Abdelwahed et al. *Boundary Value Problems* (2021) 2021:50
https://doi.org/10.1186/s13661-021-01523-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH
Open Access

Application of the topological sensitivity method to the reconstruction of plasma equilibrium domain in a tokamak

Mohamed Abdelwahed[1], Nejmeddine Chorfi[1*], Maatoug Hassine[2] and Imen Kallel[2,3]

*Correspondence:
nchorfi@ksu.edu.sa
[1]Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
The topological sensitivity method is an optimization technique used in different inverse problem solutions. In this work, we adapt this method to the identification of plasma domain in a Tokamak. An asymptotic expansion of a considered shape function is established and used to solve this inverse problem. Finally, a numerical algorithm is developed and tested in different configurations.

MSC: Primary 49Q12; secondary 35K05

Keywords: Elasticity system; Geometric inverse problem; Topological sensitivity analysis

1 Introduction

The Tokamak is a type of fusion reactor and a method for building these reactors. It is a station that uses a strong magnetic field to confine the hot plasma in the shape of a torus (see Fig. 1). This method works by raising the temperature of the plasma to the point of nuclear fusion. Since the plasma has a very high temperature, it must be separated and removed and made not to touch the components of the reactor, otherwise it will be destroyed and this will cause it to cool and interrupt the fusion reaction. In reactors based on the Tokamak method, the plasma is made to float in the middle of the reactor without touching any of its parts by means of magnets that keep the plasma in a circular path.

This method is still being researched and experimented and has not reached the stage of economic exploitation, yet. In the future, the goal is to produce energy using this technique. The main problem to reach high performance is to construct the plasma magnetic equilibrium domain. This problem has been investigated in [2–4] using parametric optimization or control theory. A new idea using the topological gradient technique is considered in [5–13]. This technique consists in reconstructing the plasma domain by inserting some holes inside a fixed initial one. The position and form of these holes are characterized by an asymptotic expansion of a considered cost function.

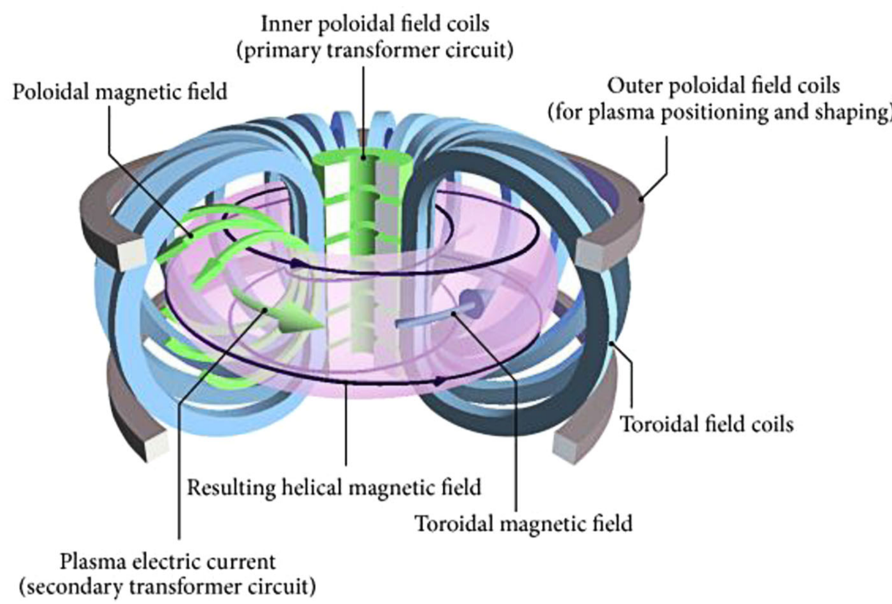

Figure 1 Tokamak [1]

The paper is organized as follows. In the next section, we present the model describing the plasma equilibrium and the formulation of the studied problem. Section 3 is devoted to the plasma reconstruction technique using the topological gradient method. In Sect. 4, the plasma reconstruction algorithm is described and some numerical results are presented.

2 Formulation of the problem

Consider the two-dimensional domain Ω representing the vacuum vessel containing a plasma domain Ω_p and $\Omega_v = \Omega\backslash\overline{\Omega_p}$. The plasma equilibrium solves in Ω_v the equation $\mathcal{G}\phi = 0$, where $\mathcal{G} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) - \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial}{\partial z})$ is the Grad–Shafranov operator and ϕ is the poloidal flux (see [14]).

We aim to determine the location of Γ_p, the boundary of Ω_p, from over-specified boundary data on $\Gamma = \partial\Omega$. In this inverse problem, ϕ satisfies

$$\begin{cases} \mathcal{G}\phi = 0 & \text{in } \Omega_v, \\ \frac{1}{r}\frac{\partial \phi}{\partial n} = \Phi & \text{on } \Gamma, \\ \phi = \phi_m & \text{on } \Gamma, \\ \phi = 0 & \text{on } \Gamma_p. \end{cases} \tag{1}$$

We remark that since Γ_p is unknown also Ω_v is unknown, which makes the problem ill posed in the sense of Hadamard.

In order to determine the location of the unknown plasma boundary Γ_p, we propose the following formulation for the considered inverse problem:

Knowing the magnetic field Φ and the poloidal flux ϕ_m on the boundary Γ, the idea consists in identifying the unknown plasma boundary $\Gamma_p = \partial\Omega_p$ where $\Omega_P \subset \Omega$ is the optimal solution of the topological optimization problem

$$\min_{D\subset\Omega} T(\Omega\backslash\overline{D}). \tag{2}$$

Here T is the boundary tracking function defined by

$$T(\Omega\backslash\overline{D}) = \int_{\Gamma} |\phi_D - \phi_m|^2 \,\mathrm{d}s \tag{3}$$

with ϕ_D being the solution to

$$\begin{cases} \mathcal{G}\phi_D = 0 & \text{in } \Omega\backslash\overline{D}, \\ \frac{1}{r}\frac{\partial\phi_D}{\partial n} = \Phi & \text{on } \Gamma, \\ \phi_D = 0 & \text{on } \partial D. \end{cases} \tag{4}$$

To solve problem (2), we use the topological gradient method. It corresponds to developing an asymptotic expansion of the function T as

$$T(\Omega\backslash\overline{\chi_{z,\varepsilon}}) - T(\Omega) = f(\varepsilon)\delta T(z) + o\big(f(\varepsilon)\big),$$

where $f(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$ and $\chi_{z,\varepsilon}$ is a geometric perturbation created near the point $z = (z_1, z_2) \in \Omega$ having a small size $\varepsilon > 0$ (chosen in such a way that $\chi_{z,\varepsilon} \subset \Omega$) and the shape $\chi_{z,\varepsilon} = z + \varepsilon\chi$ with $\chi \subset \mathbb{R}^2$ is a given, regular, and bounded domain containing the origin.

Further, δT is called the topological gradient and the minimum of T is obtained when δT is the most negative.

Using this technique, the optimal design of the unknown plasma Ω_P is constructed using a level set curve of the scalar function δT, namely

$$\Omega_P = \{x \in \Omega \text{ such that } \delta T \le c\},$$

where c is a negative constant, chosen in such a way that the shape function T decreases as most as possible.

3 Plasma reconstruction technique

Using the axisymmetric configuration and a horizontal cut, the first formulation can be rewritten as follows: find the unknown domain Ω_p occupied by the plasma as the optimal solution to the optimization problem (2), where ϕ_D is solution to the anisotropic system

$$\begin{cases} -\operatorname{div}(\gamma(x)\nabla\phi_D) = 0 & \text{in } \Omega\backslash\overline{D}, \\ \gamma(x)\nabla\phi_D \cdot \mathbf{n} = \Phi & \text{on } \Gamma, \\ \phi_D = 0 & \text{on } \partial D, \end{cases}$$

with γ being a scalar positive function defined by

$$\gamma(x) = \frac{1}{|x_1|}, \quad \forall x = (x_1, x_2) \in \Omega\backslash\overline{D}.$$

In the presence of the perturbation $\chi_{z,\varepsilon}$, we have

$$T(\Omega\backslash\overline{\chi_{z,\varepsilon}}) = \int_\Gamma |\phi_\varepsilon - \phi_m|^2 \,\mathrm{d}s$$

where ϕ_ε is the solution to

$$\begin{cases} -\operatorname{div}(\gamma(x)\nabla\phi_\varepsilon) = 0 & \text{in } \Omega_{z,\varepsilon}, \\ \gamma(x)\nabla\phi_\varepsilon.\mathbf{n} = \Phi & \text{on } \Gamma, \\ \phi_\varepsilon = 0 & \text{on } \partial\chi_{z,\varepsilon}, \end{cases} \tag{5}$$

with $\Omega_{z,\varepsilon}$ being the perturbed domain defined by $\Omega_{z,\varepsilon} = \Omega\backslash\overline{\chi_{z,\varepsilon}}$.

One can establish the following asymptotic expansion for the shape function T (see [15]).

Theorem 1 *The shape function T admits the following asymptotic expansion*:

$$T(\Omega\backslash\overline{\chi_{z,\varepsilon}}) - T(\Omega) = \frac{-1}{\log(\varepsilon)}\delta T(z) + o\left(\frac{-1}{\log(\varepsilon)}\right),$$

where δT is the topological gradient given by

$$\delta T(z) = 2\pi\gamma(z)\phi_0(z)\varphi_0(z), \quad z \in \Omega,$$

with $\phi_0(z)$ being the solution to problem (5) *for $\varepsilon = 0$ and φ_0 the solution of the associated adjoint problem.*

4 Numerical tests

For the numerical tests, we consider two cases of computational domain Ω. The unknown plasma boundary is defined by the level set curve of the topological gradient

$$\delta T(x_1,x_2) = \frac{2\pi}{x_1}\phi_0(x_1,x_2)\varphi_0(x_1,x_2), \quad \forall(x_1,x_2) \in \Omega. \tag{6}$$

4.1 Plasma reconstruction algorithm

The main steps of our numerical algorithm are the following.

The reconstruction algorithm

- Solve the direct and adjoint problems in Ω,
- Compute the topological gradient δT defined in (6),
- Determine the plasma location

  $$\Omega_P = \{(x_1,x_2) \in \Omega; \delta T(x_1,x_2) \leq (1-\varepsilon)\delta T_{\min} < 0\},$$

  where $\delta T_{\min} = \min_{(x,y)\in\Omega} \delta T(x_1,x_2)$ and $\varepsilon \in]0,1[$ is chosen in such a way that the function T decreases as much as possible.

Next, we apply this reconstruction procedure for two numerical tests.

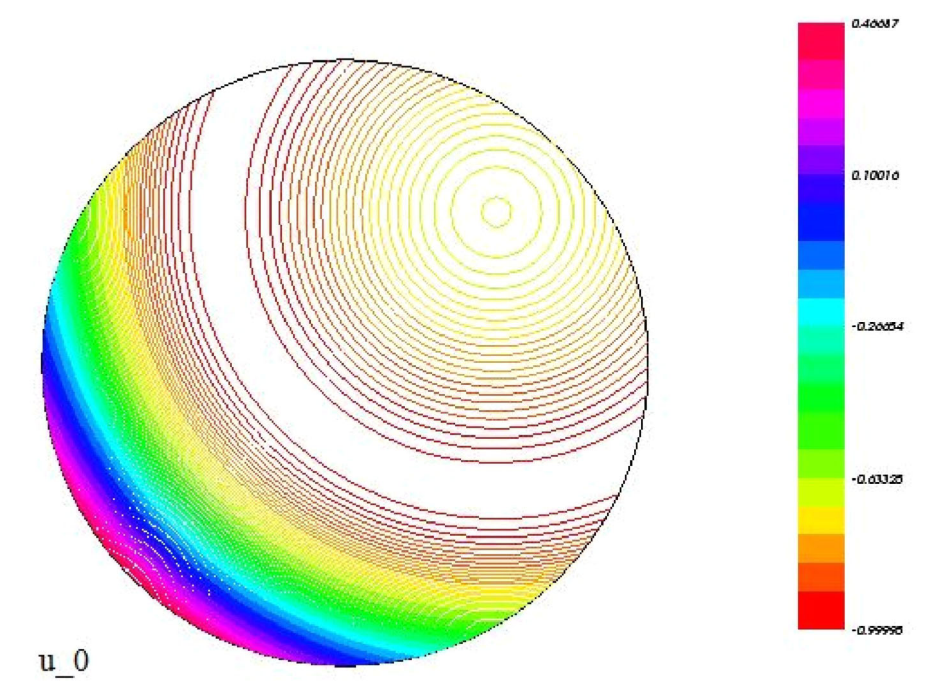

Figure 2 Isovalues of ϕ_0

4.2 First test

In this case we have used the following data:

- The vacuum vessel region is defined by the disc $\Omega = B(\mathcal{C}_1, 1)$, with $\mathcal{C}_1 = (2, 0)$.
- The exact plasma domain is defined by the disc $\Omega_P^{ex} = B(\mathcal{C}_2, 0.2)$, with $\mathcal{C}_2 = (3/2, 0)$.
- The Dirichlet and Neumann boundary data are given by

$$\phi_m(x,y) = \frac{4-x-y}{x}\sin(0.5+x+y), \quad (x,y) \in \Gamma,$$
$$\Phi(x,y) = \frac{1}{x}\frac{\partial \phi_m}{\partial n}(x,y), \quad (x,y) \in \Gamma.$$

In Fig. 2, we present the obtained isovalues of the solution ϕ_0.

The isovalues of the topological gradient δT are described in Fig. 3.

To evaluate the accuracy of our approach, we introduce the following error function which defines the Hausdorff distance between the exact Ω_P^{ex} and obtained Ω_P^{ε} plasma domain

$$er(\varepsilon) = \frac{\operatorname{mes}(\Omega_P^{ex} \cup \Omega_P^{\varepsilon}) - \operatorname{mes}(\Omega_P^{\varepsilon} \cap \Omega_P^{ex})}{\operatorname{mes}(\Omega_P^{ex})}$$

with

$$\Omega_P^{\varepsilon} = \left\{X \in \Omega; \delta T(X) < (1-\varepsilon)\delta T_{\min}\right\}.$$

The variation of the error function er is illustrated in Fig. 4. As one can see in Fig. 4, the optimal choice of the parameter ε is $\varepsilon = 0.88$. In this test, we identify the plasma domain Ω_P^{ex}.

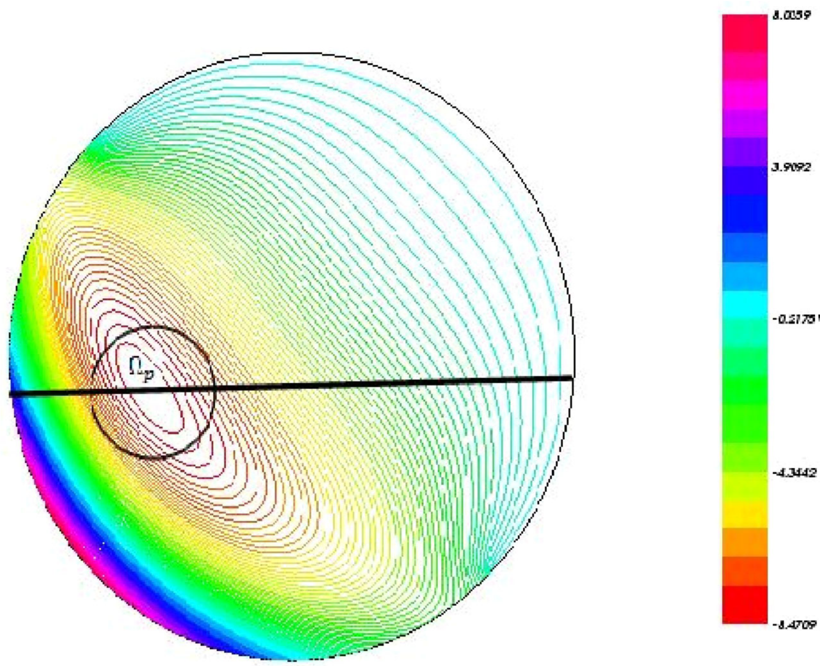

Figure 3 Isovalues of the topological gradient δT

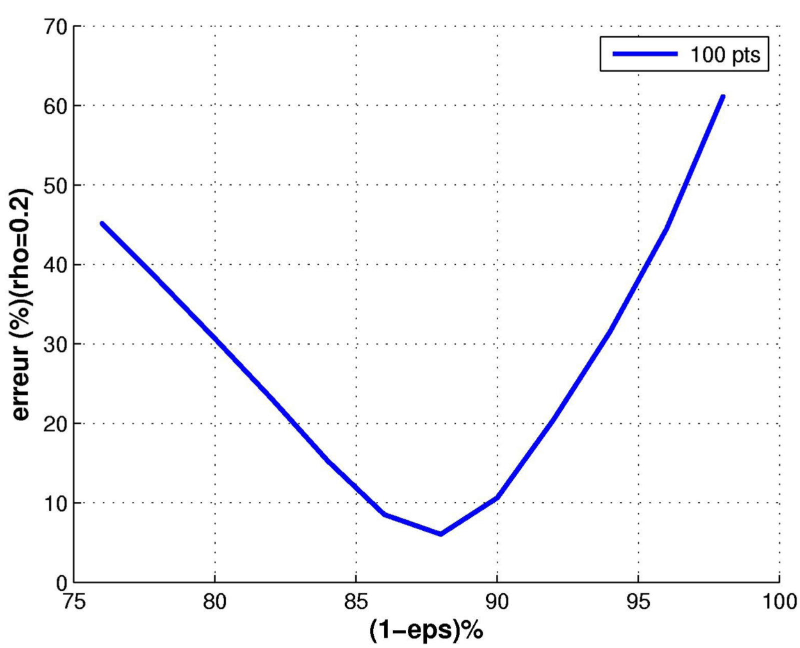

Figure 4 Variation of the error function *er*

4.3 Second test

In this case we have used the following data:

- $\Omega = [\frac{\pi}{3}, \frac{2\pi}{3}] \times [0, \frac{\pi}{3}]$.
- $\Omega_P^{ex} = B(\mathcal{C}_2, 0.2)$, with $\mathcal{C}_2 = (\frac{\pi}{2}, \frac{\pi}{6})$.

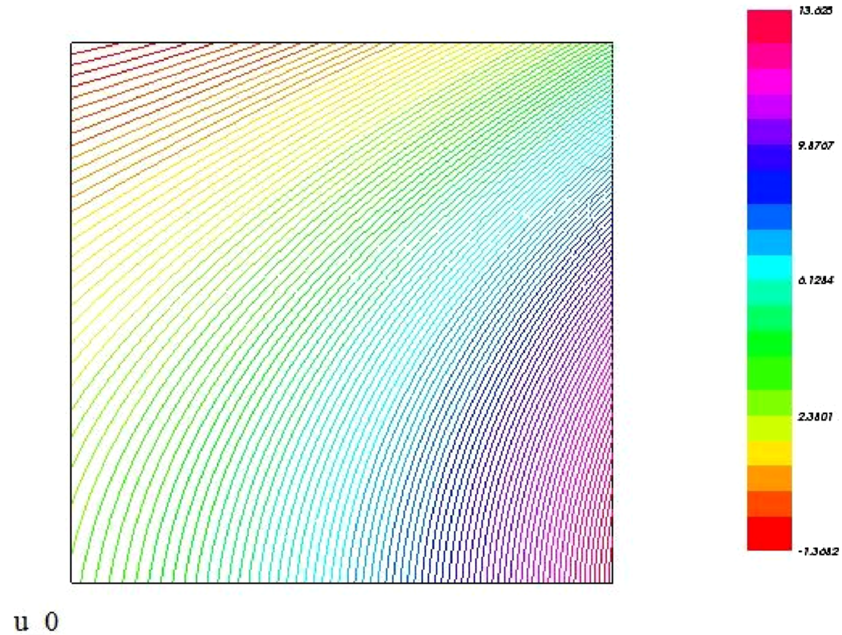

Figure 5 Isovalues of ϕ_0

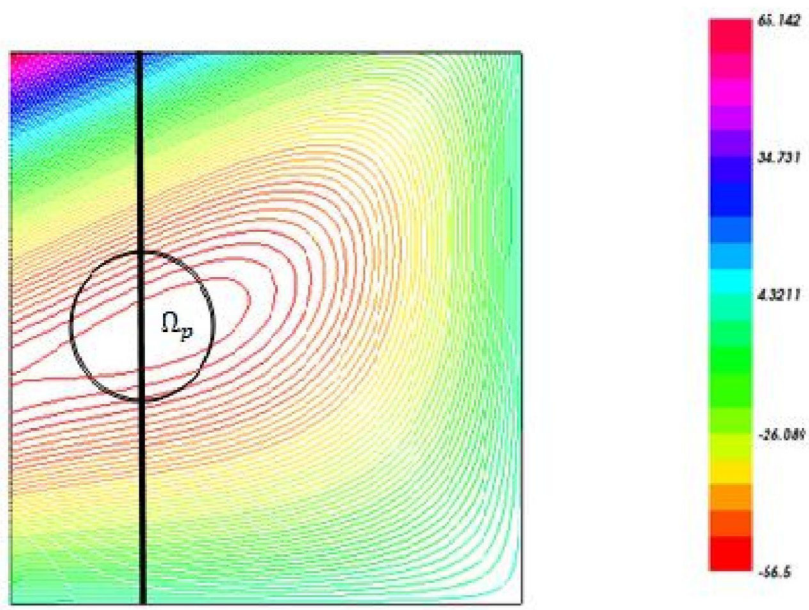

Figure 6 Isovalues of the topological gradient δT

- The Dirichlet and Neumann boundary data are given by

$$\phi_m(x,y) = 2e^x \cos(y) - e^y \sin(x) - 1.75, (x,y) \in \Gamma,$$

$$\Phi(x,y) = \frac{1}{x}\frac{\partial \phi_m}{\partial n}(x,y), (x,y) \in \Gamma.$$

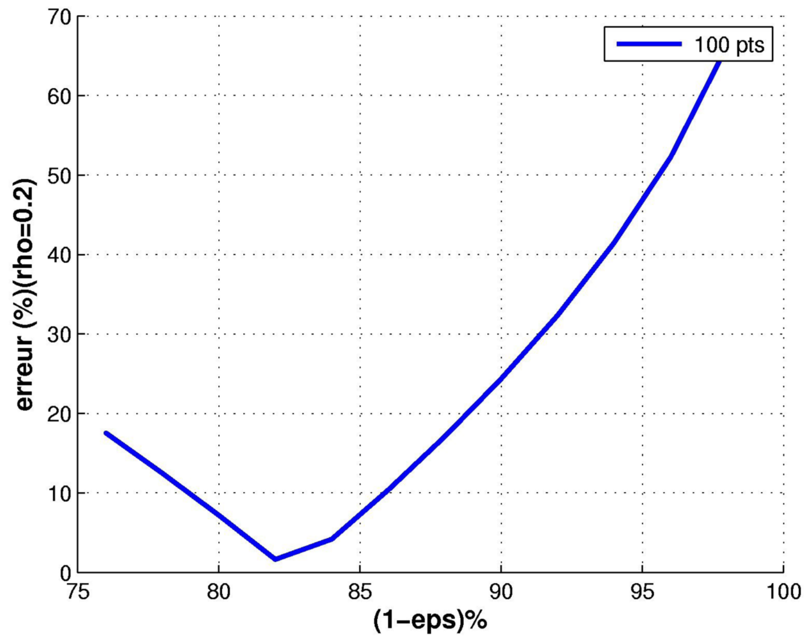

Figure 7 Variation of the error function *er*

The result of this test are described in Figs. 5, 6, and 7. In Fig. 5, we plot the isovalues of the solution ϕ_0.

Figure 6 describes the isovalues of the topological gradient δT. The variation of the error function *er* describing the Hausdorff distance between the exact Ω_P^{ex} and obtained Ω_P^{ε} plasma domain is illustrated in Fig. 7. The optimal choice of the parameter ε corresponds to $\varepsilon = 0.82$.

We remark that the plasma domain Ω_P^{ex} has been well identified in one iteration.

Acknowledgements
The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No (RG-1440-061).

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia. [2]Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, Monastir, Tunisia. [3]College of Sciences, Northern Border University, P.O. Box 1631, Dammam, Saudi Arabia.

Publisher's Note

Received: 7 February 2021 Accepted: 18 April 2021 Published online: 08 May 2021

References

1. Li, S., Jiang, Z., Xu, C.: Optimal tracking for a divergent-type parabolic PDE system in current profile control. Abstr. Appl. Anal. **2014**, Article ID 940965, 1–12 (2014)
2. Blum, J.: Numerical Simulation and Optimal Control in Plasma Physics with Applications to the Tokamaks. Gauthier-Villars Series in Modern Applied Mathematics. Wiley, New York (1989)
3. Blum, J., Boulbe, C., Faugeras, B.: Real-time plasma equilibrium reconstruction in a tokamak. In: Proceedings of ICIPE Conference, Dourdan, France (2008)
4. Blum, J., Boulbe, C., Faugeras, B.: Reconstruction of the equilibrium of the plasma in a tokamak and identification of the current density profile in real time. J. Comp. Physiol. **231**, 960–980 (2012)
5. Amstutz, S.: The topological asymptotic for the Navier–Stokes equations. ESAIM Control Optim. Calc. Var. **11**(3), 401–425 (2005)
6. Bonnet, M.: Topological sensitivity for 3d elastodynamic and acoustic inverse scattering in the time domain. Comput. Methods Appl. Mech. Eng. **195**, 5239–5254 (2006)
7. Garreau, S., Guillaume, P., Masmoudi, M.: The topological asymptotic for PDE systems: the elasticity case. SIAM J. Control Optim. **39**, 1756–1778 (2001)
8. Guillaume, P., Sid Idris, K.: Topological sensitivity and shape optimization for the Stokes equations. SIAM J. Control Optim. **43**, 1–31 (2004)
9. Hassine, M., Masmoudi, M.: The topological sensitivity analysis for the quasi-Stokes problem. ESAIM Control Optim. Calc. Var. **10**, 478–504 (2004)
10. Abdelwahed, M., Chorfi, N., Malek, R.: Reconstruction of tesla micro-valve using topological sensitivity analysis. Adv. Nonlinear Anal. **9**, 567–590 (2020)
11. Abdelwahed, M., Al Salam, A., Chorfi, N.: Topological sensitivity analysis of a time-dependent nonlinear problem. Bound. Value Probl. **2019**, 23, 1–12 (2019)
12. Malek, R., Abdelwahed, M., Chorfi, N., Hassine, M.: Obstacle detection in fluid flow using asymptotic analysis technique. Bound. Value Probl. **2020**, 45, 1–14 (2020)
13. Ghezaiel, E., Abdelwahed, M., Chorfi, N., Hassine, M.: Topological sensitivity analysis for a three-dimensional parabolic type problem. Asymptot. Anal. **120**(3–4), 249–272 (2020)
14. Alves, C., Ammar, H.: Boundary integral formulae for the reconstruction of imperfections of small diameter in an elastic medium. SIAM J. Appl. Math. **62**, 94–106 (2001)
15. Kallel, I.: Geometric identification problem for an anisotropic operator. PhD thesis, Monastir university (2019)