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Multivalued nonmonotone dynamic boundary condition

Khadija Aayadi[1], Khalid Akhlil[1*], Sultana Ben Aadi[1] and Mourad El Ouali[1]

*Correspondence:
akhlil.khalid@gmail.com
[1]Polydisciplinary Faculty of Ouarzazate, Ibn Zohr University, Agadir, Morocco

Abstract

In this paper, we introduce a new class of hemivariational inequalities, called dynamic boundary hemivariational inequalities, reflecting the fact that the governing operator is also active on the boundary. In our context, it concerns the Laplace operator with Wentzell (dynamic) boundary conditions perturbed by a multivalued nonmonotone operator expressed in terms of Clarke subdifferentials. We show that one can reformulate the problem so that standard techniques can be applied. We use the well-established theory of boundary hemivariational inequalities to prove that under growth and general sign conditions, the dynamic boundary hemivariational inequality admits a weak solution. Moreover, in the situation where the functionals are expressed in terms of locally bounded integrands, a "filling in the gaps" procedure at the discontinuity points is used to characterize the subdifferential on the product space. Finally, we prove that, under a growth condition and eventually smallness conditions, the Faedo–Galerkin approximation sequence converges to a desired solution.

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Keywords: Dynamic boundary hemivariational inequality; Wentzell boundary condition; Clarke subdifferential; Nonconvex optimization

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ with enough smooth boundary Γ, and $A = -\nabla.(a\nabla)$ be a uniformly elliptic differential operator. Then the operator A with Wentzell boundary conditions is given by the system

$$\begin{cases} Au = f & \text{in } \Omega, \\ -Au + b\partial_\nu^a u + cu = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where b and c are nonnegative bounded measurable functions on Γ and $\partial_\nu^a u = (a\nabla u).\nu$ is the co-normal derivative of u with respect to a. The fact that the boundary condition in (1) could involve the operator A (eventually with jumps) goes back to the pioneering paper of Wentzell [39]. What makes the Wentzell boundary condition relevant in the applications is the time-derivative introduced in the boundary conditions. More explicitly, the heat equation with the Wentzell boundary condition becomes

$$\begin{cases} u' + Au = f_1 & \text{in } [0, +\infty[\times \Omega, \\ u' + b\partial_\nu^a u + cu = f_2 & \text{on } [0, +\infty[\times \Gamma, \\ u(x,0) = u_0(x). \end{cases} \tag{2}$$

The heat equation (2) corresponds to the situation where there is a heat source acting on the boundary [11, 12]. Moreover, in the model of vibrating membrane, the Wentzell boundary condition arises if we assume that the boundary Γ can be affected by vibrations in Ω and thus contributes to the kinetic energy of the system [10, 21]. In [24] dynamic boundary conditions are derived for a solid in contact with a thin layer of stirred liquid with a heat exchange coefficient represented here by c. A detailed derivation of dynamic boundary conditions can also be found in [19] and the references therein. In many physical situations the exchange rate of heat diffusion can be nonlinear or nonmonotone. In this situation, the Wentzell boundary problem becomes

$$\begin{cases} u' + Au + \gamma_1(u) \ni f_1 & \text{in } [0, +\infty[\times \Omega, \\ u' + b\partial_\nu^a u + \gamma_2(u) \ni f_2 & \text{on } [0, +\infty[\times \Gamma, \\ u(x,0) = u_0(x). \end{cases} \tag{3}$$

The nonlinear, eventually nonmonotone, dynamic boundary conditions have been extensively studied in recent years. In the case where $\gamma_1 = \partial\phi_1$ and $\gamma_2 = \partial\phi_2$ are subdifferentials, in the sense of convex analysis, of proper, convex, and lower semicontinuous functionals ϕ_1 and ϕ_2 respectively (hence with maximal monotone graphs), problem (3) generates, by a result from Minty [27], a unique solution described by a strongly continuous nonlinear semigroup [11, 12, 36–38]. In the case of nonmonotone but single-valued γ_1 and γ_2 such problems were considered in [13–15] and the references therein. The functions γ_1 and γ_2 were supposed to be of class C^1 and satisfy a sign-growth conditions in the spirit of critical point theory.

The main tool in studying the above problems is to work on a product space instead of the state space itself. This trick, now a standard procedure, provides a good insight into the structure of the problem. In fact, consider functions of the form $U = (u, u_{|\Gamma})$ defined on a suitable product space and define the operator $\mathcal{A}$ by $\mathcal{A}U = (Au, b\partial_\nu^a u + cu)$, then problem (2) can be formulated as follows:

$$\begin{cases} U' + \mathcal{A}U = f, \\ U(0) = U_0, \end{cases} \tag{4}$$

where $f = (f_1, f_1)$. The idea to incorporate boundary conditions into a product space goes back to Greiner [20] and has been used by Amann and Escher [1], Arendt *et al.* [2, Chap. 6] and in [35] in the context of Dirichlet forms. In the context of heat equations, Wentzell boundary conditions were introduced by Favini *et al.* [11], see also [9, 28].

In the situation of nonlinear multivalued dynamic boundary conditions (3), the product space procedure leads to

$$\begin{cases} U' + \mathcal{A}U + \partial\phi_1(u) \times \partial\phi_2(u_{|\Gamma}) \ni f, \\ U(0) = U_0. \end{cases} \tag{5}$$

In the framework of convex functionals, the regularity, in the sense of [10], of ϕ_1 or ϕ_2 at all points allows us to write the inclusion

$$\partial\phi(U) \subset \partial\phi_1(u) \times \partial\phi_2(u_{|\Gamma}), \tag{6}$$

where the functional ϕ is defined by $\phi(U) = \phi_1(u) + \phi_2(u_{|\Gamma})$. Moreover, by Proposition 4.1, the equality holds in (6). This is due to the structure of ϕ as a variable separated functional. Problem (5) is then equivalent to the following problem:

$$\begin{cases} U' + \mathcal{A}U + \partial\phi(U) \ni f, \\ U(0) = U_0, \end{cases} \tag{7}$$

and can be solved by the nonlinear semigroups theory or the variational inequality theory. In the nonconvex functionals framework, inclusion (6) still holds but not the equality except for the regular case. Hence, generally, one can only say that the solvability of (7) implies the solvability of (5), which, in addition, cannot be expressed as a variational inequality due to a lack of monotonicity.

It is the aim of this paper to prove that problem (7) has a weak solution by using the theory of hemivariational inequality. This theory was initiated by Panagiotopoulus as a generalization of the variational inequality theory [29–31]. Hemivariational inequalities are suitable to model physical and engineering problems where multivalued and nonmonotone constitutive laws are involved (see, e.g., [22, 23] and the references therein). The main tool in these formulations is the generalized gradient of Clarke and Rockafellar [8, 10, 34]. As a subclass of this theory, we can mention boundary hemivariational inequalities, which are boundary value problems where the boundary condition is multivalued, nonmonotone, and of subdifferential form (cf. [25, 26, 33] and the references therein). Other aspects including evolution inclusions or boundary conditions which are multivalued, nonmonotone, and of subdifferential form can be found in [4, 5, 16–18].

In this paper we introduce a new class of problems within this theory. It concerns dynamic boundary conditions where the boundary condition is multivalued, nonmonotone, and of the subdifferential form. The dynamic boundary hemivariational inequalities can model problems where the boundary contains a thermostat regulating the temperature within certain specified bounds. It can also be incorporated into Navier–Stokes equations to obtain a variant of Boussinesq model describing the behavior of a heat conducting liquid with boundaries participating in the total energy. The exchange of the heat with the boundary can then be expressed, in a general way, with a multivalued, nonmonotone functional of subdifferential form.

The structure of this paper is as follows. In Sect. 2, we present the preliminary material needed later. In Sect. 3, we state our problem in suitable functional spaces, and we prove the existence of weak solutions in Sect. 4. In Sect. 5 we present a seemingly new result related to partial generalized gradient for nonregular locally Lipschitz functions, and we prove a Chang-type lemma related to locally bounded functions. Finally, we devote Sect. 6 to the convergence of the Faedo–Galerkin approximation to desired solutions.

2 Preliminaries

Let E be a reflexive Banach space with its dual E^* and $A : D(A) \subset E \to 2^{E^*}$ be a multivalued function, where

$$D(A) = \{u \in E : Au \neq \emptyset\}$$

stands for the domain of A. We say that A is *monotone* if $\langle u^* - v^*, u - v\rangle_{E^*\times E} \geq 0$ for all $u^* \in Au$, $v^* \in Av$, and $u, v \in D(A)$. If, moreover, A has a maximal graph in the sense of inclusion among all monotone operators, then we say that A is maximal monotone. In the theory of multivalued nonlinear inclusions, we may apply a multivalued perturbation. This can be made with respect to *pseudomonotone* operators, that is, operators satisfying the following properties:

(a) for each $u \in E$, the set Au is nonempty, closed, and convex in E^*.
(b) A is upper semicontinuous from each finite dimensional subspace of E into E^* endowed with its weak topology;
(c) if $u_n \to u$ weakly in E, $u_n^* \in Au_n$, and $\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{E^*\times E} \leq 0$, then for each $v \in E$ there exists $v^* \in Au$ such that $\langle v^*, u - v\rangle_{E^*\times E} \leq \liminf_{n\to\infty}\langle u_n^*, u_n - v\rangle_{E^*\times E}$.

When dealing with evolution inclusions, another concept of pseudomonotonicity should be introduced. In fact, a large class of evolution inclusions can be written as the sum of a maximal monotone operator resulting from the time derivative and a multivalued operator. This pseudomonotonicity should be defined with respect to the maximal monotone operator as follows: an operator A is *pseudomonotone with respect to* $D(L)$(or L-pseudomonotone) if for a linear, maximal monotone operator $L : D(L) \subset E \to E^*$, if (a) and (b) are satisfied and

(c′) for each sequence $\{u_n\} \subset D(L)$ and $\{u_n^*\} \subset E^*$ with $u_n \to u$ weakly in E, $Lu_n \to Lu$ weakly in E^*, $u_n^* \in Au_n$ for all $n \in \mathbb{N}$, $u_n^* \to u^*$ weakly in E^*, and $\limsup_{n\to+\infty}\langle u_n^*, u_n - u\rangle_{E^*\times E} \leq 0$, we have $u^* \in Au$ and $\lim_{n\to+\infty}\langle u_n^*, u_n\rangle_{E^*\times E} = \langle u^*, u\rangle_{E^*\times E}$.

A is *coercive* if there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$ such that $\langle u^*, u\rangle_{E^*\times E} \geq c(\|u\|_E)\|u\|_E$ for every $(u, u^*) \in \operatorname{Graph}(A)$.

For a single-valued operator $A : E \to E^*$, we say that A is *demicontinuous* if it is continuous from E to E^* endowed with weak topology and A *pseudomonotone* if for each sequence $\{u_n\} \subset E$ such that it converges weakly to $u \in E$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_{E^*\times E} \leq 0$, we have $\langle Av, u - v\rangle_{E^*\times E} \leq \liminf_{n\to\infty}\langle Au_n, u_n - v\rangle_{E^*\times E}$ for all $v \in E$.

Now let $\varphi : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous functional. The mapping $\partial_c\varphi : E \to 2^{E^*}$ defined by

$$\partial_c\varphi(u) = \left\{u^* \in E^* : \left\langle u^*, v - u\right\rangle_{E^*\times E} \leq \varphi(v) - \varphi(u) \text{ for all } v \in E\right\}$$

is called the subdifferential of φ. Any element $u^* \in \partial_c\varphi(u)$ is called a subgradient of φ at u. It is a well-known fact that $\partial\varphi_c$ is a maximal monotone operator.

Let $\Phi : E \to \mathbb{R}$ be a locally Lipschitz continuous functional and $u, v \in E$. We denote by $\Phi^\circ(u; v)$ the generalized Clarke directional derivative of Φ at the point u in the direction v defined by

$$\Phi^\circ(u; v) = \limsup_{w \to u, t \downarrow 0} \frac{\Phi(w + tv) - \Phi(w)}{t}.$$

The generalized Clarke gradient $\partial\Phi : E \to 2^{E^*}$ of Φ at $u \in E$ is defined by

$$\partial\Phi(u) = \left\{\xi \in E^* : \langle \xi, v\rangle_{E^* \times E} \le \Phi^\circ(u; v) \text{ for all } v \in E\right\}.$$

We collect the following properties:

(a) the function $v \mapsto \Phi^\circ(u; v)$ is positively homogeneous, subadditive and satisfies

$$\left|\Phi^\circ(u; v)\right| \le L_u \|v\|_E \quad \text{for all } v \in E,$$

where $L_u > 0$ is the rank of J near u.

(b) $(u, v) \mapsto \Phi^\circ(u; v)$ is upper semicontinuous.

(c) $\partial\Phi(u)$ is a nonempty, convex, and weakly* compact subset of E^* with $\|\xi\|_{E^*} \le L_u$ for all $\xi \in \partial\Phi(u)$.

(d) For all $v \in E$, we have $\Phi^\circ(u; v) = \max\{\langle \xi, v\rangle_{E^* \times E} : \xi \in \partial\Phi(u)\}$.

(e) Let F be another Banach space and $\mathfrak{t} \in \mathcal{L}(F, E)$. Then

  (i) $(\Phi \circ \mathfrak{t})^\circ(u; v) \le \Phi^\circ(\mathfrak{t}u; \mathfrak{t}v)$ for $u, v \in E$.

  (ii) $\partial(\Phi \circ \mathfrak{t})(u) \subseteq \mathfrak{t}^* \partial\Phi(\mathfrak{t}u)$ for $u \in E$ and where $\mathfrak{t}^* \in \mathcal{L}(E^*, F^*)$ denotes the adjoint operator to $\mathfrak{t}$.

The following surjectivity result for operators which are L-pseudomonotone will be used in our existence theorem in Sect. 4 (cf. [32, Theorem 2.1]).

Theorem 2.1 *If E is a reflexive strictly convex Banach space, $L : D(L) \subset E \to E^*$ is a linear maximal monotone operator, and $A : E \to 2^{E^*}$ is a multivalued operator, which is bounded, coercive, and L-pseudomonotone. Then $L + A$ is a surjective operator, i.e., for all $f \in E^*$, there exists $u \in E$ such that $Lu + Au \ni f$.*

It is worth to mention that one can drop the strict convexity of the reflexive Banach space E. It suffices to invoke the Troyanski renorming theorem to get an equivalent norm so that the space itself and its dual are strictly convex(cf. [40, Proposition 32.23, p. 862]).

3 An existence result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. Let λ_N denote the N-dimensional Lebesgue measure and σ the surface measure on Γ. For simplicity, we take $b = 1$ and a, the N-dimensional matrix identity. Define the following product space:

$$\mathbb{H} = \left\{U = (u_1, u_2) : u_1 \in L^2(\Omega), u_2 \in L^2(\Gamma)\right\},$$

endowed with the inner product

$$\langle U, V\rangle_{\mathbb{H}} = \langle u_1, v_1\rangle_{L^2(\Omega)} + \langle u_2, v_2\rangle_{L^2(\Gamma)}, \quad \forall U = (u_1, u_2), V = (v_1, v_2) \in \mathbb{H},$$

and the induced natural norm $|\cdot| := \langle\cdot,\cdot\rangle_{\mathbb{H}}^{1/2}$. Set $\mu = \lambda_N \oplus \sigma$. Then $\mathbb{H}$ can be identified with $L^2(\overline{\Omega},\mu)$. Identifying each function $u \in W^{1,2}(\Omega)$ with $U = (u_{|\Omega}, u_{|\Gamma})$, one has that $W^{1,2}(\Omega)$ is a dense subspace of $\mathbb{H}$. Define the Banach space

$$\mathbb{V} = \{U = (u, u_{|\Gamma}) : u \in W^{1,2}(\Omega)\},$$

and endow it with the norm

$$\|U\| = \|u\|_{W^{1,2}(\Omega)} + \|u\|_{L^2(\Gamma)}$$

for all $U = (u, u_{|\Gamma})$ and $V = (v, v_{|\Gamma})$ is $\mathbb{V}$. It is easy to see that we can identify $\mathbb{V}$ with $W^{1,2}(\Omega) \oplus L^2(\Gamma)$ under this norm. Moreover, we emphasize that $\mathbb{V}$ is not a product space, and since $W^{1,2}(\Omega) \hookrightarrow L^2(\Gamma)$, by trace theory $\mathbb{V}$ is topologically isomorphic to $W^{1,2}(\Omega)$ in the obvious way. It is also immediate that $\mathbb{V}$ is compactly embedded into $\mathbb{H}$. We have then the Gelfand triple

$$\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^*,$$

with continuous and compact embeddings. The embedding $\Lambda : \mathbb{V} \to \mathbb{H}$ is defined in a natural way by $\Lambda(U) = (i(u), \gamma(u))$, where $i : W^{1,2}(\Omega) \to L^2(\Omega)$ is the natural embedding and γ is the trace operator. It is obvious that Λ is continuous and compact from $\mathbb{V}$ into $\mathbb{H}$. Consider the Laplacian operator with multivalued nonmonotone dynamic boundary conditions described as follows:

$$\begin{cases} \partial_t u - \Delta u + \partial\phi_1(u) \ni f_1 & \text{in } [0,T] \times \Omega, \\ \partial_t u + \partial_\nu u + au + \partial\phi_2(u) \ni f_2 & \text{in } [0,T] \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{8}$$

where $a \in L^\infty(\partial\Omega)$ with $a \geq a_0 > 0$ for some constant a_0 and ϕ_1, ϕ_2 are locally Lipschitz functions on $\mathbb{R}$. Let $\langle\cdot,\cdot\rangle$ denote the duality between $\mathbb{V}$ and $\mathbb{V}^*$. System (8) can be written as follows:

$$\begin{cases} \partial_t U + AU + \partial\phi_1(u) \times \partial\phi_2(u) \ni (f_1, f_2), \\ U(0) = U_0, \end{cases} \tag{9}$$

where the operator $A : \mathbb{V} \to \mathbb{V}^*$ is defined by $AU = (-\Delta u, \partial_\nu u + au)$ so that

$$\langle AU, V\rangle = \int_\Omega \nabla u.\nabla v \, dx + \int_\Gamma auv \, d\sigma$$

for $U = (u, u_{|\Gamma})$, $V = (v, v_{|\Gamma}) \in \mathbb{V}$. From [6, Lemma 2.111], it is clear that the operator A is pseudomonotone. The continuity and coercivity of A can be proved in the same way as for Robin boundary conditions. Now, by using the definition of Clarke subdifferential, system (9) leads to

$$\begin{cases} \langle U' + AU - f, V - U\rangle + \int_\Omega \phi_1^\circ(u; v-u)\, dx + \int_{\partial\Omega} \phi_2^\circ(u; v-u)\, d\sigma \geq 0, \\ U(0) = U_0, \end{cases} \tag{10}$$

for all $U = (u, u_{|\Gamma})$, $V = (v, v_{|\Gamma}) \in \mathbb{V}$, where $f = (f_1, f_2) \in \mathbb{V}^*$. Define $\phi : \mathbb{V} \to \mathbb{R}$ by $\phi(U) = \phi_1(u) + \phi_2(u_{|\Gamma})$ for $U = (u, u_{|\partial\Omega})$. It follows that $\partial\phi(U) \subset \partial\phi_1(u) \times \partial\phi_2(u_{|\partial\Omega})$. Then problem (8) has a solution if the following problem has one:

$$\begin{cases} U' + AU + \Lambda^* \partial\phi(\Lambda(U)) \ni f, \\ U(0) = U_0, \end{cases} \tag{11}$$

where $\Phi(U) = \int_{\overline{\Omega}} \phi(U)\, d\mu$. It is clear that the equivalence holds, if ϕ_1 is regular at u or ϕ_2 at $u_{|\Gamma}$. An equivalent formulation to (11) reads: for every $V \in \mathbb{V}$,

$$\begin{cases} \langle U' + AU - f, V - U \rangle + \int_{\overline{\Omega}} \phi^\circ(U; V - U)\, d\mu \geq 0, \\ U(0) = U_0. \end{cases} \tag{12}$$

In what follows we need the spaces $\mathcal{V} = L^2(0, T; \mathbb{V})$, $\mathcal{H} = L^2(0, T; \mathbb{H})$ and $\mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\}$, where the time derivative involved in the definition of $\mathcal{W}$ is understood in the sense of vector-valued distributions. As usual, we equip it with the norm $\|w\|_{\mathcal{W}} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{V}^*}$, which makes the space $\mathcal{W}$ a separable Banach space. Clearly, $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. Moreover, we denote the duality for the pair $(\mathcal{V}, \mathcal{V}^*)$ as follows:

$$\langle\langle f, V \rangle\rangle = \int_0^T \bigl\langle f(t), V(t) \bigr\rangle\, dt$$

for $f \in \mathcal{V}^*$ and $V \in \mathcal{V}$. It is known [40] that the embedding $\mathcal{W} \subset C(0, T; \mathbb{H})$ is continuous. The problem under consideration is as follows: find $U \in \mathcal{W}$ such that, for all $V \in \mathbb{V}$ and a.e. $t \in (0, T)$,

$$\begin{cases} \langle U'(t) + AU(t) - f(t), V - U(t) \rangle + \int_{\overline{\Omega}} \phi^\circ(t, x, U(t); V - U(t))\, d\mu(x) \geq 0, \\ U(0) = U_0. \end{cases} \tag{13}$$

We will prove the existence of solutions to the heat problem with multivalued nonmonotone dynamic boundary condition by considering functionals defined in $L^2(\overline{\Omega})$. Define the functional $\Phi : (0, T) \times L^2(\overline{\Omega}, d\mu) \to \mathbb{R}$ by

$$\Phi(t, U) = \int_{\overline{\Omega}} \phi\bigl(t, U(x)\bigr)\, d\mu(x), \quad t \in (0, T), U \in \mathbb{V}. \tag{14}$$

Let us consider the following hypotheses:

$H(\phi)$ $\phi : (0, T) \times \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that

- (i) $\phi(t, \cdot, \xi)$ is measurable for all $t \in (0, T)$, $\xi = (\xi_1, \xi_1) \in \mathbb{R}^2$ and $\phi_1(t, ., 0) \in L^1(\overline{\Omega})$.
- (ii) $\phi(t, x, \cdot)$ is locally Lipschitz for all $\in (0, T)$, $x \in \overline{\Omega}$.
- (iii) $(\nu_1, \nu_1) \in \partial\phi(t, x, (\xi_1, \xi_2) \Rightarrow |(\nu_1, \nu_2)|_{\mathbb{R}^2} \leq c(1 + |(\xi_1, \xi_2)|_{\mathbb{R}^2})$ for all $t \in (0, T)$, $x \in \overline{\Omega}$ with $c > 0$.
- (iv) $\phi^\circ(t, x, (\xi_1, \xi_2), -(\xi_1, \xi_2)) \leq d(1 + |(\xi_1, \xi_2)|_{\mathbb{R}^2})$ for all $t \in (0, T)$, $x \in \overline{\Omega}$ with $d \geq 0$.

(H_0) $U_0 \in \mathbb{H}, f \in \mathcal{V}^*$.

One can see that if ϕ_1 and ϕ_2 satisfy assumptions similar to $H(\phi)$, then $H(\phi)$ holds. The following lemma will be proved in the same way as the similar one for functionals on Γ (cf. [26, 33]).

Lemma 3.1 *Assume that $\phi : (0,T) \times \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies hypothesis $H(\phi)$. Then the functional Φ given by (14) is well defined and locally Lipschitz in the second variable(in fact, Lipschitz in the second variable on bounded subsets of $\mathbb{H}$), its generalized gradient satisfies the linear growth condition*

$$(\xi_1,\xi_2) \in \partial\Phi(t,V) \quad \Rightarrow \quad \|(\xi_1,\xi_2\|_{\mathbb{H}} \le c'\bigl(1 + \|V\|_{\mathbb{H}}\bigr)$$

with $c' > 0$, and for its generalized directional derivative, we have

$$\Phi^\circ(t,U;V) \le \int_{\overline{\Omega}} \phi^\circ\bigl(t,x,U(x);V(x)\bigr)\,d\mu(x)$$

for $t \in (0,T)$, $U, V \in \mathbb{H}$ and

$$\Phi^\circ(t,U;-U) \le d'\bigl(1 + \|U\|_{\mathbb{H}}\bigr)$$

with $d' > 0$.

From (6), it is clear that in order to obtain the solvability of problem (13), it is enough to show that the problem

$$\begin{cases} \partial_t U + AU + \Lambda^*\partial\phi(\Lambda U) \ni f, \\ U(0) = U_0, \end{cases} \tag{15}$$

admits a solution. The proofs are similar to the ones in [26, 33].

Proposition 3.2 *Suppose that hypotheses $H(\phi)$ and H_0 hold and U is a solution of (15), then there exists a constant $C > 0$ such that*

$$\|U\|_{\mathcal{W}} \le C\bigl(1 + \|U_0\|_{\mathbb{H}} + \|f\|_{\mathcal{V}}\bigr). \tag{16}$$

Theorem 3.3 *If hypotheses $H(\phi)$ and H_0 hold, then problem (15) has a solution.*

Proof By the density of $\mathbb{V}$ in $\mathbb{H}$, we may assume that $U_0 \in \mathbb{V}$. Define the Nemytskii operators corresponding to $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ and $\mathcal{N} : \mathcal{V} \to 2^{\mathcal{V}^*}$ as follows:

$$(\mathcal{A}V)(t) = AV(t) + AU_0$$

and

$$\mathcal{N}V = \bigl\{\omega \in \mathcal{H} : \omega(t) \in \Lambda^*\partial\Phi\bigl(t, \Lambda\bigl(V(t) + U_0\bigr)\bigr), \text{a.e. } t \in (0,T)\bigr\}$$

for all $V \in \mathcal{V}$. Problem (15) reads

$$\begin{cases} Z'(t) + \mathcal{A}Z(t) + \mathcal{N}Z(t) \ni f, \\ Z(0) = 0. \end{cases} \tag{17}$$

We note that $Z \in \mathcal{W}$ is a solution to problem (17) if and only if $Z + U_0 \in \mathcal{W}$ is a solution to problem (15). Let $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{V} \to \mathcal{V}^*$ be the operator defined by $\mathcal{L}V = V'$ with $D(\mathcal{L}) = \{w \in \mathcal{W} : w(0) = 0\}$. It is well known (cf. [40]) that $\mathcal{L}$ is a linear densely defined and maximal monotone operator. As a consequence, from (17) we obtain the problem

$$\text{find } Z \in D(\mathcal{L}) : \quad \mathcal{L}Z + \mathcal{T}Z \ni f, \tag{18}$$

where $\mathcal{T} : \mathcal{V} \to 2^{\mathcal{V}^*}$ is the operator given by $\mathcal{T} = \mathcal{A} + \mathcal{N}$. It is clear that problems (18) and (15) are equivalent. Now, to prove the existence of solutions to problem (18), it suffices to use Theorem 2.1 and the standard techniques from [26, 33]. □

4 Partial generalized gradient

Let X be a Banach space. We say that a function $\phi : X \to \mathbb{R}$ is regular at x if, for all v, the usual one-sided directional derivative

$$\phi'(x, v) := \lim_{h\downarrow 0} \frac{\phi(x + hv) - \phi(x)}{h}$$

exists and is equal to the generalized directional derivative $\phi^\circ(x; v)$. Let $E = E_1 \times E_2$, where E_1, E_2 are Banach spaces, and let $\phi : E \to \mathbb{R}$ be a locally Lipschitz function. We denote by $\partial_1\phi(x_1, x_2)$ the partial generalized gradient of $\phi(\cdot, x_2)$ at x_1, and by $\partial_2\phi(x_1, x_2)$ that of $\phi(x_1, \cdot)$ at x_2. It is a fact that in general neither of the sets $\partial\phi(x_1, x_2)$ and $\partial_1\phi(x_1, x_2) \times \partial_2\phi(x_1, x_2)$ needs to be contained in the other. To be convinced, it suffices to consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \bigl(x \wedge (-y)\bigr) \vee (y - x).$$

From [8, Example 2.5.2], we have

$$\partial_x f(0, 0) \times \partial_y f(0, 0) \not\subset \partial f(0, 0) \not\subset \partial_x f(0, 0) \times \partial_y f(0, 0).$$

For regular functions, however, a general relationship does hold between these sets. From [8, Proposition 2.3.15], if ϕ is regular at $x = (x_1, x_1) \in E$, then

$$\partial\phi(x_1, x_2) \subset \partial_1\phi(x_1, x_2) \times \partial\phi_2(x_1, x_2), \tag{19}$$

and there is no reason that the equality holds even for regular functions. Next we will give a situation where inclusion (19) holds in a nonregular case, that is, for functions with separated variables.

Consider two locally Lipschitz functions $\phi_1 : E_1 \to \mathbb{R}$ and $\phi_2 : E_2 \to \mathbb{R}$ and define the function $\phi : E = E_1 \times E_2 \to \mathbb{R}$ by

$$\phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2).$$

We have the following result.

Proposition 4.1 *The function ϕ is locally Lipschitz, and for every $(x_1,x_2) \in E$, we have*

$$\partial\phi(x_1,x_2) \subset \partial\phi_1(x_1) \times \partial\phi_2(x_2).$$

Moreover, we have $\partial_k\phi = \partial\phi_k$ with $k = 1,2$. If ϕ_1 is regular at x_1 or ϕ_2 at x_2, then equality holds.

Proof Let $x = (x_1,x_2)$, $y = (y_1,y_2) \in V_1 \times V_2$, where $V_1 \times V_2$ is some neighborhood in E. Then

$$\begin{aligned}
\left|\phi(y) - \phi(x)\right| &\le \left|\phi_1(y_1) - \phi_1(x_1)\right| + \left|\phi_2(y_2) - \phi_2(x_2)\right| \\
&\le K_1\|y_1 - x_1\|_{E_1} + K_2\|y_2 - x_2\|_{E_2} \\
&\le (K_1 + K_2)\sqrt{\|y_1 - x_1\|^2_{E_1} + \|y_2 - x_2\|^2_{E_2}}.
\end{aligned}$$

Let $z = (z_1,z_2) \in \partial j(x_1,x_2)$. Then, for every $v = (v_1,v_2) \in E$, we have by definition

$$\langle z, v\rangle_{E\times E^*} = \langle z_1, v_1\rangle_{E_1\times E_1^*} + \langle z_2, v_2\rangle_{E_2\times E_2^*} \le \phi^\circ(x; v).$$

On the other hand, we get

$$\begin{aligned}
\phi^\circ(x; v) &= \limsup_{(y_1,y_2)\to(x_1,x_2),h\downarrow 0} \frac{\phi((y_1,y_2) + h(v_1,v_2)) - \phi(y_1,y_2)}{h} \\
&= \limsup_{(y_1,y_2)\to(x_1,x_2),h\downarrow 0} \frac{\phi_1(y_1 + hv_1) - \phi_2(y_1) + \phi_2(y_2 + hv_2) - \phi_2(y_2)}{h} \\
&\le \limsup_{y_1\to x_1,h\downarrow 0} \frac{\phi_1(y_1 + hv_1) - \phi_2(y_1)}{h} + \limsup_{y_2\to x_2,h\downarrow 0} \frac{\phi_2(y_2 + hv_2) - \phi_2(y_2)}{h} \\
&= \phi_1^\circ(x_1; v_1) + \phi_2^\circ(x_2; v_2).
\end{aligned}$$

It follows that, for every $(v_1,v_2) \in E$,

$$\langle z_1, v_1\rangle_{E_1\times E_1^*} + \langle z_2, v_2\rangle_{E_2\times E_2^*} \le \phi_1^\circ(x_1; v_1) + \phi_2^\circ(x_2; v_2).$$

We take $v_2 = 0$, then for every $v_1 \in E_1$ we have

$$\langle z_1, v_1\rangle_{E_1\times E_1^*} \le \phi_1^\circ(x_1; v_1),$$

which means that $z_1 \in \partial\phi_1(x_1)$. Analogously, we obtain that $z_2 \in \partial\phi_2(x_2)$. Now, if ϕ_1 is regular at x_1, we have

$$\phi^\circ(x; v) = \phi_1'(x_1; v_1) + \phi_2^\circ(x_2; v_2) = \phi_1^\circ(x_1; v_1) + \phi_2^\circ(x_2; v_2). \tag{20}$$

If $(z_1,z_2) \in \partial\phi_1(x_1) \times \partial\phi_2(x_2)$, then by (20) we get $\langle(z_1,z_2),(v_1,v_2)\rangle_{E\times E^*} = \langle z_1, v_1\rangle_{E_1\times E_1^*} + \langle z_2, v_2\rangle_{E_2\times E_2^*} \le \phi_1^\circ(x_1; v_1) + \phi_2^\circ(x_2; v_2) = \phi^\circ(x; v)$, which means that $(z_1,z_2) \in \partial\phi(x_1,x_2)$. Similarly, if ϕ_2 is regular at x_2, then equality holds. □

Remark 4.2 If ϕ_1 and ϕ_2 are convex, then j is convex and the generalized directional derivative coincides with the one-sided directional derivative, then it is clear that (19) in the general case holds and the equality in Proposition 4.1 holds too.

The well-known Chang's lemma (cf. [7, Example 1]) concerns the calculation of Clarke's gradient of a locally Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \int_0^t \gamma(\xi)\, d\xi,$$

where $\beta \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. Consider the functions

$$\overline{\gamma}_\mu(t) = \operatorname*{ess\,sup}_{|s-t|<\mu} \gamma(s) \quad \text{and} \quad \underline{\gamma}_\mu(t) = \operatorname*{ess\,inf}_{|s-t|<\mu} \gamma(s).$$

They are increasing and decreasing functions of μ, respectively. Therefore, the limits for $\mu \to 0^+$ exist. We denote them by $\overline{\gamma}(t)$ and $\underline{\gamma}(t)$, respectively. Then it is proved by Chang that

$$\partial\phi(t) \subset \bigl[\underline{\gamma}(t), \overline{\gamma}(t)\bigr].$$

If in addition $\gamma(t \pm 0)$ exists for every $t \in \mathbb{R}$, then the equality holds, i.e.,

$$\partial\phi(t) = \bigl[\underline{\gamma}(t), \overline{\gamma}(t)\bigr].$$

Now, let $\gamma_1, \gamma_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and define a locally Lipschitz function $\phi : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\phi(t,s) = \int_0^t \gamma_1(\xi)\, d\xi + \int_0^s \gamma_2(\xi)\, d\xi.$$

Theorem 4.3 (Chang-type lemma) *One has*

$$\partial\phi(t,s) \subset \widehat{\gamma_1}(t) \times \widehat{\gamma_2}(s). \tag{21}$$

If in addition $\gamma_1(t \pm 0)$ *and* $\gamma_2(s \pm 0)$ *exist for every* t, s, *then*

$$\partial\phi(t,s) = \widehat{\gamma_1}(t) \times \widehat{\gamma_2}(s).$$

Proof By Proposition 4.1 and the classical Chang's lemma discussed above, inclusion (21) holds true. If further we suppose that, for $k = 1, 2$, $\gamma_k(t \pm 0)$ exists for each $t \in \mathbb{R}$, then $\underline{\gamma}_k(t) = \min\{\gamma_k(t+0), \gamma_k(t-0)\}$, and $\overline{\gamma}_k(t) = \max\{\gamma_k(t+0), \gamma_k(t-0)\}$. By the definition of Clarke directional derivative at (t,s) in the direction (z_1, z_2), we have

$$\begin{aligned}
\phi^\circ(t,s;z_1,z_2) &= \limsup_{h_1,h_2\to 0,\lambda\downarrow 0} \frac{1}{\lambda}\bigl(\phi(t+h_1+\lambda z_1, s+h_2+\lambda z_2) \\
&\quad - \phi(t+h_1, s+h_2)\bigr) \\
&= \limsup_{h_1,h_2\to 0,\lambda\downarrow 0} \frac{1}{\lambda}\left[\int_{t+h_1}^{t+h_1+\lambda z_1} \gamma_1(\tau)\, d\tau + \int_{s+h_2}^{s+h_2+\lambda z_2} \gamma_2(\tau)\, d\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \lim_{h_1\to 0,\lambda\downarrow 0} \frac{1}{\lambda}\int_{t+h_1}^{t+h_1\lambda z_1} \gamma_1(\tau)\,d\tau + \lim_{h_2\to 0,\lambda\downarrow 0} \frac{1}{\lambda}\int_{s+h_2}^{s+h_2+\lambda z_2} \gamma_2(\tau)\,d\tau \\
&\geq \lim_{h_1\to 0,\lambda\downarrow 0} z_1 \int_0^1 \gamma_1(t+h_1+\lambda z_1\tau)\,d\tau \\
&\quad + \lim_{h_2\to 0,\lambda\downarrow 0} z_2 \int_0^1 \gamma_2(s+h_2+\lambda z_2\tau)\,d\tau \\
&\geq \gamma_1(t\pm 0)z_1 + \gamma_2(s\pm 0)z_2.
\end{aligned}
$$

It follows then that

$$
\big(\gamma_1(t\pm 0), \gamma_1(s\pm 0)\big) \in \partial\phi(t,s) \quad \text{for all } (t,s)\in\mathbb{R}^2.
$$

Since $\partial\phi(t,s)$ is convex, then the equality in (21) holds true. □

5 Galerkin approximation

The aim of this section is to consider the convergence of a numerical approximation constructed by the Galerkin method. This method can also be an alternative way to prove the existence result in Sect. 3 without making use of surjectivity results and pseudomonotone operators theory.

Let $\gamma_1, \gamma_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, and for $k = 1,2$ define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by $\phi(t,s) = \phi_1(t) + \phi_2(t)$ for all $(t,s)\in\mathbb{R}^2$, where

$$
\phi_k(t) = \int_0^t \gamma_k(s)\,ds \quad \text{for all } t\in\mathbb{R}.
$$

Let $p \in C_0^\infty(\mathbb{R})$ be a positive function with support in $[-1,1]$ such that $\int_{\mathbb{R}} p(\xi)\,d\xi = 1$. For $\xi\in\mathbb{R}$ and $\varepsilon > 0$, define the function $p_\varepsilon(\xi) = \frac{1}{\varepsilon}p(\frac{\xi}{\varepsilon})$, and for $k = 1,2$ define

$$
\gamma_{k\varepsilon}(\xi) = \int_{\mathbb{R}} p_\varepsilon(\eta)\gamma_k(\xi-\eta)\,d\eta, \quad \xi\in\mathbb{R}, \varepsilon > 0.
$$

We consider a Galerkin basis $\{Z_1, Z_2, \dots\}$ of $\mathbb{V}$, and let $\mathbb{V}_m = \operatorname{span}\{Z_1, Z_2, \dots, Z_m\}$ be the resulting m-dimensional subspaces. Let $\{U_{m0}\}_m$ be an approximation of the initial value U_0 such that $U_{m0}\in\mathbb{V}_m$, $U_{m0}\to U_0$ in $\mathbb{H}$ and $\{U_{m0}\}_m$ is bounded in $\mathbb{V}$. Let $\{\varepsilon_m\}_m$ be a sequence of real numbers converging to zero as $m\to\infty$. Instead of $\gamma_{k\varepsilon_m}$ we will write γ_{km} and we will use the notation

$$
\gamma_m(t,s) = \big(\gamma_{1m}(t), \gamma_{2m}(s)\big) \quad \forall (t,s)\in\mathbb{R}^2.
$$

We consider the following regularized Galerkin system of finite dimensional differential equations: find $U_m = (u_m, u_{m|\Gamma}) \in L^2(0,T;\mathbb{V}_m)$ with $U_m' \in L^2(0,T;\mathbb{V}_m)$ such that

$$
\begin{cases}
\langle U_m'(t) + AU_m(t), V\rangle + \langle \gamma_m(U_m), V\rangle_{\mathbb{H}} = \langle f(t), V\rangle, \\
U_m(0) = U_{m0},
\end{cases}
\tag{22}
$$

for a.e. $t \in (0,T)$ and for all $V = (v, v_{|\Gamma}) \in \mathbb{V}$. Problem (22) can be written more explicitly as follows:

$$\begin{cases} \langle U_m'(t) + AU_m(t), V\rangle + \int_\Omega \gamma_{1m}(u_m(t)).v\,dx + \int_\Gamma \gamma_{2m}(u_m(t)).v\,d\sigma = \langle f(t), V\rangle, \\ U_m(0) = U_{m0}, \end{cases} \tag{23}$$

for a.e. $t \in (0,T)$ and for all $V \in \mathbb{V}$.

For the existence of solutions, we will need the following hypothesis $H(\gamma)$: for $k = 1, 2$, assume that

(Chang condition) $\gamma_k \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, $\gamma_k(t \pm 0)$ exists for any $t \in \mathbb{R}$.

(Growth condition) for all $t \in \mathbb{R}$ we have

$$\left|\gamma_k(t)\right| \le c_k\left(1 + |t|^{\theta_k}\right),$$

with $c_k > 0$ and $0 \le \theta_k \le 1$.

Theorem 5.1 *Let $H(\gamma)$ hold. Moreover, assume that one of the following situations holds*:
(1) $\theta_1, \theta_2 < 1$,
(2) $\theta_1 < 1$ *and* $\theta_2 = 1$ *provided* $c_2 < \frac{M}{2\sqrt{2}}$,
(3) $\theta_2 < 1$ *and* $\theta_1 = 1$ *provided* $c_1 < \frac{M}{2\sqrt{2}}$,
(4) $\theta_1 = \theta_2 = 1$ *provided* $c_1 + c_2 < \frac{M}{2\sqrt{2}}$,
where M is the coercivity constant of the operator A. Then problem (15) *has at least one solution.*

Proof We substitute $U_m(t) = \sum_{k=1}^m c_{km}(t)Z_k$ in (23) to obtain an initial value problem for a system of first order ordinary differential equations for c_{km}, $k = 1, \dots, m$, where the initial values $c_{km}(0)$ are given by $U_{m0} = \sum_{k=1}^m c_{km}(0)Z_k$. From the Caratheodory theorem, the solution U_m exists on $[0, t_{\max})$, and we can extend it on the closed interval $[0,T]$ by using a priori estimates below. By replacing V with U_m in (23), we get for a.e. $t \in (0,T)$

$$\left\langle U_m'(t) + AU_m(t), U_m(t)\right\rangle + \int_\Omega \gamma_{1m}(u_m).u_m\,dx + \int_\Gamma \gamma_{2m}(u_m).u_m\,d\sigma = \left\langle f(t), U_m(t)\right\rangle.$$

Using the coercivity of A and the Young inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|U_m(t)\right|^2 + M\left\|U_m(t)\right\|^2 + \left\langle\gamma_m\left(U_m(t)\right), U_m(t)\right\rangle_{\mathbb{H}} \\ &\quad\le \frac{1}{2M}\left\|f(t)\right\|^2_{\mathbb{V}^*} + \frac{M}{2}\left\|U_m(t)\right\|^2 \end{aligned} \tag{24}$$

for a.e. $t \in (0,T)$. Consequently,

$$\frac{1}{2}\frac{d}{dt}\left|U_m(t)\right|^2 + \frac{M}{2}\left\|U_m(t)\right\|^2 + \left\langle\gamma_m\left(U_m(t)\right), U_m(t)\right\rangle_{\mathbb{H}} \le \frac{1}{2M}\left\|f(t)\right\|^2_{\mathbb{V}^*}. \tag{25}$$

Integrating over $(0,t)$, we get

$$\begin{aligned} &\frac{1}{2}\left|U_m(t)\right|^2 - \frac{1}{2}|U_{0m}|^2 + \frac{M}{2}\int_0^t \left\|U_m(s)\right\|^2 ds \\ &\quad+ \int_0^t \left\langle\gamma_m\left(U_m(s)\right), U_m(s)\right\rangle_{\mathbb{H}} ds \le \frac{1}{2M}\int_0^t \left\|f(s)\right\|^2_{\mathbb{V}^*} ds. \end{aligned} \tag{26}$$

From the growth condition, we have

$$\left|\gamma_{k\varepsilon}(s)\right| \le c_k\left(1+|s|^{\theta_k}\right)$$

for $k = 1, 2$ and $s \in \mathbb{R}$. It follows that

$$\begin{aligned}\int_\Omega \left|\gamma_{1m}\left(u_m(s,x)\right)\right|^2 dx &\le c_1 \int_\Omega \left(1+\left|u_m(s,x)\right|^{\theta_1}\right)^2 dx \\ &\le 2c_1^2 \int_\Omega \left(1+\left|u_m(s,x)\right|^{2\theta_1}\right) dx \\ &\le 2c_1^2\lambda_N(\Omega) + 2c_1^2\lambda_N(\Omega)^{1-\theta_1}\left\|u_m(t)\right\|_{L^2(\Omega)}^{2\theta_1} \\ &\le 2c_1^2\lambda_N(\Omega) + 2c_1^2\lambda_N(\Omega)^{1-\theta_1}\left\|U_m(t)\right\|^{2\theta_1}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left\|\gamma_{1m}(u_m)\right\|_{L^2(0,t;L^2(\Omega))}^2 &= \int_0^t \left\|\gamma_{1m}\left(u_m(\tau)\right)\right\|_{L^2(\Omega)}^2 d\tau \\ &\le 2tc_1^2\lambda_N(\Omega) + 2c_1^2\lambda_N(\Omega)^{1-\theta_1}\int_0^t \left\|U_m(\tau)\right\|^{2\theta_1} d\tau \\ &\le 2tc_1^2\lambda_N(\Omega) + 2c_1^2\lambda_N(\Omega)^{1-\theta_1} t^{1-\theta_1}\|U_m\|_{L^2(0,t;\mathbb{V})}^{2\theta_1}.\end{aligned}$$

It then follows that

$$\left\|\gamma_{1m}(u_m)\right\|_{L^2(0,t;L^2(\Omega))} \le a_1 + a_1'\|U_m\|_{L^2(0,t;\mathbb{V})}^{\theta_1},$$

with $a_1 = c_1\sqrt{2t\lambda_N(\Omega)}$ and $a_1' = c_1\sqrt{2\lambda_N(\Omega)^{1-\theta_1}t^{1-\theta_1}}$. This leads to

$$\begin{aligned}\left|\int_0^t \left\langle\gamma_{1m}(u_m(\tau), u_m(\tau)\right\rangle_{L^2(\Omega)\times L^2(\Omega)} d\tau\right| &\le \int_0^t \left\|\gamma_{1m}\left(u_m(\tau)\right)\right\|_{L^2(\Omega)}\left\|u_m(\tau)\right\|_{L^2(\Omega)} d\tau \\ &\le \left\|\gamma_{1m}(u_m)\right\|_{L^2(0,t;L^2(\Omega))}\|u_m\|_{L^2(0,t;L^2(\Omega))} \\ &\le \left(a_1 + a_1'\|U_m\|_{L^2(0,t;\mathbb{V})}^{\theta_1}\right)\|U_m\|_{L^2(0,t;\mathbb{V})} \\ &\le a_1\|U_m\|_{L^2(0,t;\mathbb{V})} + a_1'\|U_m\|_{L^2(0,t;\mathbb{V})}^{1+\theta_1}.\end{aligned}$$

With same calculus we obtain

$$\left|\int_0^t \left\langle\gamma_{2m}(u_m(\tau), u_m(\tau)\right\rangle_{L^2(\Gamma)\times L^2(\Gamma)} d\tau\right| \le a_2\|U_m\|_{L^2(0,t;\mathbb{V})} + a_2'\|U_m\|_{L^2(0,t;\mathbb{V})}^{1+\theta_2} \tag{27}$$

with $a_2 = c_2\sqrt{2t\sigma(\Gamma)}$ and $a_2' = c_2\sqrt{2\sigma(\Omega)^{1-\theta_2}t^{1-\theta_2}}$. It follows that

$$\begin{aligned}&\frac{1}{2}\left|U_m(t)\right|^2 + \frac{M}{2}\int_0^t \left\|U_m(s)\right\|^2 ds \\ &\quad\le \frac{1}{2M}\|f\|_{L^2(0,t;\mathbb{V}^*)}^2 + \frac{1}{2}|U_{0m}|^2 \\ &\qquad + a_1\|U_m\|_{L^2(0,t;\mathbb{V})} + a_1'\|U_m\|_{L^2(0,t;\mathbb{V})}^{1+\theta_1} + a_2\|U_m\|_{L^2(0,t;\mathbb{V})} + a_2'\|U_m\|_{L^2(0,t;\mathbb{V})}^{1+\theta_2},\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{1}{2}\big|U_m(t)\big|^2 + \frac{M}{2}\|U_m\|^2_{L^2(0,t;\mathbb{V})} \\
&\quad\leq \frac{1}{2M}\|f\|^2_{L^2(0,t;\mathbb{V}^*)} + \frac{1}{2}|U_{0m}|^2 \\
&\qquad + (a_1 + a_2)\|U_m\|_{L^2(0,t;\mathbb{V})} + a_1'\|U_m\|^{1+\theta_1}_{L^2(0,t;\mathbb{V})} + a_2'\|U_m\|^{1+\theta_2}_{L^2(0,t;\mathbb{V})}.
\end{aligned}$$

If $\theta_1, \theta_2 < 1$, it is clear that $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$ with no additional conditions. Now, if $\theta_1 < 1$ and $\theta_2 = 1$, then we have

$$\begin{aligned}
&\frac{1}{2}\big|U_m(t)\big|^2 + \left(\frac{M}{2} - a_2'\right)\|U_m\|^2_{L^2(0,t;\mathbb{V})} \\
&\quad\leq \frac{1}{2M}\|f\|^2_{L^2(0,t;\mathbb{V}^*)} + \frac{1}{2}|U_{0m}|^2 \\
&\qquad + (a_1 + a_2)\|U_m\|_{L^2(0,t;\mathbb{V})} + a_1'\|U_m\|^{1+\theta_1}_{L^2(0,t;\mathbb{V})}.
\end{aligned}$$

As $1+\theta_1 < 2$, it follows that $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$ provided $\frac{M}{2} - a_2' > 0$. Similarly, if $\theta_1 = 1$ and $\theta_2 < 1$, it then follows that $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$ provided $\frac{M}{2} - a_1' > 0$. If $\theta_1 = \theta_2 = 1$, then we get

$$\begin{aligned}
&\frac{1}{2}\big|U_m(t)\big|^2 + \left(\frac{M}{2} - a_1' - a_2'\right)\|U_m\|^2_{L^2(0,t;\mathbb{V})} \\
&\quad\leq \frac{1}{2M}\|f\|^2_{L^2(0,t;\mathbb{V}^*)} + \frac{1}{2}|U_{0m}|^2 \\
&\qquad + (a_1 + a_2)\|U_m\|_{L^2(0,t;\mathbb{V})}.
\end{aligned}$$

It then follows that $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$ provided $\frac{M}{2} - a_1' - a_2' > 0$. As a summary, we conclude that $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$ provided one of the following situations holds:

(1) $\theta_1, \theta_2 < 1$,
(2) $\theta_1 < 1$ and $\theta_2 = 1$ provided $c_2 < \frac{M}{2\sqrt{2}}$,
(3) $\theta_2 < 1$ and $\theta_1 = 1$ provided $c_1 < \frac{M}{2\sqrt{2}}$,
(4) $\theta_1 = \theta_2 = 1$ provided $c_1 + c_2 < \frac{M}{2\sqrt{2}}$.

When $\{U_m\}_m$ is bounded in $L^2(0,T;\mathbb{V})$, then it is also bounded in $L^\infty(0,T;\mathbb{H})$, so passing to a subsequence, if necessary, we have

$$U_m \to U \quad \text{weakly in } L^2(0,T;\mathbb{V}) \text{ and weakly } -^* \text{ in } L^\infty(0,T;\mathbb{H}),$$

where $U \in L^2(0,T;\mathbb{V}) \cap L^\infty(0,T;\mathbb{H})$. From the above estimates we have also that $\{U_m'\}_m$ is bounded in $L^2(0,T;\mathbb{V}^*)$. Thus, by passing to a subsequence, if necessary, we get

$$U_m \to U \quad \text{weakly in } \mathcal{W} \text{ with } U \in \mathcal{W}.$$

On the other hand, as U_m converges weakly to U in $L^2(0,T;\mathbb{V})$ and $\mathbb{V} \subset \mathbb{H}$ compactly, then U_m converges to U in $L^2(0,T;\mathbb{H})$, which means that

$$u_m \to u \quad \text{in } L^2\big(0,T;L^2(\Omega)\big) \quad \text{and} \quad u_{m|\Gamma} \to u_{|\Gamma} \quad \text{in } L^2\big(0,T;L^2(\Gamma)\big).$$

Since the mapping $\mathcal{W} \ni W \mapsto W(0) \in \mathbb{H}$ is linear and continuous, we have $U_m(0) \to U(0)$ weakly in $\mathbb{H}$, which together with $U_{m0} \to U_0$ entails $U(0) = U_0$. Let now $V \in \mathbb{V}$ and denote $\Psi_m(t,x) = \psi(t)V_m(x)$ where $\psi \in C_0^\infty(0,T)$ and $V_m \in \mathbb{V}_m$ is such that $V_m \to V$ in $\mathbb{V}$, we have $\Psi_m \to \Psi$ in $\mathcal{W}$ with $\Psi(t,x) = \psi(t)V(x)$. It follows that

$$\int_0^T \langle U'(t) + AU(t), \Psi_m(t)\rangle\, dt + \int_0^T \langle \gamma_m(U_m(t)), \Psi_m(t)\rangle_{\mathbb{H}}\, dt = \int_0^T \langle f(t), \Psi_m(t)\rangle\, dt.$$

Passing to the limit and remarking that ψ is chosen arbitrary, we deduce that

$$\langle U'(t) + AU(t), V\rangle + \langle \xi(t), V\rangle_{\mathbb{H}} = \langle f(t), V\rangle,$$

where $\xi = (\xi_1, \xi_2)$. It remains to prove that $\xi_1 \in \widehat{\gamma}_1(u(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega$ and $\xi_2 \in \widehat{\gamma}_2(u_{|\Gamma}(t,x))$ for a.e. $(t,x) \in (0,T) \times \Gamma$. We apply the convergence theorem of Aubin and Cellina [3] to the multifunctions $\partial\phi_1$ and $\partial\phi_2$. First, we observe that $\partial\phi_1, \partial\phi_2 : \mathbb{R} \to 2^{\mathbb{R}}$ are upper semicontinuous. Since $u_m \to u$ in $L^2(0,T;L^2(\Omega))$ and $u_{m|\Gamma} \to u_{|\Gamma}$ in $L^2(0,T;L^2(\Gamma))$, then by the definition of $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$, we deduce that, for a.e $(t,x) \in (0,T) \times \Omega$ and for every neighborhood $\mathcal{N}$ of zero in $\mathbb{R}^2$, there exists $n_0 = n_0(t,x,\mathcal{N}) \in \mathbb{N}$ such that

$$\bigl(u_m(t,x), \gamma_{1m}(u_m(t,x))\bigr) \in Gr\partial\phi_1 + \mathcal{N} \quad \text{for all } n \geq n_0.$$

By passing to the limit we get

$$\xi_1(t,x) \in \overline{\operatorname{conv}}\partial\phi_1\bigl(u(t,x)\bigr) = \partial\phi_1\bigl(u(t,x)\bigr)$$

for a.e. $(t,x) \in (0,T) \times \Omega$. Analogously, we get

$$\xi_2(t,x) \in \partial\phi_2\bigl(u_{|\Gamma}(t,x)\bigr)$$

for a.e. $(t,x) \in (0,T) \times \Gamma$, which completes the proof. □

6 Concluding remarks

In this paper, we introduced a new class of hemivariational inequalities, namely dynamic boundary hemivariational inequalities. It concerns dynamic boundary conditions with a Clarke subdifferential perturbation on the boundary. The suitable framework to study such problems is to work on a product space instead of the state space itself. We chose to work with a dynamic boundary condition in its simplest form, but we nevertheless could work with a general uniformly elliptic operator or even in the L^p framework with the p-Laplacian. Moreover, with some changes on the choice of the product spaces, one can incorporate the Laplace–Beltrami operator on the boundary in addition to the usual Laplacian. On the other hand, one can replace the growth condition in Sect. 6 with the Rauch condition expressing the ultimate increase of the graphs of functions β_k.

As a continuation of this paper we aim to study the abstract version of the current work and to look at hemivariational inequalities that can be formulated in terms of matrix operators on product spaces. Indeed, this formulation covers a wide range of examples, including second order problems, equations with delay, equations that are memory dependent.

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References

1. Amann, H., Escher, J.: Strongly continuous dual semigroups. Ann. Mat. Pura Appl. (4) **171**, 44–62 (1996)
2. Arendt, W., Batty, C., Hieber, M., Neubrander, F.: Vector-Valued Laplace Transforms and Cauchy Problems. Birkhäuser, Basel (2001)
3. Aubin, J.P., Cellina, A.: Differential Inclusions, Set-Valued Maps and Viability Theory. Springer, Berlin (1984)
4. Barboteu, M., Gasiński, L., Kalita, P.: Analysis of a dynamic frictional contact problem for hyperviscoelastic material with non-convex energy density. Math. Mech. Solids **23**(3), 359–391 (2018)
5. Bartosz, K., Gasiński, L., Liu, Z., Szafraniec, P.: Convergence of a time discretization for a nonlinear second-order inclusion. Proc. Edinb. Math. Soc. **61**, 93–120 (2018)
6. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications. Springer, New York (2007)
7. Chang, K.C.: The obstacle problem and partial differential equations with discontinuous non-linearities. Commun. Pure Appl. Math. **33**, 117–146 (1980)
8. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
9. Coclite, G.M., Favini, A., Gal, C.G., Goldstein, G.R., Goldstein, J.A., Obrecht, E., Romanelli, S.: The role of Wentzell boundary conditions in linear and nonlinear analysis. In: Advances in Nonlinear Analysis: Theory, Methods and Applications. Math. Probl. Eng. Aerosp. Sci., vol. 3, pp. 277–289. Camb. Sci. Publ., Cambridge (2009)
10. Crank, J.: The Mathematics of Diffusion. Clarendon, Oxford (1975)
11. Favini, A., Ruiz Goldstein, G., Goldstein, J.A., Romanelli, S.: The heat equation with general Wentzell boundary conditions. J. Evol. Equ. **2**, 1–19 (2002)
12. Favini, A., Ruiz Goldstein, G., Goldstein, J.A., Romanelli, S.: The heat equation with nonlinear general Wentzell boundary condition. Adv. Differ. Equ. **11**, 481–510 (2006)
13. Gal, C.G.: On a class of degenerate parabolic equations with dynamic boundary conditions. J. Differ. Equ. **253**(1), 126–166 (2012)
14. Gal, C.G.: The role of surface diffusion in dynamic boundary conditions: where do we stand? Milan J. Math. **83**, 237–278 (2015)
15. Gal, C.G., Warma, M.: Well-posedness and the global attractor of some quasi-linear parabolic equations with nonlinear dynamic boundary conditions. Differ. Integral Equ. **23**(3/4), 327–358 (2010)
16. Gasiński, L., Kalita, P.: On quasi-static contact problem with generalized Coulomb friction, normal compliance and damage. Eur. J. Appl. Math. **27**(4), 625–646 (2016)
17. Gasiński, L., Ochal, A., Shillor, M.: Variational-hemivariational approach to a quasistatic viscoelastic problem with normal compliance, friction and material damage. Z. Anal. Anwend. **34**(3), 251–275 (2015)
18. Gasiński, L., Ochal, A., Shillor, M.: Quasistatic thermoviscoelastic problem with normal compliance, multivalued friction and wear diffusion. Nonlinear Anal., Real World Appl. **27**, 183–202 (2016)
19. Goldstein, G.R.: Derivation of dynamical boundary conditions. Adv. Differ. Equ. **11**, 457–480 (2006)
20. Greiner, G.: Perturbing the boundary conditions of a generator. Houst. J. Math. **13**, 213–229 (1987)
21. Langer, R.E.: A problem in diffusion or in the flow of heat for a solid in contact with a fluid. Tohoku Math. J. **35**, 260–275 (1932)
22. Mahdioui, H., Ben Aadi, S., Akhlil, K.: Hemivariational inequality for Navier–Stokes equations: existence, dependence, and optimal control. Bull. Iran. Math. Soc. (2020). https://doi.org/10.1007/s41980-020-00470-x
23. Mahdioui, H., Ben Aadi, S., Akhlil, K.: Weak solutions and optimal control of hemivariational evolutionary Navier–Stokes equations under rauch condition. J. Funct. Spaces **2020**, Article ID 6573219 (2020)
24. March, H.W., Weaver, W.: The diffusion problem for a solid in contact with a stirred fluid. Phys. Rev. **31**, 1072–1082 (1928)
25. Migorski, S.: Evolution hemivariational inequalities in infinite dimension and their control. Nonlinear Anal., Theory Methods Appl. **47**(1), 101–112 (2001)
26. Migorski, S., Ochal, A.: Boundary hemivariational inequality of parabolic type. Nonlinear Anal. **57**, 579–596 (2004)
27. Minty, G.J.: Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29**, 341–346 (1962)
28. Mugnolo, D., Romanelli, S.: Dirichlet forms for general Wentzell boundary conditions, analytic semigroups, and cosine operator functions. Electron. J. Differ. Equ. **118**, 1 (2006)

29. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Birkhäuser, Basel (1985) Russian translation, MIH Publisher, Moscow, 1989
30. Panagiotopoulos, P.D.: Hemivariational inequalities and their applications. In: Moreau, J.J., Panagiotopoulos, P.D., Strang, G. (eds.) Topics in Nonsmooth Mechanics. Birkhäuser, Basel (1988)
31. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functions. Springer, Berlin (2012)
32. Papageorgiou, N.S., Papalini, F., Renzacci, F.: Existence of solutions and periodic solutions for nonlinear evolution inclusions. Rend. Circ. Mat. Palermo **48**, 341–364 (1999)
33. Park, J.Y., Ha, T.G.: Existence of antiperiodic solutions for hemivariational inequalities. Nonlinear Anal., Theory Methods Appl. **68**(4), 747–767 (2008)
34. Rockafellar, R.T.: Generalized derivatives and subgradients of nonconvex functions. Can. J. Math. **32**, 257–280 (1980)
35. Vogt, H., Voigt, J.: Wentzell boundary conditions in the context of Dirichlet forms. Adv. Differ. Equ. **8**(7), 821–842 (2013)
36. Warma, M.: Quasilinear parabolic equations with nonlinear Wentzell–Robin type boundary conditions. J. Math. Anal. Appl. **336**, 1132–1148 (2007)
37. Warma, M.: Regularity and well-posedness of some quasi-linear elliptic and parabolic problems with nonlinear general Wentzell boundary conditions on nonsmooth domains. Nonlinear Anal. **75**, 5561–5588 (2012)
38. Warma, M.: Semilinear parabolic equations with nonlinear general Wentzell boundary conditions. Discrete Contin. Dyn. Syst. **33**, 5493–5506 (2013)
39. Wentzell, A.D.: On boundary conditions for multidimensional diffusion processes. Theory Probab. Appl. **4**, 164–177 (1959)
40. Zeidler, E.: Nonlinear Functional Analysis and Applications II A/B. Springer, New York (1990)