critical Fujita exponent of the coupled equations is usually a curve which is called the critical Fujita curve. In 1991, Escobedo and Herrero [4] investigated the following coupled systems:

$$\frac{\partial u}{\partial t} = \Delta u + v^p, \qquad \frac{\partial v}{\partial t} = \Delta v + u^q, \quad x \in \mathbb{R}^n, t > 0,$$

where $p, q > 0$, and showed that the Fujita curve is

$$(pq)_c = 1 + \frac{2}{n}\max\{p+1, q+1\}.$$

In [22], the authors studied the following Newtonian filtration system:

$$\frac{\partial u}{\partial t} = \Delta u^m + v^p, \qquad \frac{\partial v}{\partial t} = \Delta v^m + u^q, \quad x \in \mathbb{R}^n, t > 0, \tag{5}$$

where $0 < m < 1$, $p, q \geq 1$, and $pq > 1$. It was proved that the critical Fujita curve is

$$(pq)_c = m^2 + \frac{2}{n}\max\{p+m, q+m\}.$$

In [9], the authors studied the Fujita type theorem for the outer problem of the following coupled nonlinear diffusion equations with convective terms:

$$\frac{\partial u}{\partial t} = \Delta u + \kappa \frac{x}{|x|^2} \cdot \nabla u + |x|^{\lambda_1} v^p, \quad x \in \mathbb{R}^n \backslash \overline{B}_1, t > 0,$$
$$\frac{\partial v}{\partial t} = \Delta v + \kappa \frac{x}{|x|^2} \cdot \nabla v + |x|^{\lambda_2} u^q, \quad x \in \mathbb{R}^n \backslash \overline{B}_1, t > 0,$$

and obtained

$$(pq)_c = \begin{cases} 1 + \frac{\max\{p(2+\lambda_2)+(2+\lambda_1), q(2+\lambda_1)+(2+\lambda_2)\}}{n+\kappa}, & \kappa > -n, \\ +\infty, & \kappa \leq -n. \end{cases}$$

In this paper, we prove that the critical Fujita exponent is

$$p_c = \begin{cases} m + \frac{\lambda+2}{n+\kappa}, & \kappa > -n, \\ +\infty, & \kappa \leq -n. \end{cases} \tag{6}$$

The main attention of this paper is to prove the global existence and blow-up properties of solutions. For the global existence of the problem solution, we use the method of constructing the self-similar solution and the comparison principle to prove our conclusion. For the blow-up properties of solutions, we adopt the integral estimation method. It is noted that when discussing the global existence of solutions, we construct the self-similar upper solution to the system. In order to let the self-similar solutions have the same compact supported set, we introduce the perturbation term $(|x| + 1)^{\mu}$. But the disturbance term has a negative impact on our results, which is the problem we need to solve.

The paper is organized as follows. In Sect. 2, we state some definitions and some theorems. Then, several useful auxiliary lemmas are given. In Sect. 4, we derive a Fujita type theorem for problem (1)–(4). At last, we study the asymptotic behavior of the solution to problem (1)–(4) in the critical case.

2 Preliminaries

In this section, we introduce the definition of the solutions to problem (1)–(4) that will be useful for the rest of the paper.

Definition 2.1 Let $0 < T \leq +\infty$. A pair of nonnegative functions (u, v) is called a super (sub) solution to problem (1)–(4) in $(0, T)$ if

$$u, v \in C\big([0, T), L^m_{\mathrm{loc}}(\mathbb{R}^n)\big) \cap L^\infty_{\mathrm{loc}}\big(0, T; L^\infty(\mathbb{R}^n)\big),$$

and the following integral inequalities

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n \setminus B_1} u(x,t) \frac{\partial \varphi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n \setminus B_1} u^m(x,t) \Delta \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \kappa \int_0^T \int_{\mathbb{R}^n \setminus B_1} u^m(x,t) \operatorname{div}\left(\frac{1}{|x|^2} \varphi(x,t) x\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^n \setminus B_1} |x|^\lambda v^p(x,t) \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\partial B_1} u^m(x,t) \left(\frac{\partial \varphi}{\partial \boldsymbol{\nu}}(x,t) - \frac{\kappa}{|x|^2} \varphi(x,t) x \cdot \boldsymbol{\nu}\right) \mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n \setminus B_1} u_0(x) \varphi(x,0)\,\mathrm{d}x \leq (\geq)\, 0, \\
&\int_0^T \int_{\mathbb{R}^n \setminus B_1} v(x,t) \frac{\partial \psi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^n \setminus B_1} v^m(x,t) \Delta \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \kappa \int_0^T \int_{\mathbb{R}^n \setminus B_1} v^m(x,t) \operatorname{div}\left(\frac{1}{|x|^2} \psi(x,t) x\right) \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^n \setminus B_1} |x|^\mu u^q(x,t) \psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\partial B_1} v^m(x,t) \left(\frac{\partial \psi}{\partial \boldsymbol{\nu}}(x,t) - \frac{\kappa}{|x|^2} \psi(x,t) x \cdot \boldsymbol{\nu}\right) \mathrm{d}\sigma\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n \setminus B_1} v_0(x) \psi(x,0)\,\mathrm{d}x \leq (\geq)\, 0
\end{aligned}$$

are fulfilled for any $0 \leq \varphi, \psi \in C^{2,1}(\mathbb{R}^n \times [0, T))$ vanishing when t is near T or $|x|$ is sufficiently large. (u, v) is called a solution to problem (1)–(4) in $(0, T)$ if it is both a supersolution and a subsolution.

Definition 2.2 A solution (u, v) to problem (1)–(4) is said to blow up in a finite time $0 < T < +\infty$ if

$$\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \big\|v(\cdot,t)\big\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \to +\infty \quad \text{as } t \to T^-,$$

which T is called the blow-up time. Otherwise, (u, v) is said to be global.

The following existence theorem and the comparison principle to problem (1)–(4) play an important role in proving our main results.

Theorem 2.1 (Local existence) *When $0 \leq u_0, v_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the Cauchy problem (1)–(4) admits at least one solution locally in time.*

Theorem 2.2 (Comparison principle) *For $0 < T \le +\infty$, assume that (u^*, v^*) and (u^{**}, v^{**}) are two solutions to system (1) and (2) with nonnegative initial data $u_0^*(x)$, $v_0^*(x)$ and $u_0^{**}(x)$, $v_0^{**}(x)$ in $(0,T)$, respectively. If $(u_0^*(x), v_0^*(x)) \le (u_0^{**}(x), v_0^{**}(x))$ a.e. in $\mathbb{R}^n$, then $(u^*, v^*) \le (u^{**}, v^{**})$ a.e. in $\mathbb{R}^n \times (0,T)$.*

The proofs of Theorem 2.1 and Theorem 2.2 are the same as the one in [23, 25, 28] and are omitted here.

3 Auxiliary lemmas

In order to research the blow-up property of solutions to problem (1)–(4), we need the following auxiliary lemmas.

Lemma 3.1 *Assume that (u,v) is a solution to problem (1)–(4). Then there exists $R_0 > 0$ depending only on n and κ such that, for any $l > R_0$,*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad\ge -C_0 l^{-2}\int_{B_{\delta l}\setminus B_l} u^m(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} |x|^\lambda v^p(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x, \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad\ge -C_0 l^{-2}\int_{B_{\delta l}\setminus B_l} v^m(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} |x|^\mu u^q(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x, \end{aligned} \tag{8}$$

where

$$\delta = \begin{cases} 2, & n+\kappa-1 \le 0, \\ \frac{\pi}{n+\kappa-1}+1, & n+\kappa-1 > 0, \end{cases} \qquad C_0 = \frac{\pi^2}{(\delta-1)^2},$$

and

$$\psi_l(r) = \begin{cases} r^\kappa, & 1 \le r \le l, \\ \frac{1}{2}r^\kappa\big(1+\cos\frac{(r-l)\pi}{(\delta-1)l}\big), & l < r < \delta l, \\ 0, & r \ge \delta l. \end{cases}$$

Proof It follows from Definition 2.1 that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad= \int_{B_{\delta l}\setminus B_1} u^m(x,t)\bigg(\Delta\psi_l\big(|x|\big) - \kappa\,\mathrm{div}\bigg(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\bigg)\bigg)\,\mathrm{d}x \\ &\qquad - \int_{\partial B_1} v^m(x,t)\bigg(\frac{\partial\psi_l(|x|)}{\partial\boldsymbol{\nu}} - \frac{\kappa}{|x|^2}\psi_l\big(|x|\big)x\cdot\boldsymbol{\nu}\bigg)\,\mathrm{d}\sigma + \int_{\mathbb{R}^n\setminus B_1} v^p(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad= \int_{B_{\delta l}\setminus B_1} u^m(x,t)\bigg(\Delta\psi_l\big(|x|\big) - \kappa\,\mathrm{div}\bigg(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\bigg)\bigg)\,\mathrm{d}x \end{aligned}$$

$$+\int_{\mathbb{R}^n\setminus B_1} v^p(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x, \quad t>0, \tag{9}$$

where $\psi_l(r)\in C^1([0,+\infty))$ satisfies $\psi_l'(0)=0$ and

$$\frac{\partial\psi_l(|x|)}{\partial\boldsymbol{\nu}}-\frac{\kappa}{|x|^2}\psi_l\big(|x|\big)x\cdot\boldsymbol{\nu}=0, \quad x\in\partial B_1.$$

For $0\le|x|\le l$, it is easily verified that

$$\begin{aligned}&\Delta\psi_l\big(|x|\big)-\kappa\operatorname{div}\biggl(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\biggr)\\&=\psi_l''\big(|x|\big)+\frac{n-\kappa-1}{|x|}\psi_l'\big(|x|\big)-\kappa\frac{n-2}{|x|^2}\psi_l\big(|x|\big)=0.\end{aligned}\tag{10}$$

While for $l\le|x|\le\delta l$, a direct calculation gives

$$\begin{aligned}&\Delta\psi_l\big(|x|\big)-\kappa\operatorname{div}\biggl(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\biggr)\\&=-\frac12(\delta-1)^{-1}\pi(n+\kappa-1)l^{-1}|x|^{\kappa-1}\sin\frac{(|x|-l)\pi}{(\delta-1)l}\\&\quad-\frac12(\delta-1)^{-2}\pi^2l^{-2}|x|^{\kappa}\cos\frac{(|x|-l)\pi}{(\delta-1)l}.\end{aligned}$$

If $n+\kappa-1\le 0$, one gets

$$\begin{aligned}\Delta\psi_l\big(|x|\big)-\kappa\operatorname{div}\biggl(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\biggr)&\ge-\frac12(\delta-1)^{-2}\pi^2l^{-2}|x|^{\kappa}\cos\frac{(|x|-l)\pi}{(\delta-1)l}\\&\ge-\frac12(\delta-1)^{-2}\pi^2l^{-2}\psi_l\big(|x|\big).\end{aligned}\tag{11}$$

If $n+\kappa-1>0$, we have

$$\begin{aligned}&\Delta\psi_l\big(|x|\big)-\kappa\operatorname{div}\biggl(\frac{1}{|x|^2}\psi_l\big(|x|\big)x\biggr)\\&\ge-\frac12(\delta-1)^{-2}\pi^2l^{-2}|x|^{\kappa}\sin\frac{(|x|-l)\pi}{(\delta-1)l}-\frac12(\delta-1)^{-2}\pi^2l^{-2}|x|^{\kappa}\cos\frac{(|x|-l)\pi}{(\delta-1)l}\\&\ge-\frac12(\delta-1)^{-2}\pi^2l^{-2}\psi_l\big(|x|\big).\end{aligned}\tag{12}$$

By (9)–(12), we obtain (7). Similarly, one can prove that (8) holds. □

To prove the existence of a nontrivial global solution to problem (1)–(4), we introduce the following form of self-similar supersolutions to system (1) and (2):

$$u(x,t)=(t+1)^{-\alpha}U\big((t+1)^{-\beta}|x|\big), \quad x\in\mathbb{R}^n\setminus B_1, t\ge 0, \tag{13}$$

$$v(x,t)=(t+1)^{-\alpha}V\big((t+1)^{-\beta}|x|\big), \quad x\in\mathbb{R}^n\setminus B_1, t\ge 0, \tag{14}$$

where

$$\alpha=\frac{\lambda+2}{\lambda(m-1)+2(p-1)}, \qquad \beta=\frac{(p-m)\alpha}{\lambda+2}.$$

By a simple calculation, we show that

$$\big(U^m\big)''(r) + \frac{n+\kappa-1}{r}\big(U^m\big)'(r) + \beta rU'(r) + \alpha U(r) + r^\lambda V^p(r) \le 0, \tag{15}$$

$$\big(V^m\big)''(r) + \frac{n+\kappa-1}{r}\big(V^m\big)'(r) + \beta rV'(r) + \alpha V(r) + r^\mu U^q(r) \le 0, \tag{16}$$

for any $r > 0$. Then the self-similar function (u, v) with the structure (13)–(14) is a supersolution to (1) and (2).

Lemma 3.2 *Assume that $m > 1$, $\kappa > -n$, $p > p_c$ and set*

$$U(r) = V(r) = \big(\eta - Ar^2\big)_+^{1/(m-1)}, \quad r \ge 0, \tag{17}$$

where $s_+ = \max\{0, s\}$, $\eta > 0$, and

$$A = \frac{(m-1)(p-m)\alpha}{m(n+\kappa)(p+p_c-2m)}.$$

Then there exists sufficiently small $\eta > 0$ such that (u, v) given by (13), (14), and (17) is a supersolution to system (1) and (2).

Proof It is clear that U^m and V^m satisfy (15) and (16) when $r \ge (\eta/A)^{1/2}$. For $0 < r < (\eta/A)^{1/2}$, a simple computation can obtain

$$\begin{aligned}&\big(U^m\big)''(r) + \frac{n+\kappa-1}{r}\big(U^m\big)'(r) + \beta rU'(r) + \alpha U(r)\\ &\quad= \left(\frac{2A}{m-1}\left(\frac{2Am}{m-1} - \beta\right)U^{1-m}(r) + \left(\alpha - \frac{2Am(n+\kappa)}{m-1}\right)\right)U(r)\end{aligned}$$

and

$$\begin{aligned}&\big(V^m\big)''(r) + \frac{n+\kappa-1}{r}\big(V^m\big)'(r) + \beta rV'(r) + \alpha V(r)\\ &\quad= \left(\frac{2A}{m-1}\left(\frac{2Am}{m-1} - \beta\right)V^{1-m}(r) + \left(\alpha - \frac{2Am(n+\kappa)}{m-1}\right)\right)V(r).\end{aligned}$$

Due to $\frac{2Am}{m-1} < \beta$, there exists sufficiently small $\eta_1 > 0$ such that, for $0 < \eta < \eta_1$,

$$\big(U^m\big)''(r) + \frac{n+\kappa-1}{r}\big(U^m\big)'(r) + \beta rU'(r) + \alpha U(r) < -\frac{(p-p_c)\alpha U(r)}{2(p+p_c-2m)}, \tag{18}$$

$$\big(V^m\big)''(r) + \frac{n+\kappa-1}{r}\big(V^m\big)'(r) + \beta rV'(r) + \alpha V(r) < -\frac{(p-p_c)\alpha V(r)}{2(p+p_c-2m)}. \tag{19}$$

Then, due to $\lambda, \mu > 0$ and the definition of U, V, there exists $\eta_2 > 0$ such that, for any $0 < \eta < \eta_2$,

$$r^\mu U^{q-1}(r) \le A^{-\mu/2}\eta^{(q-1)/(m-1)+\mu/2} < \frac{(p-p_c)\alpha}{2(p+p_c-2m)}, \quad 0 < r < \left(\frac{\eta}{A}\right)^{1/2},$$

$$r^\lambda V^{p-1}(r) \le A^{-\lambda/2}\eta^{(p-1)/(m-1)+\lambda/2} < \frac{(p-p_c)\alpha}{2(p+p_c-2m)}, \quad 0 < r < \left(\frac{\eta}{A}\right)^{1/2}.$$

Combining the above inequations with (18) and (19), we can see that for sufficiently small $0 < \eta_2 < \eta_1$ and $0 < \eta < \eta_2 < \eta_1$, one gets (15) and (16). Thus, (u, v) given by (13), (14), and (17) is a supersolution of system (1) and (2). □

4 Blow-up theorems of Fujita type

In this section, we establish the blow-up theorems of Fujita type for problem (1)–(4). First, we consider the case $\kappa \le -n$.

Theorem 4.1 *Assume that $p, q > m$, $\lambda, \mu \ge 0$, $\kappa \le -n$, and $0 \le u_0$, $v_0 \in C_0(\mathbb{R}^n \setminus B_1)$ are nontrivial, the solution to problem* (1)–(4) *blows up in a finite time.*

Proof Let (u, v) be the solution to problem (1)–(4). Denote

$$w_l(t) = \int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,t) + v(x,t)\bigr)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x, \quad t \ge 0. \tag{20}$$

For any $l > R_0$, Lemma 3.1 shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} w_l(t) \ge{}& -\frac{C_0}{l^2}\int_{B_{\delta l}\setminus B_l} u^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} |x|^\mu u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\ & - \frac{C_0}{l^2}\int_{B_{\delta l}\setminus B_l} v^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1} |x|^\lambda v^p(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x. \end{aligned} \tag{21}$$

The Hölder inequality leads to

$$\begin{aligned} & \int_{B_{\delta l}\setminus B_l} u^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\ & \quad \le C_1 l^{n+\kappa-m(n+\kappa+\mu)/q}\biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\mu u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/q}, \end{aligned} \tag{22}$$

$$\begin{aligned} & \int_{B_{\delta l}\setminus B_l} v^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\ & \quad \le C_1 l^{n+\kappa-m(n+\kappa+\lambda)/p}\biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\mu v^p(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/p}, \end{aligned} \tag{23}$$

where $C_1 > 0$ is a positive constant independent of l. Substituting (22) and (23) into (21) shows that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} w_l(t) \\ & \quad \ge \biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\mu u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/q}\biggl(\biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\mu u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{(q-m)/q} \\ & \qquad - C_0C_1 l^{-2+n+\kappa-m(n+\kappa+\mu)/q}\biggr) \\ & \qquad + \biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\lambda v^p(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/p}\biggl(\biggl(\int_{\mathbb{R}^n\setminus B_1} |x|^\lambda v^p(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{(p-m)/p} \\ & \qquad - C_0C_1 l^{-2+n+\kappa-m(n+\kappa+\lambda)/p}\biggr). \end{aligned} \tag{24}$$

Owing to the Hölder inequality, for any $t > 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\
&\quad\leq \left(\int_{B_{\delta l}\backslash B_1} |x|^{-\mu/(q-1)}\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(q-1)/q}\left(\int_{\mathbb{R}^n\backslash B_1} |x|^{\mu}u^q(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{1/q}, \\
&\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\
&\quad\leq \left(\int_{B_{\delta l}\backslash B_1} |x|^{-\lambda/(p-1)}\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(p-1)/p}\left(\int_{\mathbb{R}^n\backslash B_1} |x|^{\lambda}v^p(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{1/p},
\end{aligned}
$$

which imply

$$
\begin{aligned}
&\int_{\mathbb{R}^n\backslash B_1} |x|^{\mu}u^q(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\
&\quad\geq \begin{cases}
C_2(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l(|x|)\,\mathrm{d}x)^q l^{n+\kappa+\mu-q(n+\kappa)}, & \text{if } A(q,\mu) < 0, \\
C_2(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l(|x|)\,\mathrm{d}x)^q (\ln l)^{-(q-1)}, & \text{if } A(q,\mu) = 0, \\
C_2(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l(|x|)\,\mathrm{d}x)^q, & \text{if } A(q,\mu) > 0,
\end{cases}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n\backslash B_1} |x|^{\lambda}v^p(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\
&\quad\geq \begin{cases}
C_2(\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l(|x|)\,\mathrm{d}x)^p l^{n+\kappa+\lambda-p(n+\kappa)}, & \text{if } A(p,\lambda) < 0, \\
C_2(\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l(|x|)\,\mathrm{d}x)^p (\ln l)^{-(p-1)}, & \text{if } A(p,\lambda) = 0, \\
C_2(\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l(|x|)\,\mathrm{d}x)^p, & \text{if } A(p,\lambda) > 0,
\end{cases}
\end{aligned}
\tag{26}
$$

where $C_2 > 0$ is a positive constant independent of l and $A(q,\mu) = n + \kappa + \mu - q(n + \kappa)$, $A(p,\lambda) = n + \kappa + \lambda - p(n + \kappa)$. Here, it should be pointed out that the above discussion only requires $p, q > m$.

Due to $\kappa \leq -n$, it is easy to verify that $A(q,\mu) > 0$, $A(p,\lambda) > 0$. From (24)–(26),

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}w_l(t) \\
&\quad\geq C_2^{m/q}\left(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^m\left(C_2^{(q-m)/q}\left(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{q-m}\right. \\
&\quad\quad\left. - C_0C_1 l^{-2+n+\kappa-m(n+\kappa+\mu)/q}\right) \\
&\quad\quad + C_2^{m/p}\left(\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^m\left(C_2^{(p-m)/p}\left(\int_{\mathbb{R}^n\backslash B_1} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{p-m}\right. \\
&\quad\quad\left. - C_0C_1 l^{-2+n+\kappa-m(n+\kappa+\lambda)/p}\right).
\end{aligned}
\tag{27}
$$

For sufficiently large $l_1 > 1$, and note that $-2 + n + \kappa - m(n + \kappa + \mu)/q < 0$, $-2 + n + \kappa - m(n + \kappa + \lambda)/p < 0$, one can get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}w_{l_1}(t) \\
&\quad\geq C_2^{m/q}\left(\int_{\mathbb{R}^n\backslash B_1} u(x,t)\psi_{l_1}\big(|x|\big)\,\mathrm{d}x\right)^m
\end{aligned}
$$

$$\begin{aligned}
&\times \frac{1}{2} C_2^{(q-m)/q} \left(\int_{\mathbb{R}^n \setminus B_1} u(x,t) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \right)^{q-m} \\
&+ C_2^{m/p} \left(\int_{\mathbb{R}^n \setminus B_1} v(x,t) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \right)^{m} \\
&\times \frac{1}{2} C_2^{(p-m)/p} \left(\int_{\mathbb{R}^n \setminus B_1} v(x,t) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \right)^{p-m} \\
\geq{}& C_3 \left(\left(\int_{\mathbb{R}^n \setminus B_1} u(x,t) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \right)^{q} + \left(\int_{\mathbb{R}^n \setminus B_1} v(x,t) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \right)^{p} \right) \\
\geq{}& 2^{p+q} C_3 \cdot \min\big\{ w_{l_1}^p(t), w_{l_1}^q(t) \big\},
\end{aligned}$$

where $C_3 > 0$ is a constant depending on l_1. Since $p, q > m > 1$, there exists $0 < T < +\infty$ such that

$$w_{l_1}(t) = \int_{\mathbb{R}^n \setminus B_1} \big(u(x,t) + v(x,t)\big) \psi_{l_1}\big(|x|\big)\,\mathrm{d}x \to +\infty, \quad t \to T^-.$$

Obviously, $\operatorname{supp} \psi_{l_1}(x) = B_{2l_1}$. Then one gets

$$\big\| u(\cdot,t) \big\|_{L^\infty(\mathbb{R}^n \setminus B_1)} + \big\| v(\cdot,t) \big\|_{L^\infty(\mathbb{R}^n \setminus B_1)} \to +\infty, \quad t \to T^-.$$

That is to say, (u,v) blows up in a finite time. □

Next, we discuss the case $\kappa > -n$.

Theorem 4.2 *Assume that $p, q > m > 1$, $\lambda, \mu > 0$, $\kappa > -n$, and $0 \leq u_0, v_0 \in C_0(\mathbb{R}^n \setminus B_1)$ are nontrivial. Then, for $p < p_c$, any nontrivial solution to problem (1)–(4) blows up in a finite time.*

Proof Let (u,v) be a nontrivial solution to problem (1)–(4). Set

$$w_l(t) = \int_{\mathbb{R}^n \setminus B_1} \big(u(x,t) + l^\theta v(x,t)\big) \psi_l\big(|x|\big)\,\mathrm{d}x, \quad t \geq 0, \tag{28}$$

where θ is a constant determined below. According to Lemma 3.1, for any $l > R_0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} w_l(t) \\
&\quad \geq -C_0 l^{-2} \int_{B_{\delta l} \setminus B_l} u^m(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x + l^\theta \int_{\mathbb{R}^n \setminus B_1} |x|^\mu u^q(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \\
&\qquad - C_0 l^{-2+\theta} \int_{B_{\delta l} \setminus B_l} v^m(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x + \int_{\mathbb{R}^n \setminus B_1} |x|^\lambda v^p(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x.
\end{aligned} \tag{29}$$

Substituting (22) and (23) into (29) shows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} w_l(t) \\
&\quad \geq \left(\int_{\mathbb{R}^n \setminus B_1} |x|^\mu u^q(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^{m/q}
\end{aligned}$$

$$\cdot \left(l^{\theta} \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\mu} u^q(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^{(q-m)/q} - C_0 C_1 l^{-2+n+\kappa-m(n+\kappa+\mu)/q} \right)$$
$$+ \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\lambda} v^p(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^{m/p} \left(\left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\lambda} v^p(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^{(p-m)/p} - C_0 C_1 l^{-2+\theta+n+\kappa-m(n+\kappa+\lambda)/p} \right). \tag{30}$$

Let us discuss the classification of symbols of $A(q,\mu)$ and $A(p,\lambda)$ in (25), (26).

If $A(q,\mu) < 0$, $A(p,\lambda) < 0$, we substitute (25) and (26) into (30), and this yields that

$$\begin{aligned}\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &\geq -C_0 C_4 l^{m(\theta)} w_l^m(t) + C_2 l^{-q(n+\kappa)+n+\kappa+\mu+\theta} \left(\int_{\mathbb{R}^n \setminus B_1} u(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^q \\ &\quad + C_2 l^{-p(n+\kappa)+n+\kappa+\lambda-p\theta} \left(\int_{\mathbb{R}^n \setminus B_1} l^{\theta} v(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^p, \end{aligned} \tag{31}$$

where $C_4 = \max\{C_2^{m/p}, C_2^{m/q}\} > 0$, and

$$m(\theta) = \max\big\{(1-m)(n+\kappa) - 2, (1-m)(n+\kappa) - 2 - (m-1)\theta\big\}.$$

Set

$$\theta = \frac{q-p}{p+1} \left(n + \kappa - \frac{\lambda + 2}{p - m} \right),$$

which implies that

$$-p(n+\kappa) + n + \kappa + \lambda - p\theta = -q(n+\kappa) + n + \kappa + \mu + \theta = \Theta,$$

namely,

$$\Theta = \frac{(-p^2 q + pqm + p - m)(n+\kappa) + (\lambda + 2)(pq - p^2)}{(p+1)(p-m)} + \lambda.$$

By a simple calculation,

$$\begin{aligned}\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &\geq -C_0 C_4 l^{m(\theta)} w_l^m(t) \\ &\quad + C_2 l^{\Theta} \left(\left(\int_{\mathbb{R}^n \setminus B_1} u(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^q + \left(\int_{\mathbb{R}^n \setminus B_1} l^{\theta} v(x,t) \psi_l\big(|x|\big)\,\mathrm{d}x \right)^p \right) \\ &\geq w_l^m(t)\big(-C_0 C_4 l^{m(\theta)} + 2^{-(p+q)} C_2 l^{\Theta} \cdot \min\big\{w_l^{p-m}(t), w_l^{q-m}(t)\big\}\big). \end{aligned} \tag{32}$$

Note that if $p < p_c$, then $m(\theta) < \Theta$. Further, $w_l(0)$ is nondecreasing with respect to $l \in (0, +\infty)$ and

$$\sup\big\{w_l(0) : l \in (0, +\infty)\big\} > 0.$$

Then there exists sufficiently large $l_2 > 1$ such that

$$C_0 C_4 l_2^{m(\theta)} \le 2^{-(p+q+1)} C_2 l_2^{\Theta} \cdot \min\{w_{l_2}^{p-m}(0), w_{l_2}^{q-m}(0)\}. \tag{33}$$

Combining (32) with (33), we get

$$\frac{\mathrm{d}w_{l_2}(t)}{\mathrm{d}t} \ge 2^{-(p+q+1)} C_2 l_2^{\Theta} \cdot \min\{w_{l_2}^{p}(t), w_{l_2}^{q}(t)\}.$$

Just like the proof of Theorem 4.1, we can obtain that (u, v) blows up in a finite time.

For $A(q, \mu) = 0$, $A(p, \lambda) < 0$, we set $\theta = 0$. It follows from (25), (26), and (30) that

$$\begin{aligned}
&\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \\
&\quad\ge \left(C_2 (\ln l)^{-(q-1)} \left(\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l(|x|)\,\mathrm{d}x\right)^q\right)^{m/q} \\
&\qquad\times \left(C_2^{(q-m)/q} (\ln l)^{-(q-1)(q-m)/q} \left(\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l(|x|)\,\mathrm{d}x\right)^{q-m}\right. \\
&\qquad\left. - C_0 C_1 l^{n+\kappa-2-m(n+\kappa+\mu)/q}\right) \\
&\qquad+ \left(C_2 l^{-p(n+\kappa)+n+\kappa+\lambda} \left(\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_l(|x|)\,\mathrm{d}x\right)^p\right)^{m/p} \\
&\qquad\times \left(C_2^{(p-m)/p} l^{(-p(n+\kappa)+n+\kappa+\lambda)(p-m)/p} \left(\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_l(|x|)\,\mathrm{d}x\right)^{p-m}\right. \\
&\qquad\left. - C_0 C_1 l^{n+\kappa-2-m(n+\kappa+\lambda)/p}\right).
\end{aligned} \tag{34}$$

Here

$$\begin{aligned}
&n + \kappa - 2 - m(n + \kappa + \mu)/q < 0, \\
&n + \kappa - 2 - m(n + \kappa + \lambda)/p < \bigl(-p(n + \kappa) + n + \kappa + \lambda\bigr)(p - m)/p.
\end{aligned}$$

Then there exists sufficiently large l_3 such that

$$\begin{aligned}
&\frac{\mathrm{d}w_{l_3}(t)}{\mathrm{d}t} \\
&\quad\ge C_2^{m/q} (\ln l_3)^{m(1-q)/q} \left(\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^m \\
&\qquad\times \frac{1}{2} C_2^{(q-m)/q} (\ln l_3)^{(1-q)(q-m)/q} \left(\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^{q-m} \\
&\qquad+ C_2^{m/p} l_3^{-m(n+\kappa)+m(n+\kappa+\lambda)/p} \left(\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^m \\
&\qquad\times \frac{1}{2} C_2^{(p-m)/p} l_3^{(-p(n+\kappa)+n+\kappa+\lambda)(p-m)/p} \left(\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^{p-m} \\
&\quad\ge C_5 \left(\left(\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^q + \left(\int_{\mathbb{R}^n\setminus B_1} v(x,t)\psi_{l_3}(|x|)\,\mathrm{d}x\right)^p\right)
\end{aligned}$$

$$\geq 2^{-(p+q)} C_5 \cdot \min\left\{ w_{l_3}^{p}(t), w_{l_3}^{q}(t) \right\},$$

where $C_5 > 0$ is a positive constant depending only on l_3. Therefore, we can obtain that (u, v) blows up in a finite time by a similar proof process of Theorem 4.1.

For other cases, select $\theta = 0$. By the similar argument as $A(q, \mu) = 0$, $A(p, \lambda) < 0$, we can also prove that any nontrivial solution blows up in a finite time. □

Theorem 4.3 *Assume that $p, q > m > 1$, $\lambda, \mu > 0$, $\kappa > -n$, and $0 \leq u_0, v_0 \in C_0(\mathbb{R}^n \setminus B_1)$ are nontrivial. Then, if $p > p_c$, there exist both nontrivial global and blow-up solutions to problem* (1)–(4).

Proof The comparison principle and Lemma 3.1 can prove the existence of the nontrivial global solution to problem (1)–(4) with sufficiently small initial value. Next, we study the blow-up solution to problem (1)–(4) when the initial value is sufficiently large.

For $l > 1$ and (u, v) is the solution to problem (1)–(4), set

$$\tilde{w}_l(t) = \int_{\mathbb{R}^n \setminus B_1} \bigl(u(x,t) + v(x,t)\bigr) \psi_l\bigl(|x|\bigr)\,\mathrm{d}x, \quad t \geq 0.$$

According to the Hölder inequality and (30), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \tilde{w}_l(t) \\
&\quad \geq \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\mu} u^q(x,t) \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{m/q} \left(\left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\mu} u^q(x,t) \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{(q-m)/q} \right. \\
&\qquad \left. - C_0 C_1 l^{-2+n+\kappa-m(n+\kappa+\mu)/q} \right) \\
&\qquad + \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\lambda} v^p(x,t) \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{m/p} \left(\left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\lambda} v^p(x,t) \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{(p-m)/p} \right. \\
&\qquad \left. - C_0 C_1 l^{-2+n+\kappa-m(n+\kappa+\lambda)/p} \right) \\
&\quad \geq \tilde{w}_l^m(t) \bigl(-C_0 C_1 C_6 + 2^{-(p+q)} C_7 \cdot \min\bigl\{ \tilde{w}_l^{p-m}(t), \tilde{w}_l^{q-m}(t) \bigr\} \bigr),
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
C_6 &= \max\left\{ l^{-2+n+\kappa-m(n+\kappa+\mu)/q} \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{-\mu/(q-1)} \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{(1-q)m/q}, \right. \\
&\qquad \left. l^{-2+n+\kappa-m(n+\kappa+\lambda)/p} \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{-\lambda/(p-1)} \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{(1-p)m/p} \right\}, \\
C_7 &= \min\left\{ \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\mu/(1-q)} \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{1-q}, \left(\int_{\mathbb{R}^n \setminus B_1} |x|^{\lambda/(1-p)} \psi_l\bigl(|x|\bigr)\,\mathrm{d}x \right)^{1-p} \right\}.
\end{aligned}$$

If (u_0, v_0) is so large that

$$C_0 C_1 C_6 \leq 2^{-(p+q+1)} C_7 \cdot \min\bigl\{ \tilde{w}_l^{p-m}(0), \tilde{w}_l^{q-m}(0) \bigr\},$$

then (35) leads to

$$\frac{\mathrm{d}\tilde{w}_l(t)}{\mathrm{d}t} \geq 2^{-(p+q+1)} C_7 \cdot \min\bigl\{\tilde{w}_l^p(t), \tilde{w}_l^q(t)\bigr\}, \quad t > 0.$$

By a similar argument in the proof of Theorem 4.1, one can show that (u, v) blows up in a finite time. □

5 The critical case

In this section, we consider the critical case

$$p = p_c = m + \frac{2 + \lambda}{n + \kappa}. \tag{36}$$

Obviously, we can prove that (29), (32) still hold, and

$$n + \kappa + \mu - q(n + \kappa) = n + \kappa + \lambda - p_c(n + \kappa) = (1 - m)(n + \kappa) - 2. \tag{37}$$

The result of the critical case is based on the following three lemmas.

Lemma 5.1 *Assume that (u, v) is a nontrivial global solution to problem (1)–(4) with $p = p_c$, then there exists $M_0 > 0$ independent of t such that*

$$\int_{\mathbb{R}^n \setminus B_1} \bigl(u(x,t) + v(x,t)\bigr)|x|^\kappa \,\mathrm{d}x \leq M_0, \quad t > 0. \tag{38}$$

Proof For any sufficiently large $l > 1$, it follows from (32) that

$$\begin{aligned}&\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} \\ &\quad\geq w_l^m(t) l^{-(m-1)(n+\kappa)-2}\bigl(-C_0 C_4 + 2^{-(p_c+q)} C_2 \cdot \min\bigl\{w_l^{p_c-m}(t), w_l^{q-m}(t)\bigr\}\bigr),\end{aligned}$$

where w_l is defined by (28) with $\theta = 0$. Similar to the end of the proof of Theorem 4.1, there exists some $l_3 > 1$ such that, for any $l > l_3$,

$$2^{-(p_c+q+1)} C_2 \cdot \min\bigl\{w_l^{p_c-m}(t), w_l^{q-m}(t)\bigr\} \leq C_0 C_4,$$

which implies

$$w_l(t) \leq \max\bigl\{\bigl(C_0 C_4 C_2^{-1} 2^{p_c+q+1}\bigr)^{1/(p_c-m)}, \bigl(C_0 C_4 C_2^{-1} 2^{p_c+q+1}\bigr)^{1/(q-m)}\bigr\}.$$

Let $l \to +\infty$ in the above inequality, then we can obtain (38). □

Lemma 5.2 *Under the assumption of Lemma 5.1, there exist three positive constants $M_1, M_2, M_3 > 0$ independent of l and t such that, for any sufficiently large $l > 1$,*

$$\begin{aligned}\frac{\mathrm{d}w_l(t)}{\mathrm{d}t} &\geq M_1^{m-\tau} l^{(1-m)(n+\kappa)-2} w_l^{m-\tau}(t) \\ &\quad\times \left(-M_2\biggl(\int_{B_{\delta l}\setminus B_l} \bigl(u(x,t) + v(x,t)\bigr)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^\tau\right.\end{aligned}$$

$$+ M_1^{-(m-\tau)} M_3 \cdot \min\left\{w_l^{p_c-m+\tau}(t), w_l^{q-m+\tau}(t)\right\}\bigg), \tag{39}$$

where

$$0 < \tau < \min\left\{\frac{p_c - m}{p_c - 1}, \frac{q-m}{q-1}\right\}.$$

Proof It is easy to verify that

$$\begin{aligned} &n+\kappa-2-m(n+\kappa+\mu)/q+\tau\big(\mu-(q-1)(n+\kappa)\big)/q \\ &\quad= \big((1-q)(n+\kappa)+\mu\big)(q-m+\tau)/q, \end{aligned} \tag{40}$$

$$\begin{aligned} &n+\kappa-2-m(n+\kappa+\lambda)/p_c+\tau\big(\lambda-(p_c-1)(n+\kappa)\big)/p_c \\ &\quad= \big((1-p_c)(n+\kappa)+\lambda\big)(p_c-m+\tau)/p_c. \end{aligned} \tag{41}$$

For any sufficiently large $l > 1$, it follows from the Hölder inequality that

$$\begin{aligned} &\int_{B_{\delta l}\setminus B_l} u^m(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad\le \left(\int_{B_{\delta l}\setminus B_l} |x|^{-(m-\tau)\mu/(q-m-(q-1)\tau)}\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(q-m-(q-1)\tau)/q} \\ &\qquad\times \left(\int_{B_{\delta l}\setminus B_l} |x|^{\mu} u^q(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(m-\tau)/q}\left(\int_{B_{\delta l}\setminus B_l} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{\tau} \\ &\quad\le C_8 l^{n+\kappa-(n+\kappa+\mu)m/q+\tau(\mu-(q-1)(n+\kappa))/q} \\ &\qquad\times \left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\mu} u^q(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(m-\tau)/q}\left(\int_{B_{\delta l}\setminus B_l} u(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{\tau}, \\ &\int_{B_{\delta l}\setminus B_l} v^m(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x \\ &\quad\le \left(\int_{B_{\delta l}\setminus B_l} |x|^{-(m-\tau)\lambda/(p_c-m-(p_c-1)\tau)}\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(p_c-m-(p_c-1)\tau)/p_c} \\ &\qquad\times \left(\int_{B_{\delta l}\setminus B_l} |x|^{\lambda} v^{p_c}(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(m-\tau)/p_c}\left(\int_{B_{\delta l}\setminus B_l} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{\tau} \\ &\quad\le C_8 l^{n+\kappa-(n+\kappa+\lambda)m/p_c+\tau(\lambda-(p_c-1)(n+\kappa))/p_c} \\ &\qquad\times \left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda} v^{p_c}(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{(m-\tau)/p_c}\left(\int_{B_{\delta l}\setminus B_l} v(x,t)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{\tau}, \end{aligned}$$

where $C_8 > 0$ is a constant independent of l. Substituting the above two inequalities into (29) with $\theta = 0$, it follows from (25), (26), (37), (40), and (41) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} w_l(t) \\ &\quad\ge -C_0 C_8 l^{-(m-1)(n+\kappa)-2}\big(M_1 w_l(t)\big)^{m-\tau}\left(\int_{B_{2l}\setminus B_l} \big(u(x,t)+v(x,t)\big)\psi_l\big(|x|\big)\,\mathrm{d}x\right)^{\tau} \\ &\qquad+ 2^{-(p_c+q)} C_2 l^{-(m-1)(n+\kappa)-2}\cdot\min\left\{w_l^{p_c}(t), w_l^{q}(t)\right\}, \end{aligned}$$

which yields (39) by choosing

$$M_1 = \max\{C_2^{1/p_c}, C_2^{1/q}\}, \qquad M_2 = C_0C_8, \qquad M_3 = 2^{-(p_c+q)}C_2. \qquad \square$$

Lemma 5.3 *Under the assumption of Lemma 5.1, there exists a constant $M_4 > 0$ independent of l and t such that, for any sufficiently large $l > 1$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n\setminus B_1}\bigl(u(x,t)+v(x,t)\bigr)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \ge -M_4 l^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}. \tag{42}$$

Proof Owing to the Hölder inequality, one obtains

$$\begin{aligned}
&\int_{B_{2l}\setminus B_l} u^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\
&\quad\le \biggl(\int_{B_{2l}\setminus B_l}|x|^{-\frac{m\mu}{q-m}}\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{(q-m)/q}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\mu}u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/q} \\
&\quad\le C_9 l^{n+\kappa-m(n+\kappa+\mu)/q}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\mu}u^q(x,t)\,\mathrm{d}x\biggr)^{m/q}, \\
&\int_{B_{2l}\setminus B_l} v^m(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\
&\quad\le \biggl(\int_{B_{2l}\setminus B_l}|x|^{-\frac{m\lambda}{p_c-m}}\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{(p_c-m)/p_c}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\mu}v^{p_c}(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{m/p_c} \\
&\quad\le C_9 l^{n+\kappa-m(n+\kappa+\lambda)/p_c}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\lambda}v^{p_c}(x,t)\,\mathrm{d}x\biggr)^{m/p_c},
\end{aligned}$$

where $C_9 > 0$, independent of l. Substitute the above results into (29) and

$$\frac{q(n+\kappa-2)-m(n+\kappa+\mu)}{q-m} = \frac{p_c(n+\kappa-2)-m(n+\kappa+\lambda)}{p_c-m},$$

then it follows from the Young inequality that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}w_l(t) &\ge -C_0C_9 l^{n+\kappa-2-m(n+\kappa+\mu)/q}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\mu}u^q(x,t)\,\mathrm{d}x\biggr)^{m/q} \\
&\quad + \int_{\mathbb{R}^n\setminus B_1}|x|^{\mu}u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\
&\quad - C_0C_9 l^{n+\kappa-2-m(n+\kappa+\lambda)/p_c}\biggl(\int_{B_{2l}\setminus B_l}|x|^{\lambda}v^{p_c}(x,t)\,\mathrm{d}x\biggr)^{m/p_c} \\
&\quad + \int_{\mathbb{R}^n\setminus B_1}|x|^{\lambda}v^{p}(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\
&\ge -\frac{m}{q}\int_{\mathbb{R}^n\setminus B_1}|x|^{\mu}u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1}|x|^{\mu}u^q(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x \\
&\quad - \frac{q-m}{q}(C_0C_9)^{q/(q-m)}l^{(q(n+\kappa-2)-m(n+\kappa+\mu))/(q-m)} \\
&\quad - \frac{m}{p_c}\int_{\mathbb{R}^n\setminus B_1}|x|^{\lambda}v^{p}(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_1}|x|^{\lambda}v^{p}(x,t)\psi_l\bigl(|x|\bigr)\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}&- \frac{p_c - m}{p_c}(C_0C_9)^{p_c/(p_c-m)} l^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}\\ &\geq -M_4 l^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)},\end{aligned}$$

where

$$M_4 = \max\left\{\frac{q-m}{q}(C_0C_9)^{q/(q-m)}, \frac{p_c-m}{p_c}(C_0C_9)^{p_c/(p_c-m)}\right\}. \qquad \square$$

Now we prove the following theorem.

Theorem 5.1 *Assume that $\kappa > -n$. Then any nontrivial solution to problem (1)–(4) with $p = p_c$ blows up in a finite time.*

Proof We prove the theorem by contradiction. Assume that (u, v) is a nontrivial global solution to problem (1)–(4) with $p = p_c$. Set

$$\Lambda = \sup_{l>0,t>0} w_l(t) = \sup_{t>0} \int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,t) + v(x,t)\bigr)|x|^\kappa \,\mathrm{d}x. \tag{43}$$

It follows from (38) and the nontriviality of (u, v) that $0 < \Lambda < +\infty$. Owing to (43) and the monotonicity of $w_l(t)$ with respect to $l \in (0, +\infty)$, there exist $l_0 > 1$ and $t_0 > 0$ such that, for any $0 < \varepsilon < \Lambda$,

$$w_{l_0/\delta}(t_0) \geq \Lambda - \varepsilon.$$

From Lemma 5.3, for $s \geq t_0$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,s) + v(x,s)\bigr)\psi_{l_0/\delta}\bigl(|x|\bigr)\,\mathrm{d}x\\ &\quad\geq \int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,t_0) + v(x,t_0)\bigr)\psi_{l_0/\delta}\bigl(|x|\bigr)\,\mathrm{d}x\\ &\qquad - M_4(l_0/\delta)^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}(s-t_0)\\ &\quad\geq \Lambda - \varepsilon - M_4(l_0/\delta)^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}(s-t_0),\end{aligned}$$

which yields that

$$\begin{aligned}&\int_{B_{\delta l_0}\setminus B_{l_0}} \bigl(u(x,s) + v(x,s)\bigr)\psi_{l_0}\bigl(|x|\bigr)\,\mathrm{d}x\\ &\quad\leq \int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,t) + v(x,t)\bigr)|x|^\kappa\,\mathrm{d}x - \int_{\mathbb{R}^n\setminus B_1} \bigl(u(x,s) + v(x,s)\bigr)\psi_{l_0/\delta}\bigl(|x|\bigr)\,\mathrm{d}x\\ &\quad\leq \varepsilon + M_4(l_0/\delta)^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}(s-t_0), \quad s \geq t_0.\end{aligned}$$

Let $l = l_0$ in (39), from the above inequality, one gets that

$$\frac{\mathrm{d}w_{l_0}(t)}{\mathrm{d}t} \geq M_1^{m-\tau} l_0^{(1-m)(n+\kappa)-2} w_{l_0}^{m-\tau}(t)$$

$$
\begin{aligned}
&\times \left(-M_2\left(\int_{B_{\delta l_0}\setminus B_{l_0}}\big(u(x,t)+v(x,t)\big)\psi_{l_0}\,\mathrm{d}x\right)^{\tau}\right.\\
&\left.\quad + M_1^{-(m-\tau)}M_3\cdot\min\bigl\{w_{l_0}^{p_c-m+\tau}(t), w_{l_0}^{q-m+\tau}(t)\bigr\}\right)\\
&\geq M_1^{m-\tau}l_0^{(1-m)(n+\kappa)-2}w_{l_0}^{m-\tau}(t)\\
&\quad\times\bigl(-M_2\bigl(\varepsilon+M_4(l_0/\delta)^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}(s-t_0)\bigr)^{\tau}\\
&\quad + M_1^{-(m-\tau)}M_3\cdot\min\bigl\{w_{l_0}^{p_c-m+\tau}(t), w_{l_0}^{q-m+\tau}(t)\bigr\}\bigr).
\end{aligned}
$$

Take $\varepsilon_0\in(0,\Lambda)$ and $M_5>0$ to get

$$
M_2(\varepsilon_0+M_5)^{\tau}\leq\frac{1}{2}M_1^{-(m-\tau)}M_3\cdot\min\bigl\{(\Lambda-\varepsilon)^{p_c-m+\tau}(t),(\Lambda-\varepsilon)^{q-m+\tau}(t)\bigr\},
$$

where ε_0 and M_5 are independent of l_0, $0<\tau<\min\{\frac{p_c-m}{p_c-1},\frac{q-m}{q-1}\}$. Then we obtain

$$
\frac{\mathrm{d}w_{l_0}(t)}{\mathrm{d}t}\geq\frac{1}{2}M_3l_0^{(1-m)(n+\kappa)-2}\cdot\min\bigl\{w_{l_0}^{p_c}(t),w_{l_0}^{q}(t)\bigr\},\quad t_0<t<t_1, \tag{44}
$$

where

$$
t_1=t_0+\frac{M_5}{M_4}(l_0/\delta)^{(-p_c(n+\kappa-2)+m(n+\kappa+\lambda))/(p_c-m)}.
$$

Integrating (44) over (t_0,t_1) with respect to t and using

$$
\bigl(p_c(n+\kappa-2)-m(n+\kappa+\lambda)\bigr)/(p_c-m)=(1-m)(n+\kappa)-2,
$$

one gets that

$$
\begin{aligned}
&w_{l_0}(t_1)\\
&\quad\geq w_{l_0}(t_0)+\frac{1}{2}M_3l_0^{(1-m)(n+\kappa)-2}\cdot\min\bigl\{(\Lambda-\varepsilon_0)^{p_c},(\Lambda-\varepsilon_0)^{q}\bigr\}(t_1-t_0)\\
&\quad\geq w_{l_0/\delta}(t_0)+\frac{1}{2}M_3l_0^{(1-m)(n+\kappa)-2}\cdot\min\bigl\{(\Lambda-\varepsilon_0)^{p_c},(\Lambda-\varepsilon_0)^{q}\bigr\}\\
&\qquad\times\frac{M_5}{M_4}(l_0/\delta)^{(-p_c(n+\kappa-2)+m(n+\kappa+\lambda))/(p_c-m)}\\
&\quad= w_{l_0/\delta}(t_0)\\
&\qquad+\frac{M_3M_5}{2M_4}\delta^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}\cdot\min\bigl\{(\Lambda-\varepsilon_0)^{p_c},(\Lambda-\varepsilon_0)^{q}\bigr\}.
\end{aligned}
$$

That is to say,

$$
\int_{\mathbb{R}^n\setminus B_1}\bigl(u(x,t_1)+v(x,t_1)\bigr)|x|^{\kappa}\,\mathrm{d}x\geq w_{l_0}(t_1)\geq w_{l_0/\delta}(t_0)+\gamma_0\geq\Lambda-\varepsilon_0+\gamma_0,
$$

where

$$
\gamma_0=\frac{M_3M_5}{2M_4}\delta^{(p_c(n+\kappa-2)-m(n+\kappa+\lambda))/(p_c-m)}\cdot\min\bigl\{(\Lambda-\varepsilon_0)^{p_c},(\Lambda-\varepsilon_0)^{q}\bigr\}
$$

is a positive constant independent of l_0. It is obviously verified that

$$w_{(\delta l_0)/\delta}(t_1) = w_{l_0}(t_1) \geq \Lambda - \varepsilon_0 + \gamma_0 \geq \Lambda - \varepsilon_0.$$

Using the same method, one gets

$$\int_{\mathbb{R}^n \setminus B_1} \big(u(x,t_2) + v(x,t_2)\big)|x|^\kappa \,\mathrm{d}x \geq w_{\delta l_0}(t_2) \geq w_{l_0}(t_1) + \gamma_0 \geq \Lambda - \varepsilon_0 + 2\gamma_0,$$

where

$$t_2 = t_1 + \frac{M_5}{M_4} l_0^{(-p_c(n+\kappa-2)+m(n+\kappa+\lambda))/(p_c-m)}.$$

Similarly, for any positive integer i, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B_1} \big(u(x,t_i) + v(x,t_i)\big)|x|^\kappa \,\mathrm{d}x \\ &\quad \geq w_{\delta^{i-1} l_0}(t_i) \geq w_{\delta^{i-2} l_0}(t_{i-1}) + \delta_0 \geq \Lambda - \varepsilon_0 + i\gamma_0, \end{aligned} \tag{45}$$

where

$$t_i = t_{i-1} + \frac{M_5}{M_4}\big(\delta^{i-2} l_0\big)^{-(p_c(n+\kappa-2)+m(n+\kappa+\lambda))/(p_c-m)}.$$

Letting $i \to +\infty$ in (45) implies

$$\sup_{t>0} \int_{\mathbb{R}^n \setminus B_1} \big(u(x,t) + v(x,t)\big)|x|^\kappa \,\mathrm{d}x = +\infty,$$

which contradicts (38). □

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Authors' contributions
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