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Linear stability of blowup solution of incompressible Keller–Segel–Navier–Stokes system

Yan Yan[1*] and Hengyan Li[2*]

*Correspondence:
yanyan_1984@163.com;
lihengyan@ncwu.edu.cn
[1]School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou, P.R. China
[2]School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, P.R. China

Abstract

In this paper, we consider the linear stability of blowup solution for incompressible Keller–Segel–Navier–Stokes system in whole space $\mathbb{R}^3$. More precisely, we show that, if the initial data of the three dimensional Keller–Segel–Navier–Stokes system is close to the smooth initial function $(0, 0, \mathbf{u}_s(0, x))^T$, then there exists a blowup solution of the three dimensional linear Keller–Segel–Navier–Stokes system satisfying the decomposition

$$(n(t,x), c(t,x), \mathbf{u}(t,x))^T = (0, 0, \mathbf{u}_s(t,x))^T + \mathcal{O}(\varepsilon), \quad \forall (t,x) \in (0, T^*) \times \mathbb{R}^3,$$

in Sobolev space $H^s(\mathbb{R}^3)$ with $s = \frac{3}{2} - 5a$ and constant $0 < a \ll 1$, where T^* is the maximal existence time, and $\mathbf{u}_s(t,x)$ given in (Yan 2018) is the explicit blowup solution admitted smooth initial data for three dimensional incompressible Navier–Stokes equations.

MSC: 35K15; 74H35; 35B35

Keywords: Keller–Segel–Navier–Stokes system; Blowup solution; Stability

1 Introduction and main results

The Keller–Segel system coupled to the incompressible Navier–Stokes equations arises from a biological process in which cells move towards a chemically more favorable environment [20]. In this paper, we consider the blowup analysis for the three dimensional Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + \mathbf{u} \cdot \nabla n = \triangle n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, \\ c_t + \mathbf{u} \cdot \nabla c = \triangle c - c + n, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \triangle \mathbf{u} - \nabla P + n \nabla \Phi, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.1}$$

where $\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $\sigma \in \mathbb{R}$ and $\mu > 0$ are given parameters, the constant χ is the chemotactic sensitivity. We assume $\chi > 0$ presuming that cells move toward increasing concentrations of the signal substance which is produced by themselves. The unknown scalar functions $n(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ and $c(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ denote the population density and the signal concentration, respectively, the vector function $\mathbf{u}(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ denotes the 3 D velocity field of the fluid, $P(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ stands for the pressure in the fluid. Moreover, the pressure $P(t,x)$ is determined by the formula

$$P(t,x)=-\triangle^{-1}\left(\sum_{i,j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}-\nabla\cdot(n\nabla\Phi)\right). \tag{1.2}$$

$\Phi(x)$ is a given potential function accounting for the effects of external forces such as gravity. The divergence free condition in the last equation of (1.1) guarantees the incompressibility of the fluid.

Chemotaxis, as a mechanism of the partially-oriented movement of cells in response to a chemical signal, plays an important role in various biological process [3, 11]. Tuval et al. [26] observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid–air interface. They established a mathematical model, the so-called chemotaxis-fluid system, to describe the dynamics of swimming bacteria, Bacillus subtilis. It has the form

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\triangle n-\nabla\cdot(n\chi(c)\nabla c),\\ c_t+\mathbf{u}\cdot\nabla c=\triangle c-n\kappa(c),\\ \mathbf{u}_t+\kappa(\mathbf{u}\cdot\nabla\mathbf{u})=\triangle\mathbf{u}-\nabla P+n\nabla\Phi,\\ \nabla\cdot\mathbf{u}=0,\quad\forall(t,x)\in\mathbb{R}^+\times\mathcal{D},\end{cases}$$

where $\mathcal{D}$ is a smooth bounded domain or $\mathbb{R}^3$. Lorz [19] first showed the local existence of weak solutions for the two and three dimensional system (1.1) with non-flux and inhomogeneous Dirichlet boundary conditions, respectively. Chae-Kang-Lee [6] gave the local existence of the smooth solution in the cases of two and three dimensions and global existence of classical solution for the two dimensional case for system (1.1). Winkler [31] asserted that a solution of the two dimensional chemotaxis-Navier–Stokes system in a bounded convex domain Ω stabilizes to the spatially uniform equilibrium $(\frac{1}{|\Omega|}\int_\Omega n_0(x)dx,0,0)^T$ in $\mathbb{L}^\infty(\Omega)$. Zhang–Zheng [40] used a microlocal analysis to obtain the global existence and uniqueness of weak solutions for a two dimensional chemotaxis-Navier–Stokes system in $\mathbb{R}^3$ for a large class of initial data. One can also refer to [7, 18, 28, 29] for more results on this two dimensional model. For the spatially three dimensional case, Tao–Winkler [25] constructed locally bounded global solutions in a chemotaxis-Stokes system with nonlinear diffusion. Winkler [33] established the global weak existence theory to a chemotaxis-Navier–Stokes system in a smooth bounded convex domain under a homogeneous Neumann boundary condition. With some relaxation time, Winkler [34] has shown this model admitted eventual energy solutions, meanwhile, he got such eventual energy solutions $(n,c,\mathbf{u})\to(\bar{n}_0,0,0)$ uniformly in a smooth bounded convex domain after the waiting time. Due to spatially limitation, we do not list all of the interesting papers on this kind of models.

Since the Keller–Segel–Navier–Stokes system plays an important role in bioconvection processes, it has attracted great interest also at the level of mathematical theory [2, 3].

Wang [27] showed the global existence of weak solutions for a three dimensional Keller–Segel–Navier–Stokes system with subcritical sensitivity. Winkler [35] proved that the three dimensional model with logistic source admits global weak solutions and asymptotic stabilization in a smooth bounded convex domain with homogeneous Dirichlet boundary conditions. At present, there are extreme difficulties to study the properties of solutions for system (1.1). One of the reasons is the question of finite time singularity/global regularity for three dimensional incompressible Navier–Stokes equations, one of the most important open problems in mathematical fluid mechanics [9]. On the one hand, even if we get rid of the effect of a fluid ($\mathbf{u} \equiv 0$), the question whether blowup may occur in the three dimensional Keller–Segel system

$$\begin{cases} n_t = \triangle n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ c_t = \triangle c - c + n, & (t,x) \in \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.3}$$

with small positive constant μ is still open [35]. Here Ω is a smooth bounded domain in $\mathbb{R}^3$. Lankeit [15] got the global existence of weak solutions for system (1.3). Winkler [30] proved that system (1.3) can lead to a finite time explosion even for the case $\sigma = \mu = 0$. One can refer to [12–14, 32] for more results on this kind of models.

On the other hand, if we set $n(t,x) = c(t,x) \equiv 0$, then the Keller–Segel–Navier–Stokes system (1.1) is reduced to the $3D$ incompressible Navier–Stokes equations

$$\begin{aligned} &\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \triangle \mathbf{u} - \nabla P, \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{1.4}$$

For these equations, Leray [16] showed there is global-forward-in-time weak solution of the initial value problem. After that, non-existence of finite energy self-similar blowup solutions in $\mathbb{L}^3(\mathbb{R}^3)$ was obtained in [22]. As is well known, Eq. (1.4) admits a simplest nontrivial stationary solution with infinite energy, i.e. the Couette flow $(y,0,0)^T$. Bedrossian–Germain–Masmoudi [1] proved the nonlinear stability of this flow. One can refer to [4, 5, 8, 10, 17, 21, 24, 39] for more related results. Yan [36, 37] found that the three dimensional Navier–Stokes equation (1.4) admits a family of stable explicit blowup solutions with infinite energy,

$$\begin{aligned} &\mathbf{u}_s(t,x) = \left(\frac{ax_1}{T^* - t} + kx_2 \left(T^* - t\right)^{2a}, \frac{ax_2}{T^* - t} - kx_1 \left(T^* - t\right)^{2a}, -\frac{2ax_3}{T^* - t} \right)^T, \\ &\quad (t,x) \in [0, T^*) \times \mathbb{R}^3, \end{aligned} \tag{1.5}$$

with the smooth initial data

$$\mathbf{u}_s(0,x) = \left(\frac{ax_1}{T^*} + kx_2 \left(T^*\right)^{2a}, \frac{ax_2}{T^*} - kx_1 \left(T^*\right)^{2a}, -\frac{2ax_3}{T^*} \right)^T,$$

where we have the constants $a, k \in \mathbb{R}/\{0\}$, and the positive constant T^* is the maximal existence time. Moreover, the pressure P is determined by the formula

$$P(t,x) = -\triangle^{-1} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}.$$

We supplement the 3 D incompressible Keller–Segel–Navier–Stokes equations (1.1) with the initial data

$$\begin{cases} n(0,x) = n_0(x), & x \in \mathbb{R}^3, \\ c(0,x) = c_0(x), & x \in \mathbb{R}^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.6}$$

We now state the main result.

Theorem 1.1 *Let $0 < a \ll \frac{1}{8}$ and $0 < s < \frac{3}{2} - 5a$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+3}(\mathbb{R}^3)} \lesssim R \ll 1$. The three dimensional Keller–Segel–Navier–Stokes system (1.1) admits a family of linearly stable blowup solutions*

$$\big(n(t,x), c(t,x), \boldsymbol{u}(t,x)\big)^T = \big(0, 0, \boldsymbol{u}_s(t,x)\big)^T. \tag{1.7}$$

Throughout this paper, we denote the usual norm of $\mathbb{L}^2(\mathbb{R}^3)$ and Sobolev space $\mathbb{H}^s(\mathbb{R}^3)$ by $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$ for $s \in \mathbb{R}^+$, respectively. In fact, the Sobolev space can be equivalent to be defined via Fourier transformation, thus the fractional case is contained. The norm of the Sobolev space $H^s(\mathbb{R}^3) := (\mathbb{H}^s(\mathbb{R}^3))^3$ is denoted by $\|\cdot\|_{H^s}$. The space $\mathbb{L}^2([0,T^*); H^s(\mathbb{R}^3))$ is equipped with the norm

$$\|u\|^2_{\mathbb{L}^2((0,T^*);H^s(\mathbb{R}^3))} := \int_0^{T^*} \big\|u(t,\cdot)\big\|^2_{H^s}\, dt.$$

We also introduce the function spaces

$$\mathcal{C}^s_0 := \bigcap_{i=0}^{1} \mathbb{C}^i\big((0,T^*); \mathbb{H}^{s-i}\big(\mathbb{R}^3\big)\big),$$
$$\overline{\mathcal{C}}^s_0 := \bigcap_{i=0}^{1} \mathbb{C}^i\big((0,T^*); H^{s-i}\big(\mathbb{R}^3\big)\big),$$

with the norm

$$\|u\|^2_{\mathcal{C}^s_1} := \sup_{t\in(0,T^*)} \sum_{i=0}^{1} \big\|\partial_t^i u\big\|^2_{\mathbb{H}^{s-i}},$$
$$\|u\|^2_{\overline{\mathcal{C}}^s_1} := \sup_{t\in(0,T^*)} \sum_{i=0}^{1} \big\|\partial_t^i u\big\|^2_{H^{s-i}},$$

respectively. The symbol $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. $(a,b,c)^T$ denotes the column vector in $\mathbb{R}^3$. The letter C with subscripts to denote dependencies stands for a positive constant that might change its value at each occurrence.

2 Proof of Theorem 1.1

The linear stability of a blowup solution is equivalent to the well-posedness of linearized equations in $\mathbb{R}^3$. For any $(t,x) \in (0,T^*) \times \mathbb{R}^3$, we recall the perturbation equations

$$\begin{cases} n_t + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla n = \triangle n - \chi \nabla \cdot (n \nabla c) + \sigma n - \mu n^2, \\ c_t + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla c = \triangle c - c + n, \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \overline{P} + \nu \triangle \mathbf{v} + n \nabla \Phi, \\ \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{2.1}$$

with initial data

$$\begin{cases} n(0,x) = n_0(x), \quad x \in \mathbb{R}^3, \\ c(0,x) = c_0(x), \quad x \in \mathbb{R}^3, \\ \mathbf{v}(0,x) = \mathbf{v}_0(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{2.2}$$

and the boundary condition

$$\begin{cases} \lim_{|x| \to +\infty} n(t,x) = 0, \\ \lim_{|x| \to +\infty} c(t,x) = 0, \\ \lim_{|x| \to +\infty} \mathbf{v}(t,x) = 0, \end{cases} \tag{2.3}$$

where $(t,x) \in (0,T^*) \times \mathbb{R}^3$, the pressure $\overline{P}$ satisfies

$$\begin{aligned} \overline{P}(t,x) = & -\triangle^{-1} \Bigg(\sum_{k=1}^{3} \left(\frac{\partial v_k}{\partial x_k} \right)^2 + 2k\big(T^* - t\big)^{2a} \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) + 2 \frac{\partial v_1}{\partial x_2} \frac{\partial v_2}{\partial x_1} \\ & + 2 \frac{\partial v_1}{\partial x_3} \frac{\partial v_3}{\partial x_1} + 2 \frac{\partial v_2}{\partial x_3} \frac{\partial v_3}{\partial x_2} - \nabla \cdot (n \nabla \Phi) \Bigg), \end{aligned}$$

and we have

$$\nabla \overline{\mathbf{u}} = \begin{pmatrix} \frac{a}{T^*-t} & -k(T^*-t)^{2a} & 0 \\ k(T^*-t)^{2a} & \frac{a}{T^*-t} & 0 \\ 0 & 0 & -\frac{2a}{T^*-t} \end{pmatrix}.$$

Let $R \in (0,1)$ be a fixed constant. We define

$$\mathcal{B}_R := \Big\{ \big(n(t,x), c(t,x), \mathbf{v}(t,x)\big)^T : \|n\|_{C_1^{s+\frac{11}{2}}} + \|c\|_{C_1^{s+\frac{11}{2}}} + \|\mathbf{v}\|_{C_1^{s+\frac{11}{2}}} \le R < 1 \Big\} \tag{2.4}$$

with a constant $s > 0$.

Assume that fixed functions $(n(t,x), c(t,x), \mathbf{v}(t,x))^T \in \mathcal{B}_R$. We linearize the nonlinear equations (2.1) around fixed functions $(n(t,x), c(t,x), \mathbf{v}(t,x))^T$ to get the linearized equations on the unknown variables $(\Gamma(t,x), \Lambda(t,x), \mathbf{h}(t,x))^T$ with an external force

$(f_1(t,x), f_2(t,x), \mathbf{g}(t,x))^T$ as follows:

$$\begin{cases} \Gamma_t - \triangle\Gamma + (2\mu n - \sigma)\Gamma + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Gamma + \mathbf{h} \cdot \nabla n \\ \quad + \chi\nabla \cdot [\Gamma\nabla c + n\nabla\Lambda] = f_1(t,x), \\ \Lambda_t - \triangle\Lambda + \Lambda + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Lambda + \mathbf{h} \cdot \nabla c - \Gamma = f_2(t,x), \\ \mathbf{h}_t - \nu\triangle\mathbf{h} + \mathbf{h} \cdot \nabla(\overline{\mathbf{u}} + \mathbf{v}) + (\overline{\mathbf{u}} + \mathbf{v}) \cdot \nabla\mathbf{h} - (\mathcal{F}_{\mathbf{v}}\nabla\overline{P})\mathbf{h} \\ \quad - (\mathcal{F}_n\nabla\overline{P})\Gamma - \Gamma\nabla\Phi = \mathbf{g}(t,x), \\ \nabla \cdot \mathbf{h} = 0, \quad \forall(t,x) \in (0,T^*) \times \Omega_t, \end{cases} \tag{2.5}$$

where $\mathcal{F}_{\mathbf{v}}$ denotes the Fréchet derivative on $\mathbf{v}$ and $\mathbf{h}(t,x) = (h_1(t,x), h_2(t,x), h_3(t,x))^T$.

In order to get some suitable prior estimates, we rewrite the linearized equations (2.5) as a coupled system,

$$\Gamma_t - \triangle\Gamma + (2\mu n - \sigma)\Gamma + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Gamma + \mathbf{h} \cdot \nabla n + \chi\nabla \cdot [\Gamma\nabla c + n\nabla\Lambda] = f_1(t,x), \tag{2.6}$$

$$\Lambda_t - \triangle\Lambda + \Lambda + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Lambda + \mathbf{h} \cdot \nabla c - \Gamma = f_2(t,x), \tag{2.7}$$

$$\begin{aligned} &\partial_t h_1 - \nu\triangle h_1 + \frac{a}{T^*-t}h_1 + k\big(T^*-t\big)^{2a}h_2 + \left(\frac{ax_1}{T^*-t} + kx_2\big(T^*-t\big)^{2a}\right)\partial_{x_1}h_1 \\ &\quad + \left(\frac{ax_2}{T^*-t} + kx_1\big(T^*-t\big)^{2a}\right)\partial_{x_2}h_1 - \frac{2ax_3}{T^*-t}\partial_{x_3}h_1 + h_1\partial_{x_1}w_1 + w_1\partial_{x_1}h_1 \\ &\quad + h_2\partial_{x_2}w_1 + w_2\partial_{x_2}h_1 + h_3\partial_{x_3}w_1 + w_3\partial_{x_3}h_1 - \Gamma\partial_{x_1}\Phi = \partial_{x_1}f(t,x) + g_1(t,x), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\partial_t h_2 - \nu\triangle h_2 - k\big(T^*-t\big)^{2a}h_1 + \frac{a}{T^*-t}h_2 + \left(\frac{ax_1}{T^*-t} + kx_2\big(T^*-t\big)^{2a}\right)\partial_{x_1}h_2 \\ &\quad + \left(\frac{ax_2}{T^*-t} - kx_1\big(T^*-t\big)^{2a}\right)\partial_{x_2}h_2 - \frac{2ax_3}{T^*-t}\partial_{x_3}h_2 + h_1\partial_{x_1}w_2 + w_1\partial_{x_1}h_2 \\ &\quad + h_2\partial_{x_2}w_2 + w_2\partial_{x_2}h_2 + h_3\partial_{x_3}w_2 + w_3\partial_{x_3}h_2 - \Gamma\partial_{x_2}\Phi = \partial_{x_2}f(t,x) + g_2(t,x), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\partial_t h_3 - \nu\triangle h_3 - \frac{a}{T^*-t}h_3 + \left(\frac{ax_1}{T^*-t} + kx_2\big(T^*-t\big)^{2a}\right)\partial_{x_1}h_3 \\ &\quad + \left(\frac{ax_2}{T^*-t} - kx_1\big(T^*-t\big)^{2a}\right)\partial_{x_2}h_3 - \frac{2ax_3}{T^*-t}\partial_{x_3}h_3 + h_1\partial_{x_1}w_3 + w_1\partial_{x_1}h_3 \\ &\quad + h_2\partial_{x_2}w_3 + w_2\partial_{x_2}h_3 + h_3\partial_{x_3}w_3 + w_3\partial_{x_3}h_3 - \Gamma\partial_{x_3}\Phi = \partial_{x_3}f(t,x) + g_3(t,x), \end{aligned} \tag{2.10}$$

with the incompressibility condition

$$\nabla \cdot \mathbf{h} = 0,$$

where

$$\begin{aligned} f(t,x) = -2\triangle^{-1}\Bigg[&\sum_{i=1}^{3}\big(\partial_{x_i}w_i\partial_{x_i}h_i - \partial_{x_i}(\Gamma\partial_{x_i}\Phi)\big) + k\big(T^*-t\big)^{2a}(\partial_{x_1}h_2 - \partial_{x_2}h_1) \\ &+ \partial_{x_2}h_1\partial_{x_1}w_2 + \partial_{x_1}h_2\partial_{x_2}w_1 + \partial_{x_3}w_1\partial_{x_1}h_3 + \partial_{x_3}h_1\partial_{x_1}w_3 \\ &+ \partial_{x_3}w_2\partial_{x_2}h_3 + \partial_{x_3}h_2\partial_{x_2}w_3\Bigg]. \end{aligned} \tag{2.11}$$

We introduce the similarity coordinates

$$\begin{aligned}
&\tau = -\ln\left(T^* - t\right) + \ln T^*, \\
&y = \frac{x}{\sqrt{T^* - t}},
\end{aligned} \tag{2.12}$$

where one can see the blowup time $T^* > 0$ has been transformed into $+\infty$ in the similarity coordinates (2.12). So the local existence of linearized coupled system (2.6)–(2.10) with the incompressibility condition in some Sobolev space is equivalent to the global existence of linearized coupled system (2.13)–(2.17) in a related Sobolev space. This means the key point is to get the decay in time of solutions for system (2.13)–(2.17).

The linearized coupled system (2.6)–(2.10) under these coordinates is transformed into

$$\begin{aligned}
&\partial_\tau \Gamma - \triangle_y \Gamma - \frac{y}{2} \cdot \nabla_y \Gamma + T^* e^{-\tau} (2\mu n - \sigma)\Gamma + a y_1 \partial_{y_1} \Gamma + a y_2 \partial_{y_2} \Gamma - 2a y_3 \partial_{y_3} \Gamma \\
&\quad + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau} (y_2 \partial_{y_1} \Gamma + y_1 \partial_{y_2} \Gamma) + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \mathbf{h} \cdot \nabla_y n + \chi \sum_{i=1}^{3} \partial_{y_i} (\Gamma \partial_{y_i} c) \\
&\quad + \chi \nabla_y \cdot (n \nabla_y \Lambda) = T^* e^{-\tau} f_1\left(T^*\left(1 - e^{-\tau}\right), \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} y\right),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\partial_\tau \Lambda - \triangle_y \Lambda - \frac{y}{2} \cdot \nabla_y \Lambda + T^* e^{-\tau} \Lambda + a y_1 \partial_{y_1} \Lambda + a y_2 \partial_{y_2} \Lambda - 2a y_3 \partial_{y_3} \Lambda \\
&\quad + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau} (y_2 \partial_{y_1} \Lambda + y_1 \partial_{y_2} \Lambda) + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \left(\sum_{i=1}^{2} h_i \partial_{y_i} c_i - \Gamma\right) \\
&= T^* e^{-\tau} f_2\left(T^*\left(1 - e^{-\tau}\right), \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} y\right),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&\partial_\tau h_1 - \nu \triangle_y h_1 - \frac{y}{2} \cdot \nabla_y h_1 + a h_1 + a \partial_{y_1} h_1 + a y_2 \partial_{y_2} h_1 - 2a y_3 \partial_{y_3} h_1 \\
&\quad + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau} (h_2 + y_2 \partial_{y_1} h_1 + y_1 \partial_{y_2} h_1) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \Gamma \partial_{y_1} \Phi \\
&\quad + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_1 + w_i \partial_{y_i} h_1) \\
&= \left(T^*\right)^{\frac{1}{2}} \partial_{y_1} \bar{f} + T^* e^{-\tau} g_1\left(T^*\left(1 - e^{-\tau}\right), \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} y\right),
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&\partial_\tau h_2 - \nu \triangle_y h_2 - \frac{y}{2} \cdot \nabla_y h_2 + a h_2 + a y_1 \partial_{y_1} h_2 + a y_2 \partial_{y_2} h_2 - 2a y_3 \partial_{y_3} h_2 \\
&\quad + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau} (-h_1 + y_2 \partial_{y_1} h_2 - y_1 \partial_{y_2} h_2) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \Gamma \partial_{y_2} \Phi \\
&\quad + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_2 + w_i \partial_{y_i} h_2) \\
&= \left(T^*\right)^{\frac{1}{2}} \partial_{y_2} \bar{f} + T^* e^{-\tau} g_2\left(T^*\left(1 - e^{-\tau}\right), \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} y\right),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&\partial_\tau h_3 - \nu \triangle_y h_3 - \frac{y}{2} \cdot \nabla_y h_3 - h_3 + a y_1 \partial_{y_1} h_3 + a y_2 \partial_{y_2} h_3 - 2a y_3 \partial_{y_3} h_3 \\
&\quad + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau} (y_2 \partial_{y_1} h_3 - y_1 \partial_{y_2} h_3) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \Gamma \partial_{y_3} \Phi
\end{aligned}$$

$$
\begin{aligned}
&+ \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \sum_{i=1}^{3} (h_i \partial_{y_i} w_3 + w_i \partial_{y_i} h_3) \\
&= \left(T^*\right)^{\frac{1}{2}} \partial_{y_3} \overline{f} + T^* e^{-\tau} g_3\left(T^*\left(1-e^{-\tau}\right), \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} y\right),
\end{aligned}
\tag{2.17}
$$

with the incompressibility condition

$$
\nabla_y \cdot \mathbf{h} = 0,
$$

where

$$
\begin{aligned}
\overline{f} = -2\triangle_y^{-1} \Bigg[&\sum_{i=1}^{3} \left(\partial_{y_i} w_i \partial_{y_i} h_i - \partial_{y_i}(\Gamma \partial_{y_i} \Phi)\right) + k\left(T^*\right)^{2a} e^{-2a\tau} (\partial_{y_1} h_2 - \partial_{y_2} h_1) \\
&+ \partial_{y_2} h_1 \partial_{y_1} w_2 + \partial_{y_1} h_2 \partial_{y_2} w_1 + \partial_{y_3} w_1 \partial_{y_1} h_3 + \partial_{y_3} h_1 \partial_{y_1} w_3 \\
&+ \partial_{y_3} w_2 \partial_{y_2} h_3 + \partial_{y_3} h_2 \partial_{y_2} w_3 \Bigg].
\end{aligned}
\tag{2.18}
$$

We supplement the linearized system (2.13)–(2.17) with the initial data

$$
\begin{cases}
\Gamma(0,y) = \Gamma_0(y) \in H^s(\mathbb{R}^3), \\
\Lambda(0,y) = \Lambda_0(y) \in H^s(\mathbb{R}^3), \\
\mathbf{h}(0,y) = \mathbf{h}_0(y) \in H^s(\mathbb{R}^3),
\end{cases}
\tag{2.19}
$$

and the boundary condition

$$
\begin{cases}
\lim_{|y|\to\infty} \Gamma(\tau,y) = 0, \\
\lim_{|y|\to\infty} \Lambda(\tau,y) = 0, \\
\lim_{|y|\to\infty} \mathbf{h}(\tau,y) = 0.
\end{cases}
\tag{2.20}
$$

We first derive prior estimates of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20).

Lemma 2.1 *Let $s > 0$, $0 < a \ll \frac{1}{8}$ and $T^* \in (0,1)$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+3}(\mathbb{R}^3)} \lesssim R \ll 1$, $f_i \in \mathbb{C}^1((0,+\infty), \mathbb{H}^s(\mathbb{R}^3))$ $(i = 1,2)$, $\boldsymbol{g} \in \mathbb{C}^1((0,+\infty), H^s(\mathbb{R}^3))$ and $(n,c,\boldsymbol{v})^T \in \mathcal{B}_R$. Then, for any $\tau > 0$, the solution $(\Gamma, \Lambda, \boldsymbol{h})^T$ of linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies*

$$
\begin{aligned}
\int_{\mathbb{R}^3} \left(|\Gamma|^2 + |\Lambda|^2 + |\boldsymbol{h}|^2\right) dy \lesssim e^{-C\tau} \Bigg(&\int_{\mathbb{R}^3} \left(|\Gamma_0|^2 + |\Lambda_0|^2 + |\boldsymbol{h}_0|^2\right) dy \\
&+ \int_0^{+\infty} \int_{\mathbb{R}^3} \left(|f_1|^2 + |f_2|^2 + |\boldsymbol{g}|^2\right) dy\, d\tau \Bigg),
\end{aligned}
$$

where C is a positive constant.

Proof Multiplying both sides of (2.13)–(2.17) by Γ, Λ, h_1, h_2 and h_3, respectively, then integrating by parts (using the boundary condition (2.20)), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\Gamma\|_{\mathbb{L}^2}^2+\|\nabla_y\Gamma\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}-T^*e^{-\tau}\sigma\right)\|\Gamma\|_{\mathbb{L}^2}^2+2\mu T^*e^{-\tau}\int_{\mathbb{R}^3}n\Gamma^2\,dy\\
&\quad+\chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\Gamma\partial_{y_i}c\partial_{y_i}\Gamma+n\partial_{y_i}^2\Lambda\right)dy+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}(\mathbf{h}\cdot\nabla_y n)\Gamma\,dy\\
&\quad=T^*e^{-\tau}\int_{\mathbb{R}^3}\Gamma f_1\,dy,
\end{aligned}\tag{2.21}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\Lambda\|_{\mathbb{L}^2}^2+\|\nabla_y\Lambda\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}+T^*e^{-\tau}\right)\|\Lambda\|_{\mathbb{L}^2}^2\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{2}h_i\partial_{y_i}c_i-\Gamma\right)\Lambda\,dy=T^*e^{-\tau}\int_{\mathbb{R}^3}\Lambda f_2\,dy,
\end{aligned}\tag{2.22}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|h_1\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\|\partial_{y_i}h_j\|_{\mathbb{L}^2}^2+\left(a+\frac{3}{4}\right)\|h_1\|_{\mathbb{L}^2}^2+k\left(T^*\right)^{2a+1}e^{-2(a+1)\tau}\int_{\mathbb{R}^3}h_1h_2\,dy\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_1(h_i\partial_{y_i}w_1+w_i\partial_{y_i}h_1)\,dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\Gamma\partial_{y_1}\Phi h_1\,dy=\left(T^*\right)^{\frac{1}{2}}\int_{\mathbb{R}^3}h_1\partial_{y_1}\bar{f}\,dy+\int_{\mathbb{R}^3}h_1g_1\,dy,
\end{aligned}\tag{2.23}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|h_2\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\|\partial_{y_i}h_j\|_{\mathbb{L}^2}^2+\left(a+\frac{3}{4}\right)\|h_2\|_{\mathbb{L}^2}^2-k\left(T^*\right)^{2a+1}e^{-2(a+1)\tau}\int_{\mathbb{R}^3}h_1h_2\,dy\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_2(h_i\partial_{y_i}w_2+w_i\partial_{y_i}h_2)\,dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\Gamma\partial_{y_2}\Phi h_2\,dy=\left(T^*\right)^{\frac{1}{2}}\int_{\mathbb{R}^3}h_2\partial_{y_2}\bar{f}\,dy+\int_{\mathbb{R}^3}h_2g_2\,dy,
\end{aligned}\tag{2.24}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|h_3\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\|\partial_{y_i}h_j\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}-2a\right)\|h_3\|_{\mathbb{L}^2}^2\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_3(h_i\partial_{y_i}w_3+w_i\partial_{y_i}h_3)\,dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\Gamma\partial_{y_3}\Phi h_3\,dy=\left(T^*\right)^{\frac{1}{2}}\int_{\mathbb{R}^3}h_3\partial_{y_3}\bar{f}\,dy+\int_{\mathbb{R}^3}h_3g_3\,dy.
\end{aligned}\tag{2.25}$$

Summing up (2.21)–(2.25), then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{3}\frac{d}{d\tau}\left(\|h_i\|_{\mathbb{L}^2}^2+\|\Gamma\|_{\mathbb{L}^2}^2+\|\Lambda\|_{\mathbb{L}^2}^2\right)+\|\nabla_y\Gamma\|_{\mathbb{L}^2}^2+\|\nabla_y\Lambda\|_{\mathbb{L}^2}^2+3\nu\sum_{i,j=1}^{3}\|\partial_{y_i}h_j\|_{\mathbb{L}^2}^2\\
&\quad+\left(\frac{3}{4}-T^*e^{-\tau}\sigma\right)\|\Gamma\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}+T^*e^{-\tau}\right)\|\Lambda\|_{\mathbb{L}^2}^2+\left(a+\frac{3}{4}\right)\left(\|h_1\|_{\mathbb{L}^2}^2+\|h_2\|_{\mathbb{L}^2}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{3}{4}-2a\right)\|h_3\|_{\mathbb{L}^2}^2+2\mu T^*e^{-\tau}\int_{\mathbb{R}^3} n\Gamma^2\,dy-\chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\Gamma\partial_{y_i}c\partial_{y_i}\Gamma+n\partial_{y_i}^2\Lambda\right)dy\\
&+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}(\mathbf{h}\cdot\nabla_y n)\Gamma\,dy+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{3}h_i\partial_{y_i}c_i-\Gamma\right)\Lambda\,dy\\
&-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Gamma\partial_{y_i}\Phi h_i\,dy+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_1(h_i\partial_{y_i}w_1+w_i\partial_{y_i}h_1)\,dy\\
&+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_2(h_i\partial_{y_i}w_2+w_i\partial_{y_i}h_2)\,dy\\
&+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_3(h_i\partial_{y_i}w_3+w_i\partial_{y_i}h_3)\,dy\\
=&\left(T^*\right)^{\frac{1}{2}}\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_i\partial_{y_i}\overline{f}\,dy+T^*e^{-\tau}\int_{\mathbb{R}^3}\left(\Gamma f_1+\Lambda f_2+\sum_{i=1}^{3}h_ig_i\right)dy,\quad\forall\tau>0.
\end{aligned}
\tag{2.26}
$$

We now estimate each coupled nonlinear term in (2.26). Note that $(n,c,\mathbf{v})^T\in\mathcal{B}_R$ and $H^{\frac{5}{2}}(\mathbb{R}^3)\subset L^\infty(\mathbb{R}^3)$. We use Young's inequality to derive

$$
\begin{aligned}
&\left|2\mu T^*e^{-\tau}\int_{\mathbb{R}^3}n\Gamma^2\,dy\right|\lesssim C_R\|\Gamma\|_{\mathbb{L}^2}^2,\\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\Gamma\partial_{y_i}c\partial_{y_i}\Gamma+n\partial_{y_i}^2\Lambda\right)dy\right|\lesssim C_R\left(\|\Gamma\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\left(\|\partial_{y_i}\Gamma\|_{\mathbb{L}^2}^2+\|\partial_{y_i}\Lambda\|_{\mathbb{L}^2}^2\right)\right),\\
&\left|\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}(\mathbf{h}\cdot\nabla_y n)\Gamma\,dy\right|\lesssim C_R\left(\|\Gamma\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\|h_i\|_{\mathbb{L}^2}^2\right),\\
&\left|\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{3}h_i\partial_{y_i}d_i-\Gamma\right)\Lambda\,dy\right|\\
&\quad\lesssim\frac{(T^*)^{\frac{1}{2}}}{2}e^{-\frac{1}{2}\tau}\left(2\|\Lambda\|_{\mathbb{L}^2}^2+\|\Gamma\|_{\mathbb{L}^2}^2\right)+C_R\sum_{i=1}^{3}\|h_i\|_{\mathbb{L}^2}^2,\\
&\left|\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\Gamma\partial_{y_i}\Phi h_i\,dy\right|\lesssim C_R\left(\|\Gamma\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\|h_i\|_{\mathbb{L}^2}^2\right),
\end{aligned}
\tag{2.27}
$$

and

$$
\begin{aligned}
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_1(h_i\partial_{x_i}w_1+w_i\partial_{y_i}h_1)\,dy\right|\lesssim C_R\sum_{i=1}^{3}\left(\|h_i\|_{\mathbb{L}^2}^2+\|\partial_{y_i}h_1\|_{\mathbb{L}^2}\right),\\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_2(h_i\partial_{y_i}w_2+w_i\partial_{y_i}h_2)\,dy\right|\lesssim C_R\sum_{i=1}^{3}\left(\|h_i\|_{\mathbb{L}^2}^2+\|\partial_{y_i}h_2\|_{\mathbb{L}^2}\right),\\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}h_3(h_i\partial_{y_i}w_3+w_i\partial_{y_i}h_3)\,dy\right|\lesssim C_R\sum_{i=1}^{3}\left(\|h_i\|_{\mathbb{L}^2}^2+\|\partial_{y_i}h_3\|_{\mathbb{L}^2}\right),
\end{aligned}
\tag{2.28}
$$

where C_R, $C_{a,R}$, $C_{\kappa,R}$ are three positive constants depending on R, a, R and κ, R, respectively.

On the other hand, by (2.18) and the standard Calderon–Zygmund theory, i.e. for the Riesz operator $\mathcal{R}$, we have $\|\mathcal{R}w\|_{\mathbb{L}^p} \leq \|w\|_{\mathbb{L}^p}$ with $1 < p < \infty$, we also use Young's inequality to get

$$\left|\sum_{i=1}^{3}\int_{\Omega} h_i \partial_{y_i} \bar{f}\, dy\right| \lesssim C_R \sum_{i,j=1}^{3}\left(\|\Gamma\|_{\mathbb{L}^2}^2 + \|h_i\|_{\mathbb{L}^2}^2 + \|\partial_{y_i} h_j\|_{\mathbb{L}^2}^2\right) \tag{2.29}$$

and

$$\begin{aligned}
&\left|T^* e^{-\tau}\int_{\Omega}\left(\Gamma f_1 + \Lambda f_2 + \sum_{i=1}^{3} h_i g_i\right) dy\right| \\
&\quad\leq T^* e^{-\tau}\Bigg[b\left(\|\Gamma\|_{\mathbb{L}^2}^2 + \|\Lambda\|_{\mathbb{L}^2}^2\right) + b^{-1}\left(\|f_1\|_{\mathbb{L}^2}^2 + \|f_2\|_{\mathbb{L}^2}^2\right) \\
&\qquad + \sum_{i=1}^{3}\left(b\|h_i\|_{\mathbb{L}^2}^2 + b^{-1}\|g_i\|_{\mathbb{L}^2}^2\right)\Bigg],
\end{aligned} \tag{2.30}$$

where the positive constant $b < 1$.

Thus by (2.27)–(2.30), it follows from (2.26) that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{3}\frac{d}{d\tau}\|h_i\|_{\mathbb{L}^2}^2 + (1 - C_R)\|\nabla_y \Gamma\|_{\mathbb{L}^2}^2 + (1 - C_R)\|\nabla_y \Lambda\|_{\mathbb{L}^2}^2 + (3\nu - C_R)\sum_{i,j=1}^{3}\|\partial_{y_i} h_j\|_{\mathbb{L}^2}^2 \\
&\quad + \left(\frac{3}{4} - T^* e^{-\tau}(b + \sigma) - \frac{(T^*)^{\frac{1}{2}}}{2} e^{-\frac{1}{2}\tau} - C_R\right)\|\Gamma\|_{\mathbb{L}^2}^2 \\
&\quad + \left(\frac{3}{4} + T^* e^{-\tau}(1 - b) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} - C_R\right)\|\Lambda\|_{\mathbb{L}^2}^2 \\
&\quad + \left(a + \frac{3}{4} - T^* e^{-\tau} b - C_R\right)\left(\|h_1\|_{\mathbb{L}^2}^2 + \|h_2\|_{\mathbb{L}^2}^2\right) + \left(\frac{3}{4} - 2a - T^* e^{-\tau} b - C_R\right)\|h_3\|_{\mathbb{L}^2}^2 \\
&\lesssim b^{-1}\left(\|f_1\|_{\mathbb{L}^2}^2 + \|f_2\|_{\mathbb{L}^2}^2 + \sum_{i=1}^{3}\|g_i\|_{\mathbb{L}^2}^2\right),
\end{aligned} \tag{2.31}$$

where C_R is a positive constant depending on R, which can be very small if constants R is small.

There exists a sufficiently small positive constant $b \in (0, 1)$ such that

$$\begin{aligned}
&1 - C_R > 0, \qquad 1 - C_R > 0, \quad 3\nu - C_R > 0, \\
&\frac{3}{4} - T^* e^{-\tau}(b + \sigma) - \frac{(T^*)^{\frac{1}{2}}}{2} e^{-\frac{1}{2}\tau} - C_R > 0, \\
&\frac{3}{4} + T^* e^{-\tau}(1 - b) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} - C_R > 0, \\
&a + \frac{3}{4} - T^* e^{-\tau} b - C_R > 0, \qquad \frac{3}{4} - 2a - T^* e^{-\tau} b - C_R > 0.
\end{aligned}$$

Hence, applying Gronwall's inequality to (2.31), there exists a positive constant C such that

$$\|\Gamma\|_{\mathbb{L}^2}^2 + \|\Gamma\|_{\mathbb{L}^2}^2 + \sum_{i=1}^{3}\|h_i\|_{\mathbb{L}^2}^2 \lesssim e^{-C\tau}\int_{\mathbb{R}^3}\left(\Gamma_0^2 + \Lambda_0^2 + \sum_{i=1}^{3}h_{0i}^2\right)dy$$
$$+ e^{-C\tau}\int_0^{+\infty}\int_{\mathbb{R}^3}\left(f_1^2 + f_2^2 + \sum_{i=1}^{3}g_i^2\,d\tau\right)dy, \quad \forall\tau > 0. \qquad \square$$

In what follows, we plan to carry out higher order derivative estimates to the solutions of linearized system (2.8)–(2.10). For a fixed constant $s > 0$, applying $\nabla^s = \partial_{y_i}^s$ to both sides of (2.22)–(2.10), we obtain

$$\begin{aligned}
&\partial_\tau\nabla_y^s\Gamma - \triangle_y\nabla_y^s\Gamma - \frac{y}{2}\cdot\nabla_y^{s+1}\Gamma + \left(T^*e^{-\tau}(2\mu n - \sigma) - \frac{s}{2}\right)\nabla_y^s\Gamma + ay_1\partial_{y_1}\nabla_y^s\Gamma + ay_2\partial_{y_2}\nabla_y^s\Gamma \\
&\quad - 2ay_3\partial_{y_3}\nabla_y^s\Gamma + k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\left(2s\nabla_y^s\Gamma + y_2\partial_{y_1}\nabla_y^s\Gamma + y_1\partial_{y_2}\nabla_y^s\Gamma\right) \\
&\quad + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\nabla_y^s(\mathbf{h}\cdot\nabla_y n) + \chi\sum_{i=1}^{3}\partial_{y_i}\nabla_y^s(\Gamma\partial_{y_i}c) + \chi\nabla_y^s\left(\nabla_y\cdot(n\nabla_y\Lambda)\right) \\
&= T^*e^{-\tau}\nabla_y^s f_1,
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
&\partial_\tau\nabla_y^s\Lambda - \triangle_y\nabla_y^s\Lambda - \frac{y}{2}\cdot\nabla_y^{s+1}\Lambda + \left(T^*e^{-\tau} - \frac{s}{2}\right)\nabla_y^s\Lambda + ay_1\partial_{y_1}\nabla_y^s\Lambda + ay_2\partial_{y_2}\nabla_y^s\Lambda \\
&\quad - 2ay_3\partial_{y_3}\nabla_y^s\Lambda + k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\left(2s\nabla_y^s\Lambda + y_2\partial_{y_1}\nabla_y^s\Lambda + y_1\partial_{y_2}\nabla_y^s\Lambda\right) \\
&\quad + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\left(\sum_{i=1}^{2}\nabla_y^s(h_i\partial_{y_i}c_i) - \Gamma\right) = T^*e^{-\tau}\nabla_y^s f_2,
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
&\partial_\tau\nabla_y^s h_1 - \nu\triangle_y\nabla_y^s h_1 - \frac{y}{2}\cdot\partial_y\nabla_y^s h_1 + \left(a - \frac{s}{2}\right)\nabla_y^s h_1 + ay_1\partial_{y_1}\nabla_y^s h_1 + ay_2\partial_{y_2}\nabla_y^s h_1 \\
&\quad - 2ay_3\partial_{y_3}\nabla_y^s h_1 + k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\left(\nabla_y^s h_2 + 2s\nabla_y^s h_1 + y_2\partial_{y_1}\nabla_y^s h_1 + y_1\partial_{y_2}\nabla_y^s h_1\right) \\
&\quad - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\nabla_y^s(\Gamma\partial_{y_1}\Phi) + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\left(\nabla_y^s h_i\partial_{y_i}w_1 + w_i\partial_{y_i}\nabla_y^s h_1\right) = \tilde{g}_1,
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
&\partial_\tau\nabla_y^s h_2 - \nu\triangle_y\nabla_y^s h_2 - \frac{y}{2}\cdot\partial_y\nabla_y^s h_2 + \left(a - \frac{s}{2}\right)\nabla_y^s h_2 + ay_1\partial_{y_1}\nabla_y^s h_2 + ay_2\partial_{y_2}\nabla_y^s h_2 \\
&\quad - 2ay_3\partial_{y_3}\nabla_y^s h_2 + k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\left(-\nabla_y^s h_1 + y_2\partial_{y_1}\nabla_y^s h_2 - y_1\partial_{y_2}\nabla_y^s h_2\right) \\
&\quad - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\nabla_y^s(\Gamma\partial_{y_2}\Phi) + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\left(\nabla_y^s h_i\partial_{y_i}w_2 + w_i\partial_{y_i}\nabla_y^s h_2\right) = \tilde{g}_2,
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
&\partial_\tau\nabla_y^s h_3 - \nu\triangle_y\nabla_y^s h_3 - \frac{y}{2}\cdot\partial_y\nabla_y^s h_3 - \left(2a + \frac{s}{2}\right)\nabla_y^s h_3 + ay_1\partial_{y_1}\nabla_y^s h_3 + ay_2\partial_{y_2}\nabla_y^s h_3 \\
&\quad - 2ay_3\partial_{y_3}\nabla_y^s h_3 + k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\left(y_2\partial_{y_1}\nabla_y^s h_3 - y_1\partial_{y_2}\nabla_y^s h_3\right) \\
&\quad + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\left(\nabla_y^s h_i\partial_{y_i}w_3 + w_i\partial_{y_i}\nabla_y^s h_3\right) - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\nabla_y^s(\Gamma\partial_{y_3}\Phi) = \tilde{g}_3,
\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}\tilde{g}_1 &:= \left(T^*\right)^{\frac{1}{2}}\partial_{y_1}\nabla_y^s\overline{f} + T^*e^{-\tau}\nabla_y^s g_1\\&\quad - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{j=1}^{3}\sum_{i_1+i_2=s,\ 0\le i_2\le s-1}\left(\nabla_y^{i_2}h_j\partial_{y_j}\nabla_y^{i_1}w_1 + \nabla_y^{i_1}w_j\partial_{y_j}\nabla_y^{i_2}h_1\right),\end{aligned} \tag{2.37}$$

$$\begin{aligned}\tilde{g}_2 &:= \left(T^*\right)^{\frac{1}{2}}\partial_{y_2}\nabla_y^s\overline{f} + T^*e^{-\tau}\nabla_y^s g_2\\&\quad - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{j=1}^{3}\sum_{i_1+i_2=s,\ 0\le i_2\le s-1}\left(\nabla_y^{i_2}h_j\partial_{y_j}\nabla_y^{i_1}w_2 + \nabla_y^{i_1}w_j\partial_{y_j}\nabla_y^{i_2}h_2\right),\end{aligned} \tag{2.38}$$

$$\begin{aligned}\tilde{g}_3 &:= \left(T^*\right)^{\frac{1}{2}}\partial_{y_3}\nabla_y^s\overline{f} + T^*e^{-\tau}\nabla_y^s g_3\\&\quad - \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{j=1}^{3}\sum_{i_1+i_2=s,\ 0\le i_2\le s-1}\left(\nabla_y^{i_2}h_j\partial_{y_j}\nabla_y^{i_1}w_3 + \nabla_y^{i_1}w_j\partial_{y_j}\nabla_y^{i_2}h_3\right).\end{aligned} \tag{2.39}$$

We now have the following higher order derivatives estimates.

Lemma 2.2 *Let $0 < a \ll \frac{1}{8}$ and $0 < s < \frac{3}{2} - 5a$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+3}(\mathbb{R}^3)} \lesssim R \ll 1$, $f_i \in \mathbb{C}^1((0,+\infty), \mathbb{H}^s(\mathbb{R}^3))$ $(i = 1,2)$, $\boldsymbol{g} \in \mathbb{C}^1((0,+\infty), H^s(\mathbb{R}^3))$ and $(n,c,\boldsymbol{v})^T \in \mathcal{B}_R$. Then, for any $\tau > 0$, the solution $(\Gamma,\Lambda,\boldsymbol{h})^T$ of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies*

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\left|\nabla_y^s\Gamma\right|^2 + \left|\nabla_y^s\Lambda\right|^2 + \sum_{i=1}^{3}\left|\nabla_y^s h_i\right|^2\right)dy\\&\quad\lesssim e^{-C_{R,T^*}\tau}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s\Gamma_0\right|^2 + \left|\nabla_y^s\Lambda_0\right|^2 + \sum_{i=1}^{3}\left|\nabla_y^s h_{0i}\right|^2\right)dy\\&\qquad + e^{-C_{R,T^*}\tau}\int_0^{+\infty}\left(\left\|\nabla_y^s f_1\right\|_{\mathbb{L}^2}^2 + \left\|\nabla_y^s f_2\right\|_{\mathbb{L}^2}^2 + \sum_{i=1}^{3}\left\|\nabla_y^s g_i\right\|_{\mathbb{L}^2}^2\right)d\tau,\quad\forall\tau > 0,\end{aligned}$$

where C_{R,T^*} is a positive constant depending on constants R, T^*.

Proof Taking the inner product of both sides of (2.32)–(2.36) by $\nabla_y^s\Gamma$, $\nabla_y^s\Lambda$, $\nabla_y^s h_1$, $\nabla_y^s h_2$ and $\nabla_y^s h_3$, respectively, then integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2 + \left\|\nabla_y^{s+1}\Gamma\right\|_{\mathbb{L}^2}^2 + \left(\frac{3}{4} - \frac{s}{2} - T^*e^{-\tau}\sigma + +2s\kappa\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\right)\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2\\&\quad + 2\mu T^*e^{-\tau}\int_{\mathbb{R}^3}n|\nabla_y\Gamma|^2\,dy + \left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\nabla_y^s(\mathbf{h}\cdot\nabla_y n)\cdot\nabla_y^s\Gamma\,dy\\&\quad + \chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_{y_i}\nabla_y^s(\Gamma\partial_{y_i}c)\cdot\nabla_y^s\Gamma\,dy + \chi\int_{\mathbb{R}^3}\nabla_y^s\left(\nabla_y\cdot(n\nabla_y\Lambda)\right)\cdot\nabla_y^s\Gamma\,dy\\&= T^*e^{-\tau}\int_{\mathbb{R}^3}\nabla_y^s f_1\cdot\nabla_y^s\Gamma\,dy,\end{aligned} \tag{2.40}$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^{s+1}\Lambda\right\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}-\frac{s}{2}+T^*e^{-\tau}+2s\kappa\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\right)\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{2}\nabla_y^s(h_i\partial_{y_i}d_i)-\nabla_y^s\Gamma\right)\cdot\nabla_y^s\Lambda\,dy\\
&\quad=T^*e^{-\tau}\int_{\mathbb{R}^3}\nabla_y^sf_2\cdot\nabla_y^s\Lambda\,dy, \qquad (2.41)\\
&\frac{1}{2}\frac{d}{d\tau}\left\|\nabla_y^sh_1\right\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\left\|\partial_{y_i}\nabla_y^sh_j\right\|_{\mathbb{L}^2}^2+\left(a+\frac{3}{4}-\frac{s}{2}\right)\left\|\nabla_y^sh_1\right\|_{\mathbb{L}^2}^2\\
&\quad+k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\int_{\mathbb{R}^3}\nabla_y^sh_1\cdot\nabla_y^sh_2\,dy\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^sh_1\cdot\left(\nabla_y^sh_i\partial_{y_i}w_1+w_i\partial_{y_i}\nabla_y^sh_1\right)dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\mathbb{R}^3}\nabla_y^sh_1\cdot\nabla_y^s(\Gamma\partial_{y_1}\Phi)\,dy\\
&\quad=\int_{\mathbb{R}^3}\nabla_y^sh_1\cdot\tilde{g}_1\,dy, \qquad (2.42)\\
&\frac{1}{2}\frac{d}{d\tau}\left\|\nabla_y^sh_2\right\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\left\|\partial_{y_i}\nabla_y^sh_j\right\|_{\mathbb{L}^2}^2+\left(a+\frac{3}{4}-\frac{s}{2}\right)\left\|\nabla_y^sh_2\right\|_{\mathbb{L}^2}^2\\
&\quad-k\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\int_{\Omega}\nabla_y^sh_1\cdot\nabla_y^sh_2\,dy\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\Omega}\nabla_y^sh_2\cdot\left(\nabla_y^sh_i\partial_{y_i}w_2+w_i\partial_{y_i}\nabla_y^sh_2\right)dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\Omega}\nabla_y^sh_2\cdot\nabla_y^s(\Gamma\partial_{y_2}\Phi)\,dy\\
&\quad=\int_{\Omega}\nabla_y^sh_2\cdot\tilde{g}_2\,dy, \qquad (2.43)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\left\|\nabla_y^sh_3\right\|_{\mathbb{L}^2}^2+\nu\sum_{i,j=1}^{3}\left\|\partial_{y_i}\nabla_y^sh_j\right\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}-2a-\frac{s}{2}\right)\left\|\nabla_y^sh_3\right\|_{\mathbb{L}^2}^2\\
&\quad+\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\Omega}\nabla_y^sh_3\cdot\left(\nabla_y^sh_i\partial_{y_i}w_3+w_i\partial_{y_i}\nabla_y^sh_3\right)dy\\
&\quad-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\int_{\Omega}\nabla_y^sh_3\cdot\nabla_y^s(\Gamma\partial_{y_3}\Phi)\,dy=\int_{\Omega}\nabla_y^sh_3\cdot\tilde{g}_3\,dy. \qquad (2.44)
\end{aligned}
$$

Summing up (2.42)–(2.44), we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{3}\frac{d}{d\tau}\left(\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^sh_i\right\|_{\mathbb{L}^2}^2\right)+\left\|\nabla_y^{s+1}\Gamma\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^{s+1}\Lambda\right\|_{\mathbb{L}^2}^2\\
&\quad+3\nu\sum_{i,j=1}^{3}\left\|\partial_{y_i}\nabla_y^sh_j\right\|_{\mathbb{L}^2}^2+\left(\frac{3}{4}-\frac{s}{2}-T^*e^{-\tau}\sigma+2s\kappa\left(T^*\right)^{2a+1}e^{-(2a+1)\tau}\right)\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\mu T^* e^{-\tau} \int_{\mathbb{R}^3} n|\nabla_y \Gamma|^2 \, dy \\
&+ \left(\frac{3}{4} - \frac{s}{2} + T^* e^{-\tau} + 2s\kappa \left(T^*\right)^{2a+1} e^{-(2a+1)\tau} \right) \| \nabla_y^s \Lambda \|_{\mathbb{L}^2}^2 \\
&+ \left(a + \frac{3}{4} - \frac{s}{2} \right) \left(\| \nabla_y^s h_1 \|_{\mathbb{L}^2}^2 + \| \nabla_y^s h_2 \|_{\mathbb{L}^2}^2 \right) + \left(\frac{3}{4} - 2a - \frac{s}{2} \right) \| \nabla_y^s h_3 \|_{\mathbb{L}^2}^2 \\
&+ \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \int_{\mathbb{R}^3} \nabla_y^s (\mathbf{h} \cdot \nabla_y n) \cdot \nabla_y^s \Gamma \, dy + \chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_{y_i} \nabla_y^s (\Gamma \partial_{y_i} c) \cdot \nabla_y^s \Gamma \, dy \\
&+ \chi \int_{\mathbb{R}^3} \nabla_y^s \big(\nabla_y \cdot (n \nabla_y \Lambda) \big) \cdot \nabla_y^s \Gamma \, dy \\
&+ \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \int_{\mathbb{R}^3} \left(\sum_{i=1}^{2} \nabla_y^s (h_i \partial_{y_i} d_i) - \nabla_y^s \Gamma \right) \cdot \nabla_y^s \Lambda \, dy \\
&+ \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \big[\nabla_y^s h_1 \cdot \big(\nabla_y^s h_i \partial_{y_i} w_1 + w_i \partial_{y_i} \nabla_y^s h_1 \big) \\
&+ \nabla_y^s h_2 \cdot \big(\nabla_y^s h_i \partial_{y_i} w_2 + w_i \partial_{y_i} \nabla_y^s h_2 \big) + \nabla_y^s h_3 \cdot \big(\nabla_y^s h_i \partial_{y_i} w_3 + w_i \partial_{y_i} \nabla_y^s h_3 \big) \, dy \big] \\
&- \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \nabla_y^s h_i \cdot \nabla_y^s (\Gamma \partial_{y_i} \Phi) \, dy \\
= {}& T^* e^{-\tau} \int_{\mathbb{R}^3} \big(\nabla_y^s f_1 \cdot \nabla_y^s \Gamma + \nabla_y^s f_2 \cdot \nabla_y^s \Lambda \big) \, dy + \sum_{i=1}^{3} \int_{\mathbb{R}^3} \big(\nabla_y^s h_i \cdot \tilde{g}_i \big) \, dy.
\end{aligned}
\tag{2.45}
$$

We now estimate each nonlinear term in (2.45). On the one hand, note that $(n, c, \mathbf{v})^T \in \mathcal{B}_R$. We employ Young's inequality, $H^{\frac{5}{2}}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ and integrating by parts to derive

$$
\begin{aligned}
&\left| 2\mu T^* e^{-\tau} \int_{\mathbb{R}^3} n|\nabla_y \Gamma|^2 \, dy \right| \lesssim C_R \|\nabla_y \Gamma\|_{\mathbb{L}^2}^2, \\
&\left| \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \int_{\mathbb{R}^3} \nabla_y^s (\mathbf{h} \cdot \nabla_y n) \cdot \nabla_y^s \Gamma \, dy \right| \\
&\quad \lesssim C_R \left(\| \nabla_y^s \Gamma \|_{\mathbb{L}^2}^2 + \sum_{i=1}^{3} \big(\|h_i\|_{\mathbb{L}^2}^2 + \| \nabla_y^s h_i \|_{\mathbb{L}^2}^2 \big) \right), \\
&\left| \chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_{y_i} \nabla_y^s (\Gamma \partial_{y_i} c) \cdot \nabla_y^s \Gamma \, dy \right| \lesssim C_R \big(\| \nabla_y^{s+1} \Gamma \|_{\mathbb{L}^2}^2 + \| \nabla_y^s \Gamma \|_{\mathbb{L}^2}^2 \big), \\
&\left| \chi \int_{\mathbb{R}^3} \nabla_y^s \big(\nabla_y \cdot (n \nabla_y \Lambda) \big) \cdot \nabla_y^s \Gamma \, dy \right| \\
&\quad \lesssim C_R \big(\| \nabla_y^{s+1} \Gamma \|_{\mathbb{L}^2}^2 + \| \nabla_y^{s+1} \Lambda \|_{\mathbb{L}^2}^2 + \| \nabla_y^s \Gamma \|_{\mathbb{L}^2}^2 + \| \nabla_y^s \Lambda \|_{\mathbb{L}^2}^2 \big), \\
&\left| \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau} \int_{\mathbb{R}^3} \left(\sum_{i=1}^{2} \nabla_y^s (h_i \partial_{y_i} d_i) - \nabla_y^s \Gamma \right) \cdot \nabla_y^s \Lambda \, dy \right| \\
&\quad \lesssim \frac{(T^*)^{\frac{1}{2}}}{2} e^{-\frac{1}{2}\tau} \big(2 \| \nabla_y^s \Lambda \|_{\mathbb{L}^2}^2 + \| \nabla_y^s \Gamma \|_{\mathbb{L}^2}^2 \big) + C_R \sum_{i=1}^{2} \big(\|h_i\|_{\mathbb{L}^2}^2 + \| \nabla_y^s h_i \|_{\mathbb{L}^2}^2 \big),
\end{aligned}
\tag{2.46}
$$

and

$$\begin{aligned}
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^s h_1\cdot\left(\nabla_y^s h_i\partial_{y_i}w_1+w_i\partial_{y_i}\nabla_y^s h_1\right)dy\right|\\
&\quad\lesssim\left(\sum_{k=1}^{3}\left(\|\partial_{y_i}w_1\|_{\mathbb{L}^\infty}+\|w_i\|_{\mathbb{L}^\infty}\right)\right)\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_1\right|^2\right)dy\\
&\quad\lesssim C_R\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_1\right|^2\right)dy, \qquad (2.47)\\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^s h_2\cdot\left(\nabla_y^s h_i\partial_{y_i}w_2+w_i\partial_{y_i}\nabla_y^s h_2\right)dy\right|\\
&\quad\lesssim\left(\sum_{i=1}^{3}\left(\|\partial_{y_i}w_2\|_{\mathbb{L}^\infty}+\|w_i\|_{\mathbb{L}^\infty}\right)\right)\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_2\right|^2\right)dy\\
&\quad\lesssim C_R\sum_{k=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_2\right|^2\right)dy, \qquad (2.48)\\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^s h_3\cdot\left(\nabla_y^s h_i\partial_{y_i}w_3+w_i\partial_{y_i}\nabla_y^s h_3\right)dy\right|\\
&\quad\lesssim\left(\sum_{i=1}^{3}\left(\|\partial_{y_i}w_3\|_{\mathbb{L}^\infty}+\|w_i\|_{\mathbb{L}^\infty}\right)\right)\\
&\qquad\times\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_3\right|^2\right)dy\\
&\quad\lesssim C_R\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s h_i\right|^2+\left|\partial_{y_i}\nabla_y^s h_3\right|^2\right)dy, \qquad (2.49)\\
&\left|\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^s h_i\cdot\nabla_y^s(\Gamma\partial_{y_i}\Phi)\,dy\right|\\
&\quad\lesssim C_R\left(\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\left\|\nabla_y^s h_i\right\|_{\mathbb{L}^2}^2\right), \qquad (2.50)
\end{aligned}$$

where the C_R are a positive constants depending on R, which are small constants as R is small.

On the other hand, by (2.18), we know the highest order derivatives on h_i of $\partial_{y_1}\nabla_y^s\overline{f}$ is s. So we can use the standard Calderon–Zygmund theory, Young's inequality and integrating by parts to derive

$$\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla_y^s h_i\cdot\partial_{y_i}\nabla_y^s\overline{f}\,dy\right|\lesssim C_R\left(\sum_{i=1}^{3}\left\|\nabla_y^s h_i\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2\right), \qquad (2.51)$$

furthermore, by (2.37)–(2.39), we have

$$\begin{aligned}
&\left|T^*e^{-\tau}\int_{\mathbb{R}^3}\left(\nabla_y^s f_1\cdot\nabla_y^s\Gamma+\nabla_y^s f_2\cdot\nabla_y^s\Lambda\right)\right| \\
&\quad\lesssim\frac{1}{2}\left(\sum_{i=1}^{2}\left\|\nabla_y^s f_i\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2\right), \\
&\left|\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left(\nabla_y^s h_i\cdot\tilde{g}_i\right)dy\right| \\
&\quad\lesssim\left(C_{R,T^*}+\frac{1}{2}\right)\sum_{i=1}^{3}\left(\left\|\nabla_y^s h_i\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2\right)+4\sum_{i=1}^{3}\left\|\nabla_y^s g_i\right\|_{\mathbb{L}^2}^2,
\end{aligned}\tag{2.52}$$

where C_{R,T^*} is a positive constant depending on R, T^*, which is a small constant as R small.

Hence we can apply estimates (2.46)–(2.52) to (2.45), then

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{3}\frac{d}{d\tau}\left(\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s h_i\right\|_{\mathbb{L}^2}^2\right) \\
&\qquad+(1-C_R)\left\|\nabla_y^{s+1}\Gamma\right\|_{\mathbb{L}^2}^2+(1-C_R)\left\|\nabla_y^{s+1}\Lambda\right\|_{\mathbb{L}^2}^2+(3\nu-C_R)\sum_{i,j=1}^{3}\left\|\partial_{y_i}\nabla_y^s h_j\right\|_{\mathbb{L}^2}^2 \\
&\qquad+\left(\frac{3}{4}-\frac{s}{2}-T^*e^{-\tau}(b+\sigma)-\frac{(T^*)^{\frac{1}{2}}}{2}e^{-\frac{1}{2}\tau}-C_R\right)\|\Gamma\|_{\mathbb{L}^2}^2 \\
&\qquad+\left(\frac{3}{4}-\frac{s}{2}+T^*e^{-\tau}(1-b)-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}-C_R\right)\|\Lambda\|_{\mathbb{L}^2}^2 \\
&\qquad+\left(a+\frac{3}{4}-\frac{s}{2}-T^*e^{-\tau}b-C_R\right)\left(\left\|\nabla_y^s h_1\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s h_2\right\|_{\mathbb{L}^2}^2\right) \\
&\qquad+\left(\frac{3}{4}-\frac{s}{2}-2a-T^*e^{-\tau}b-C_R\right)\left\|\nabla_y^s h_3\right\|_{\mathbb{L}^2}^2 \\
&\quad\lesssim\frac{1}{2}\left(\|f_1\|_{\mathbb{L}^2}^2+\|f_2\|_{\mathbb{L}^2}^2\right)+4\sum_{i=1}^{3}\left\|\nabla_y^s g_i\right\|_{\mathbb{L}^2}^2.
\end{aligned}\tag{2.53}$$

Since $0<a\ll\frac{1}{8}$ and $0<s<\frac{3}{2}-5a$ are constants, there exists a sufficient small positive constant R such that

$$\begin{aligned}
&1-C_R>0,\qquad 1-C_R>0,\qquad 3\nu-C_R>0, \\
&\frac{3}{4}-\frac{s}{2}-T^*e^{-\tau}(b+\sigma)-\frac{(T^*)^{\frac{1}{2}}}{2}e^{-\frac{1}{2}\tau}-C_R>0, \\
&\frac{3}{4}-\frac{s}{2}+T^*e^{-\tau}(1-b)-\left(T^*\right)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}-C_R>0, \\
&a+\frac{3}{4}-\frac{s}{2}-T^*e^{-\tau}b-C_R>0, \\
&\frac{3}{4}-\frac{s}{2}-2a-T^*e^{-\tau}b-C_R>0.
\end{aligned}$$

Hence, applying Gronwall's inequality to (2.53), there exists a positive constant C_{R,T^*} depending on R and T^* such that

$$\begin{aligned}
&\sum_{i=1}^{3}\left(\left\|\nabla_y^s\Gamma\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Lambda\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s h_i\right\|_{\mathbb{L}^2}^2\right)\\
&\quad\lesssim e^{-C_{R,T^*}\tau}\sum_{i=1}^{3}\left(\left\|\nabla_y^s\Gamma_0\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\Lambda_0\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s h_{0i}\right\|_{\mathbb{L}^2}^2\right)\\
&\qquad+e^{-C_{R,T^*}\tau}\int_0^{+\infty}\left(\|f_1\|_{\mathbb{L}^2}^2+\|f_2\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\left\|\nabla_y^s g_i\right\|_{\mathbb{L}^2}^2\right)d\tau,\quad\forall\tau>0.
\end{aligned}$$

□

Furthermore, we have the following result.

Lemma 2.3 *Let $0<a\ll\frac{1}{8}$ and $0<s<\frac{3}{2}-5a$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+5}(\mathbb{R}^3)}\lesssim R\ll 1$, $f_i\in\mathbb{C}^1((0,+\infty),\mathbb{H}^s(\mathbb{R}^3))$ $(i=1,2)$, $\boldsymbol{g}\in\mathbb{C}^1((0,+\infty),H^s(\mathbb{R}^3))$ and $(n,c,\boldsymbol{v})^T\in\mathcal{B}_R$. Then, for any $\tau>0$, the solution $(\Gamma,\Lambda,\boldsymbol{h})^T$ of the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) satisfies*

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\left|\nabla_y^s\partial_\tau\Gamma\right|^2+\left|\nabla_y^s\partial_\tau\Lambda\right|^2+\sum_{i=1}^{3}\left|\nabla_y^s\partial_\tau h_i\right|^2\right)dy\\
&\quad\lesssim e^{-C_{a,R,\kappa,\nu,\mu,\delta}\tau}\int_{\mathbb{R}^3}\left(\left|\nabla_y^s\partial_\tau\Gamma_0\right|^2+\left|\nabla_y^s\partial_\tau\Lambda_0\right|^2+\sum_{i=1}^{3}\left|\nabla_y^s\partial_\tau h_{0i}\right|^2\right)dy\\
&\qquad+e^{-C_{a,R,\kappa,\nu,\mu,\delta}\tau}\int_0^{+\infty}\left(\left\|\nabla_y^s\partial_\tau f_1\right\|_{\mathbb{L}^2}^2+\left\|\nabla_y^s\partial_\tau f_2\right\|_{\mathbb{L}^2}^2+\sum_{i=1}^{3}\left\|\nabla_y^s\partial_\tau g_i\right\|_{\mathbb{L}^2}^2\right)d\tau,\quad\forall\tau>0,
\end{aligned}$$

where $C_{a,R,\kappa,\nu,\mu,\delta}$ is a positive constant depending on the constants a, R, κ, ν, μ, δ.

Proof Similar to getting the estimate in Lemma 2.2, we apply the operator $\partial_\tau\nabla_y^s$ to both sides of (2.8)–(2.10), then using a similar process to the proof of Lemma 2.2, we can obtain this result. □

Proposition 2.1 *Let $0<a\ll\frac{1}{8}$ and $0<s<\frac{3}{2}-5a$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+5}(\mathbb{R}^3)}\lesssim R\ll 1$, $f_i\in\mathbb{C}^1((0,+\infty),\mathbb{H}^s(\mathbb{R}^3))$ $(i=1,2)$, $\boldsymbol{g}\in\mathbb{C}^1((0,+\infty),H^s(\mathbb{R}^3))$ and $(n,c,\boldsymbol{v})^T\in\mathcal{B}_R$. Then, for any $\tau>0$, the linearized coupled system (2.13)–(2.17) with the initial data (2.19) and condition (2.20) admits a solution*

$$\Gamma\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);\mathbb{H}^{s-i}\big),$$

$$\Lambda\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);\mathbb{H}^{s-i}\big),$$

$$\boldsymbol{h}\in\overline{\mathcal{C}}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);H^{s-i}\big).$$

Moreover,

$$\|\Gamma\|^2_{C^s_0} + \|\Lambda\|^2_{C^s_0} + \|\boldsymbol{h}\|^2_{C^s_0} \lesssim \|\Gamma_0\|^2_{C^s_0} + \|\Lambda_0\|^2_{C_0} + \|\boldsymbol{h}_0\|^2_{C^s_0} + \|f_1\|^2_{C^s_0} + \|f_2\|^2_{C^s_0} + \|\boldsymbol{g}\|^2_{C^s_0},$$
$$\forall \tau > 0. \tag{2.54}$$

Proof Let $\mathbb{P}$ be the Leray projector onto the space of divergence free functions. We apply the Leray projector to system (2.5), we have

$$\begin{cases} \Gamma_t - \triangle\Gamma + (2\mu n - \sigma)\Gamma + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Gamma + \mathbf{h} \cdot \nabla n + \chi\nabla \cdot [\Gamma\nabla c + n\nabla\Lambda] = f_1(t,x), \\ \Lambda_t - \triangle\Lambda + \Lambda + (\mathbf{v} + \overline{\mathbf{u}}) \cdot \nabla\Lambda + \mathbf{h} \cdot \nabla c - \Gamma = f_2(t,x), \\ \mathbf{h}_t - \nu\mathbb{P}\triangle\mathbf{h} + \mathbb{P}(\mathbf{h} \cdot \nabla(\overline{\mathbf{u}} + \mathbf{v}) + (\overline{\mathbf{u}} + \mathbf{v}) \cdot \nabla\mathbf{h} - \Gamma\nabla\Phi) = \mathbb{P}\mathbf{g}(t,x). \end{cases} \tag{2.55}$$

In the similarity coordinates (2.12), we can rewrite the linear system (2.55) as

$$\partial_\tau U + (\mathcal{A} + \mathcal{N})U = T^* e^{-\tau} F,$$

where $U := (\Gamma, \Lambda, h_1, h_2, h_3)^T$, $\mathcal{N}(U) := (\mathbb{M}_1, \mathbb{M}_2, \mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3)^T$, $F := (f_1, f_2, \mathbb{P}g_1, \mathbb{P}g_2, \mathbb{P}g_3)^T$ and the matrix operator is given by

$$\mathcal{A} := \begin{pmatrix} -\mu\triangle_y & 0 & 0 & 0 & 0 \\ 0 & -\delta\triangle_y & 0 & 0 & 0 \\ 0 & 0 & -\nu\mathbb{P}\triangle_y & 0 & 0 \\ 0 & 0 & 0 & -\nu\mathbb{P}\triangle_y & 0 \\ 0 & 0 & 0 & 0 & -\nu\mathbb{P}\triangle_y \end{pmatrix}_{5\times 5},$$

and

$$\begin{aligned} \mathbb{M}_1(\Gamma, \Lambda, \mathbf{h}) :=& -\triangle_y\Gamma - \frac{y}{2} \cdot \nabla_y\Gamma + T^* e^{-\tau}(2\mu n - \sigma)\Gamma + a y_1\partial_{y_1}\Gamma + a y_2\partial_{y_2}\Gamma \\ & - 2a y_3\partial_{y_3}\Gamma + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau}(y_2\partial_{y_1}\Gamma + y_1\partial_{y_2}\Gamma) \\ & + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau}\mathbf{h} \cdot \nabla_y n + \chi\sum_{i=1}^{3}\partial_{y_i}(\Gamma\partial_{y_i}c) + \chi\nabla_y \cdot (n\nabla_y\Lambda), \\ \mathbb{M}_2(\Gamma, \Lambda, \mathbf{h}) :=& -\frac{y}{2} \cdot \nabla_y\Lambda + T^* e^{-\tau}\Lambda + a y_1\partial_{y_1}\Lambda + a y_2\partial_{y_2}\Lambda - 2a y_3\partial_{y_3}\Lambda \\ & + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau}(y_2\partial_{y_1}\Lambda + y_1\partial_{y_2}\Lambda) + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau}\left(\sum_{i=1}^{2} h_i\partial_{y_i}c_i - \Gamma\right), \\ \mathbb{N}_1(\Gamma, \Lambda, \mathbf{h}) :=& -\frac{y}{2} \cdot \nabla_y h_1 + a h_1 + a y_1\partial_{y_1}h_1 + a y_2\partial_{y_2}h_1 - 2a y_3\partial_{y_3}h_1 \\ & + k\left(T^*\right)^{2a+1} e^{-(2a+1)\tau}(h_2 + y_2\partial_{y_1}h_1 + y_1\partial_{y_2}h_1) - \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau}\Gamma\partial_{y_1}\Phi \\ & + \left(T^*\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}(h_i\partial_{y_i}w_1 + w_i\partial_{y_i}h_1), \end{aligned}$$

$$\begin{aligned}\mathbb{N}_2(\Gamma,\Lambda,\mathbf{h}) &:= -\frac{y}{2}\cdot\nabla_y h_2 + ah_2 + ay_1\partial_{y_1}h_2 + ay_2\partial_{y_2}h_2 - 2ay_3\partial_{y_3}h_2\\ &\quad + k\big(T^*\big)^{2a+1}e^{-(2a+1)\tau}(-h_1 + y_2\partial_{y_1}h_2 - y_1\partial_{y_2}h_2) - \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\Gamma\partial_{y_2}\Phi\\ &\quad + \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}(h_i\partial_{y_i}w_2 + w_i\partial_{y_i}h_2),\\ \mathbb{N}_3(\Gamma,\Lambda,\mathbf{h}) &:= -\frac{y}{2}\cdot\nabla_y h_3 - ah_3 + ay_1\partial_{y_1}h_3 + ay_2\partial_{y_2}h_3 - 2ay_3\partial_{y_3}h_3\\ &\quad + k\big(T^*\big)^{2a+1}e^{-(2a+1)\tau}(y_2\partial_{y_1}h_3 - y_1\partial_{y_2}h_3) - \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\Gamma\partial_{y_3}\Phi\\ &\quad + \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau}\sum_{i=1}^{3}(h_i\partial_{y_i}w_3 + w_i\partial_{y_i}h_3).\end{aligned}$$

Obviously, there is no singular coefficient in the linear operator $\mathcal{A}+\mathcal{N}$ by noticing (2.55). We follow the idea of [38] to show the linear operator $\mathcal{A}+\mathcal{N}$ generate a strongly continuous semigroup $e^{(\mathcal{A}+\mathcal{N})\tau}$ in Sobolev space $\mathbb{H}^s(\mathbb{R}^3)\times\mathbb{H}^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3)$. To see this, by the same process as getting (2.53), for the constants $0 < a \ll \frac{1}{8}$ and $0 < s < \frac{3}{2} - 5a$, we can obtain

$$\begin{aligned}&\int_\Omega \nabla_y^s U\cdot\nabla_y^s\big((\mathcal{A}+\mathcal{N})U\big)\,dy\\ &\quad\lesssim -\bigg(\frac{3}{4} - \frac{s}{2} - T^*e^{-\tau}(b+\sigma) - \frac{(T^*)^{\frac{1}{2}}}{2}e^{-\frac{1}{2}\tau} - C_R\bigg)\|\Gamma\|_{\mathbb{L}^2}^2\\ &\qquad - \bigg(\frac{3}{4} - \frac{s}{2} + T^*e^{-\tau}(1-b) - \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau} - C_R\bigg)\|\Lambda\|_{\mathbb{L}^2}^2\\ &\qquad - \bigg(a + \frac{3}{4} - \frac{s}{2} - T^*e^{-\tau}b - C_R\bigg)\big(\big\|\nabla_y^s h_1\big\|_{\mathbb{L}^2}^2 + \big\|\nabla_y^s h_2\big\|_{\mathbb{L}^2}^2\big)\\ &\qquad - \bigg(\frac{3}{4} - \frac{s}{2} - 2a - T^*e^{-\tau}b - C_R\bigg)\big\|\nabla_y^s h_3\big\|_{\mathbb{L}^2}^2\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}&\frac{3}{4} - \frac{s}{2} - T^*e^{-\tau}(b+\sigma) - \frac{(T^*)^{\frac{1}{2}}}{2}e^{-\frac{1}{2}\tau} - C_R > 0,\\ &\frac{3}{4} - \frac{s}{2} + T^*e^{-\tau}(1-b) - \big(T^*\big)^{\frac{1}{2}}e^{-\frac{1}{2}\tau} - C_R > 0,\\ &a + \frac{3}{4} - \frac{s}{2} - T^*e^{-\tau}b - C_R > 0,\\ &\frac{3}{4} - \frac{s}{2} - 2a - T^*e^{-\tau}b - C_R > 0.\end{aligned}$$

Hence by (2.56), we get

$$\int_\Omega \nabla_y^s U\cdot\nabla_y^s\big((\mathcal{A}+\mathcal{N})U\big)\,dy \le 0.$$

Hence the linear operator $\mathcal{A}+\mathcal{N}$ is a linear dissipative operator in $\mathbb{H}^s(\mathbb{R}^3)\times\mathbb{H}^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3)$. Moreover, if we set

$$(\mathcal{A}+\mathcal{N})U = 0,$$

then, by (2.56), we know the linear operator $\mathcal{A}+\mathcal{N}$ is injective. Furthermore, we can verify that this linear operator is surjective by using the standard theory of elliptic-type equations of the general order. Thus the linear operator $\mathcal{A}+\mathcal{N}$ can generate a strongly continuous semigroup $e^{(\mathcal{A}+\mathcal{N})\tau}$ in Sobolev space $\mathbb{H}^s(\mathbb{R}^3)\times\mathbb{H}^s(\mathbb{R}^3)\times H^s(\mathbb{R}^3)$ by the Lumer–Phillips theorem [23]. Therefore, the linear system (2.55) admits a solution

$$\Gamma\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);\mathbb{H}^{s-i}\big),$$

$$\Lambda\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);\mathbb{H}^{s-i}\big),$$

$$\mathbf{h}\in\overline{\mathcal{C}}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,+\infty);H^{s-i}\big).$$

Furthermore, it follows from Lemmas 2.2–2.3 that (2.54) holds. $\square$

Recalling the self-similarity coordinates (2.12), the original coordinate can be expressed by the self-similarity coordinates as follows:

$$t=T\big(1-e^{-\tau}\big),\qquad x=y\sqrt{T^*-t},$$

so we can directly use Proposition 2.1 to get the following result.

Proposition 2.2 *Let $0<a\ll\frac{1}{8}$ and $0<s<\frac{3}{2}-5a$ be constants. Assume that $\|\Phi\|_{\mathbb{H}^{s+5}(\mathbb{R}^3)}\lesssim R\ll 1$, $f_i\in\mathbb{C}^1((0,T^*),\mathbb{H}^s(\mathbb{R}^3))$ $(i=1,2)$, $\boldsymbol{g}\in\mathbb{C}^1((0,T^*),H^s(\mathbb{R}^3))$ and $(n,c,\boldsymbol{v})^T\in\mathcal{B}_R$. Then the linearized coupled system (2.5) with the initial data (2.2) and condition (2.3) admits a local solution*

$$\Gamma\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,T^*);\mathbb{H}^{s-i}(\mathbb{R}^3)\big),$$

$$\Lambda\in\mathcal{C}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,T^*);\mathbb{H}^{s-i}(\mathbb{R}^3)\big),$$

$$\boldsymbol{h}\in\overline{\mathcal{C}}_0^s:=\bigcap_{i=0}^{1}\mathbb{C}^i\big((0,T^*);H^{s-i}(\mathbb{R}^3)\big).$$

Moreover,

$$\|\Gamma\|_{\mathcal{C}_0^s}^2+\|\Lambda\|_{\mathcal{C}_0^s}^2+\|\boldsymbol{h}\|_{\overline{\mathcal{C}}_0^s}^2\lesssim\|\Gamma_0\|_{\mathcal{C}_0^s}^2+\|\Lambda_0\|_{\mathcal{C}_0}^2+\|\boldsymbol{h}_0\|_{\overline{\mathcal{C}}_0^s}^2+\|f_1\|_{\mathcal{C}_0^s}^2+\|f_2\|_{\mathcal{C}_0^s}^2+\|\boldsymbol{g}\|_{\overline{\mathcal{C}}_0^s}^2,$$
$$\forall t\in\big(0,T^*\big).$$

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