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Structural stability for the Forchheimer equations interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^3$

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Abstract

The structural stability for the Forchheimer fluid interfacing with a Darcy fluid in a bounded region in $\mathbb{R}^3$ was studied. We assumed that the nonlinear fluid was governed by the Forchheimer equations in Ω_1, while in Ω_2, we supposed that the flow satisfies the Darcy equations. With the aid of some useful a priori bounds, we were able to demonstrate the continuous dependence results for the Forchheimer coefficient λ.

MSC: 35B40; 35Q30; 76D05

Keywords: Structural stability; Forchheimer equations; Darcy equations; Interface boundary condition

1 Introduction

Many papers in the literature studied the structural stability for the partial differential equations. They obtained the results of continuous dependence or convergence on the equations. Unlike the traditional stability study, they focused on the changes of the coefficients of the equations. This is to say, the structural stability mainly focuses on changes in the model itself, while the traditional stability focuses on the initial data. For a review of the nature of the structural stability, one could see the monograph of Ames and Straughan [3]. In continuum mechanics problems, it is important to obtain the continuous dependence result on the model itself. This problem is discussed for several different partial differential equations by Hirsch and Smale [8]. We usually want to know if a small change in the constructive coefficient in the equations themselves will lead to drastic changes in the solutions. If the answer is no, we can do further studies. It is very important for us to study the structural stability for the model.

There are many models that have been studied in a porous medium. Nield and Beijan [14] and Straughan [27, 28] discussed these models in their books. The authors of [2, 16, 17]studied these models in an unbounded domain and obtained some Saint-Venant-type results. They mainly focused on the studies of the Brinkman, Darcy, and Forchheimer equations in porous media.

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Recently, some authors began to study the structural stability for equations in porous media. They obtained some continuous dependence results. For a review of these papers, one could see Payne and Straughan [19–22], Scott [23], Scott and Straughan [24], Straughan [26], Ames and Payne [1], Celebi, Kalantarov and Ugurlu [4, 5], Franchi and Straughan [6], Harfash [7], Kaloni and Guo [9], Li and Lin [10], Lin and Payne [11, 12], Payne, Song and Straughan [18], and Straughan and Hutter [30]. The Brinkman, Forchheimer, and Darcy equations are widely studied in these papers. They consider only one fluid in the domain. In reality, there typically exists more than one fluid in a domain. It is interesting to study two fluids interfacing with each other in one domain.

In [21], Payne and Straughan established the structural stability result for the Brinkman–Darcy interfacing equations. They studied the continuous dependence result for the interface boundary coefficient α_1. We change the Brinkman equations to the Forchheimer equations. However, if we use the same method as in [21], we cannot obtain a similar result. Since the equations do not contain the term $\triangle u$, it is difficult to deal with the nonlinear term $|u|u_i$. Recently, in [13] and [25], the authors studied the structural stability for the Forchheimer–Darcy interfacing problems in a bounded domain. In order to obtain their results, the authors obtained the results $\sup_{[0,\tau]} \|T\|_\infty \leq T_M$ and $\sup_{[0,\tau]} \|S\|_\infty \leq S_M$ for the temperatures T and S using the method proposed by Payne, Rodrigues, and Straughan in [15]. In the present paper, the equations for the temperatures are not the same as in [13] and [25]. We cannot get the same results by using the method proposed in [15]. We must seek a new method to get the results. How to get the maximum estimates and the related bounds for T and S is the biggest innovation of this paper. In our opinion, it is of great significance to study the structural stability for the Forchheimer–Darcy interfacing fluids.

The purpose of this paper is to study the manner in which a solution to a flow in a fluid which borders a porous medium depends on a coefficient in the Forchheimer equation. Thus, let an appropriate part of the plane $z = x_3 = 0$ denote the boundary between a porous medium occupying a bounded region Ω_2 in $\mathbb{R}^3$ and a nonlinear fluid occupying a bounded region Ω_1 in $\mathbb{R}^3$, and the governing equations be Forchheimer equations. We denote the interface by L, and further denote the remaining parts of the boundaries of Ω_1 and Ω_2 by Γ_1 and Γ_2. We also denote $\partial\Omega_1 = \Gamma_1 \cup L$ and $\partial\Omega_2 = \Gamma_2 \cup L$.

We are interested in the solution of the following initial-boundary value problem. The governing equations for Forchheimer flow are (see [29])

$$\begin{cases} \frac{\partial u_i}{\partial t} = -\lambda |u| u_i - p_{,i} + g_i T, \\ \frac{\partial u_i}{\partial x_i} = 0, \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa \Delta T + Q, \end{cases} \tag{1.1}$$

where u_i, p, and T are the velocity, pressure, and temperature, κ is the thermal diffusivity. Here $g_i(x)$ are gravity vector functions, and $Q(x,t)$ is a prescribed heat source. We assume that g_i satisfy $|g| \leq G_1$. Here also Δ is the Laplace operator.

Equations (1.1) hold in the region $\Omega_1 \times [0,\tau]$, where Ω_1 is a bounded, simply connected, and star-shaped domain with boundary $\partial\Omega_1$ in $\mathbb{R}^3$, and τ is a given number satisfying $0 \leq \tau < \infty$.

The Darcy equations governing the flow are (see [27])

$$\begin{cases} v_i = -q_{,i} + g_i S, \\ \frac{\partial v_i}{\partial x_i} = 0, \\ \frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} = \kappa \Delta S + Q_s, \end{cases} \tag{1.2}$$

where v_i, q, and S are the velocity, pressure, and temperature, while $Q_s(x,t)$ is a prescribed heat source.

Equations (1.2) hold in the region $\Omega_2 \times [0,\tau]$, where Ω_2 is a bounded, simply connected, and star-shaped domain with boundary $\partial\Omega_2$ in $\mathbb{R}^3$, and τ is a given number satisfying $0 \le \tau < \infty$.

We impose the boundary and initial conditions as follows:

$$\begin{cases} u_i = 0, T = T_U(x,t), & (x,t) \in \Gamma_1 \times [0,\tau], \\ v_i n_i = 0, S = S_U(x,t), & (x,t) \in \Gamma_2 \times [0,\tau]. \end{cases} \tag{1.3}$$

We assume further that

$$\begin{cases} u_i(x,0) = f_i(x), \qquad T(x,0) = T_0(x), & x \in \Omega_1, \\ S(x,0) = S_0(x), & x \in \Omega_2. \end{cases} \tag{1.4}$$

Finally, the interfacing conditions are taken from [21] as

$$\begin{cases} u_3 = v_3, \qquad T = S, \qquad T_{,3} = S_{,3}, \\ q = p \end{cases} \tag{1.5}$$

on $L \times \{t > 0\}$.

In the next section, we will derive some a priori bounds which will be used in deriving our main results. In Sect. 3, the convergence results for the Forchheimer coefficient are obtained.

In this present paper, the comma is used to indicate differentiation, and the differentiation with respect to the direction x_k is denoted as "$,k$", thus $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$. Hence, $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$.

2 A priori bounds

We now begin to derive a priori bounds for both T and S.

First, we introduce the function H, which takes the same boundary values as T:

$$\begin{cases} \Delta H = H_{,t}, & (x,t) \in \Omega_1 \times [0,\tau], \\ H(x,0) = T_0(x), & x \in \Omega_1, \\ H(x,t) = T_U(x,t), & (x,t) \in \Gamma_1 \times [0,\tau], \\ H_{,t}(x,t) = T_{U,t}(x,t), & (x,t) \in \Gamma_1 \times [0,\tau]. \end{cases} \tag{2.1}$$

Next, we introduce the function I, which takes the same boundary values as S:

$$\begin{cases} \Delta I = I_{,t}, & (x,t) \in \Omega_2 \times [0,\tau], \\ I(x,0) = S_0(x), & x \in \Omega_2, \\ I(x,t) = S_U(x,t), & (x,t) \in \Gamma_2 \times [0,\tau], \\ I_{,t}(x,t) = S_{U,t}(x,t), & (x,t) \in \Gamma_2 \times [0,\tau]. \end{cases} \tag{2.2}$$

On $L \times \{t > 0\}$,we let

$$\begin{cases} H = I, \\ H_{,i} = I_{,i}, \qquad H_{,t} = I_{,t}. \end{cases} \tag{2.3}$$

If we let

$$F = \begin{cases} H, & (x,t) \in \Omega_1 \times [0,\tau], \\ I, & (x,t) \in \Omega_2 \times [0,\tau], \end{cases} \tag{2.4}$$

we get

$$\begin{cases} \Delta F = F_{,t}, \quad (x,t) \in \Omega \times [0,\tau], \\ F(x,t) = \begin{cases} T_U(x,t), & (x,t) \in \Gamma_1 \times [0,\tau], \\ S_U(x,t), & (x,t) \in \Gamma_2 \times [0,\tau], \end{cases} \\ F(x,0) = \begin{cases} T_0(x), & x \in \Omega_1, \\ S_0(x), & x \in \Omega_2. \end{cases} \end{cases} \tag{2.5}$$

If we let

$$F_M = \max\Big\{ \sup_{\Omega_1} T_0, \sup_{\Omega_2} S_0, \sup_{\Gamma_1 \times [0,\tau]} T_U, \sup_{\Gamma_2 \times [0,\tau]} S_U \Big\}, \tag{2.6}$$

we know by maximum principle that $|F| \leq F_M$.

The following lemmas will be used in deriving our main result.

Lemma 1 *For the temperatures T and S, we have the following estimates*:

$$\begin{aligned} &\int_{\Omega_1} T^2\,dx + \int_{\Omega_2} S^2\,dx + \kappa \int_0^t \int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta + \kappa \int_0^t \int_{\Omega_2} S_{,i}S_{,i}\,dx\,d\eta \\ &\quad \leq 4 \int_0^t \int_{\Omega_1} T^2\,dx\,d\eta + 4 \int_0^t \int_{\Omega_2} S^2\,dx\,d\eta + \frac{4F_M^2}{\kappa} \int_0^t \int_{\Omega_1} u_i u_i\,dx\,d\eta \\ &\qquad + \frac{4F_M^2}{\kappa} \int_0^t \int_{\Omega_2} v_i v_i\,dx\,d\eta + 4 \int_{\Omega_1} H^2\,dx + 2 \int_0^t \int_{\Omega_1} H^2\,dx\,d\eta \\ &\qquad + 2\kappa \int_0^t \int_{\Omega_1} H_{,i}H_{,i}\,dx\,d\eta + 2 \int_0^t \int_{\Omega_1} H_{,\eta}H_{,\eta}\,dx\,d\eta + 4 \int_0^t \int_{\Omega_1} Q^2\,dx\,d\eta \end{aligned}$$

$$+ 4\int_{\Omega_2} I^2\,dx + 2\int_0^t\int_{\Omega_2} I^2\,dx\,d\eta + 2\kappa\int_0^t\int_{\Omega_2} I_{,i}I_{,i}\,dx\,d\eta$$
$$+ 2\int_0^t\int_{\Omega_2} I_{,\eta}I_{,\eta}\,dx\,d\eta + 4\int_0^t\int_{\Omega_2} Q_s^2\,dx\,d\eta. \tag{2.7}$$

Proof Multiplying (1.1)$_3$ by $2(T-H)$ and integrating over $\Omega_1\times[0,t]$, we find

$$\begin{aligned}
&2\int_0^t\int_{\Omega_1} u_iTT_{,i}\,dx\,d\eta - 2\int_0^t\int_{\Omega_1} u_iHT_{,i}\,dx\,d\eta\\
&\quad= 2\kappa\int_0^t\int_{\Omega_1}(T-H)\Delta T\,dx\,d\eta + 2\int_0^t\int_{\Omega_1}(T-H)Q\,dx\,d\eta\\
&\qquad- 2\int_0^t\int_{\Omega_1}(T-H)T_{,\eta}\,dx\,d\eta.
\end{aligned}\tag{2.8}$$

For the first function on the left-hand side of (2.8), using the divergence theorem and equations (1.4), (2.1), we find

$$\begin{aligned}
2\int_0^t\int_{\Omega_1} u_iTT_{,i}\,dx\,d\eta &= \int_0^t\int_{\Omega_1} u_i\big(T^2\big)_{,i}\,dx\,d\eta = \int_0^t\int_L T^2u_3n_3^{(1)}\,dS\,d\eta\\
&= -\int_0^t\int_L S^2v_3n_3^{(2)}\,dS\,d\eta.
\end{aligned}\tag{2.9}$$

For the second function on the left-hand side of (2.8), we have

$$2\left|\int_0^t\int_{\Omega_1} u_iHT_{,i}\,dx\,d\eta\right| \le \frac{2F_M^2}{\kappa}\int_0^t\int_{\Omega_1} u_iu_i\,dx\,d\eta + \frac{\kappa}{2}\int_0^t\int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta. \tag{2.10}$$

For the first function on the right-hand side of (2.8), using the divergence theorem and equations (1.3), (1.5), and (2.1), we get

$$\begin{aligned}
&2\kappa\int_0^t\int_{\Omega_1}(T-H)\Delta T\,dx\,d\eta\\
&\quad= 2\kappa\int_0^t\int_L TT_{,3}n_3^{(1)}\,dS\,d\eta - 2\kappa\int_0^t\int_L HT_{,3}n_3^{(1)}\,dS\,d\eta\\
&\qquad- 2\kappa\int_0^t\int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta + 2\kappa\int_0^t\int_{\Omega_1} H_{,i}T_{,i}\,dx\,d\eta\\
&\quad\le -2\kappa\int_0^t\int_L SS_{,3}n_3^{(2)}\,dS\,d\eta + 2\kappa\int_0^t\int_L IS_{,3}n_3^{(2)}\,dS\,d\eta\\
&\qquad- \kappa\int_0^t\int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta + \kappa\int_0^t\int_{\Omega_1} H_{,i}H_{,i}\,dx\,d\eta.
\end{aligned}\tag{2.11}$$

For the second function on the right-hand side of (2.8), we get

$$\begin{aligned}
&2\int_0^t\int_{\Omega_1}(T-H)Q\,dx\,d\eta\\
&\quad\le 2\int_0^t\int_{\Omega_1} Q^2\,dx\,d\eta + \int_0^t\int_{\Omega_1} T^2\,dx\,d\eta + \int_0^t\int_{\Omega_1} H^2\,dx\,d\eta.
\end{aligned}\tag{2.12}$$

For the third function on the right-hand side of (2.8), using equations (1.4) and (2.1), we find

$$\begin{aligned}
&-2\int_0^t\int_{\Omega_1}(T-H)T_{,\eta}\,dx\,d\eta \\
&\quad=2\int_0^t\int_{\Omega_1}(T-H)_{,\eta}T\,dx\,d\eta-2\int_{\Omega_1}(T-H)T\,dx \\
&\quad\le-\int_{\Omega_1}T^2\,dx-\int_{\Omega_1}T_0^2\,dx+2\int_{\Omega_1}HTdx-2\int_0^t\int_{\Omega_1}H_{,\eta}T\,dx\,d\eta \\
&\quad\le-\int_{\Omega_1}T_0^2\,dx-\frac{1}{2}\int_{\Omega_1}T^2\,dx+2\int_{\Omega_1}H^2\,dx+\int_0^t\int_{\Omega_1}H_{,\eta}H_{,\eta}\,dx\,d\eta \\
&\qquad+\int_0^t\int_{\Omega_1}T^2\,dx\,d\eta.
\end{aligned}\tag{2.13}$$

Combining (2.8)–(2.13), we obtain

$$\begin{aligned}
&\int_{\Omega_1}T^2\,dx+\kappa\int_0^t\int_{\Omega_1}T_{,i}T_{,i}\,dx\,d\eta+4\kappa\int_0^t\int_L SS_3n_3^{(2)}\,dS\,d\eta \\
&\qquad-4\kappa\int_0^t\int_L IS_{,3}n_3^{(2)}\,dS\,d\eta \\
&\quad\le 2\int_0^t\int_L S^2v_3n_3^{(2)}\,dx\,d\eta+4\int_0^t\int_{\Omega_1}T^2\,dx\,d\eta+\frac{4F_M^2}{\kappa}\int_0^t\int_{\Omega_1}u_iu_i\,dx\,d\eta \\
&\qquad+4\int_0^t\int_{\Omega_1}Q^2\,dx\,d\eta+4\int_{\Omega_1}H^2\,dx+2\int_0^t\int_{\Omega_1}H^2\,dx\,d\eta \\
&\qquad+2\kappa\int_0^t\int_{\Omega_1}H_{,i}H_{,i}\,dx\,d\eta+2\int_0^t\int_{\Omega_1}H_{,\eta}H_{,\eta}\,dx\,d\eta.
\end{aligned}\tag{2.14}$$

Similarly, we get

$$\begin{aligned}
&\int_{\Omega_2}S^2\,dx+\kappa\int_0^t\int_{\Omega_2}S_{,i}S_{,i}\,dx\,d\eta-4\kappa\int_0^t\int_L SS_3n_3^{(2)}\,dS\,d\eta \\
&\qquad+4\kappa\int_0^t\int_L IS_{,3}n_3^{(2)}\,dS\,d\eta \\
&\quad\le -2\int_0^t\int_L S^2v_3n_3^{(2)}\,dx\,d\eta+4\int_0^t\int_{\Omega_2}S^2\,dx\,d\eta+\frac{4F_M^2}{\kappa}\int_0^t\int_{\Omega_2}v_iv_i\,dx\,d\eta \\
&\qquad+4\int_0^t\int_{\Omega_2}Q_s^2\,dx\,d\eta+4\int_{\Omega_2}I^2\,dx+2\int_0^t\int_{\Omega_2}I^2\,dx\,d\eta \\
&\qquad+2\kappa\int_0^t\int_{\Omega_2}I_{,i}I_{,i}\,dx\,d\eta+2\int_0^t\int_{\Omega_2}I_{,\eta}I_{,\eta}\,dx\,d\eta.
\end{aligned}\tag{2.15}$$

Combining (2.14) and (2.15), we can get the desired result (2.7). □

Lemma 2 *If*

$$F_1(t)=\int_{\Omega_1}H_{,i}H_{,i}\,dx+\int_{\Omega_2}I_{,i}I_{,i}\,dx,$$

$$\begin{aligned}D_1(t) &= \left(\int_{\Omega_1} T_{0,i}T_{0,i}\,dx + \int_{\Omega_2} S_{0,i}S_{0,i}\,dx\right)\\&\quad + \left(\frac{4}{d}+\frac{8}{m}\right)\left(\int_0^t\int_{\Gamma_1}|\nabla_s H|^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2}|\nabla_s I|^2\,dS\,d\eta\right)\\&\quad + \frac{d^2}{4m}\left(\int_0^t\int_{\Gamma_1}(T_{U,t})^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2}(S_{U,t})^2\,dS\,d\eta\right),\end{aligned}$$

we have

$$F_1(t) \le D_1(t) + \frac{4}{d^2}\int_0^t D_1(\eta)e^{\frac{4}{d^2}(t-\eta)}\,d\eta = m_1(t), \tag{2.16}$$

where m and d are positive constants to be defined later.

Proof Start with the identity

$$2\int_{\Omega_1} x_iH_{,i}\Delta H\,dx = 2\int_{\Omega_1} x_iH_{,i}H_{,t}\,dx. \tag{2.17}$$

For the first function on the left-hand side of (2.17), using the divergence theorem and equations (2.1), (2.3), we get

$$\begin{aligned}&2\int_{\Omega_1} x_iH_{,i}\Delta H\,dx\\&\quad = 2\int_{\Gamma_1} x_iH_{,i}H_{,j}n_j\,dS + 2\int_L x_iH_{,i}H_{,3}n_3^{(1)}\,dS\\&\qquad - 2\int_{\Omega_1} H_{,i}H_{,i}\,dx - 2\int_{\Omega_1} x_iH_{,ij}H_{,j}\,dx\\&\quad = 2\int_{\Gamma_1} x_iH_{,i}H_{,j}n_j\,dS - 2\int_L x_iI_{,i}I_{,3}n_3^{(2)}\,dS\\&\qquad - 2\int_{\Omega_1} H_{,i}H_{,i}\,dx - 2\int_{\Omega_1} x_iH_{,ij}H_{,j}\,dx.\end{aligned} \tag{2.18}$$

For the fourth function on the right-hand side of (2.18), using the divergence theorem and equations (2.1), (2.3), we get

$$\begin{aligned}-2\int_{\Omega_1} x_iH_{,ij}H_{,j}\,dx &= 3\int_{\Omega_1} H_{,i}H_{,i}\,dx - \int_{\Gamma_1} x_iH_{,j}H_{,j}n_i\,dS - \int_L H_{,j}H_{,j}x_3n_3^{(1)}\,dS\\&= 3\int_{\Omega_1} H_{,i}H_{,i}\,dx - \int_{\Gamma_1} x_iH_{,j}H_{,j}n_i\,dS + \int_L I_{,j}I_{,j}x_3n_3^{(2)}\,dS.\end{aligned} \tag{2.19}$$

For the first function on the right-hand side of (2.17), we get

$$\begin{aligned}2\int_{\Omega_1} x_iH_{,i}H_{,t}\,dx &\le 2\int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{1}{2}\int_{\Omega_1} x_ix_iH_{,t}H_{,t}\,dx\\&\le 2\int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{d^2}{2}\int_{\Omega_1} H_{,t}H_{,t}\,dx,\end{aligned} \tag{2.20}$$

where $d^2 = \max_\Omega x_ix_i$.

Combining (2.17)–(2.20), we obtain

$$
\begin{aligned}
&2\int_{\Gamma_1} x_iH_{,i}H_{,j}n_j\,dS - \int_{\Gamma_1} x_iH_{,j}H_{,j}n_i\,dS + \int_L I_{,j}I_{,j}x_3n_3^{(2)}\,dS \\
&\quad\le \int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{d^2}{2}\int_{\Omega_1} H_{,t}H_{,t}\,dx + 2\int_L x_iI_{,i}I_{,3}n_3^{(2)}\,dS.
\end{aligned}
\tag{2.21}
$$

Similarly, we get

$$
\begin{aligned}
&2\int_{\Gamma_2} x_iI_{,i}I_{,j}n_j\,dS - \int_{\Gamma_2} x_iI_{,j}I_{,j}n_i\,dS - \int_L I_{,j}I_{,j}x_3n_3^{(2)}\,dS \\
&\quad\le \int_{\Omega_2} I_{,i}I_{,i}\,dx + \frac{d^2}{2}\int_{\Omega_2} I_{,t}I_{,t}\,dx - 2\int_L x_iI_{,i}I_{,3}n_3^{(2)}\,dS.
\end{aligned}
\tag{2.22}
$$

Combining (2.21) and (2.22), we obtain

$$
\begin{aligned}
&2\int_{\Gamma_1} x_iH_{,i}H_{,j}n_j\,dS + 2\int_{\Gamma_2} x_iI_{,i}I_{,j}n_j\,dS - \int_{\Gamma_1} x_iH_{,j}H_{,j}n_i\,dS - \int_{\Gamma_2} x_iI_{,j}I_{,j}n_i\,dS \\
&\quad\le \int_{\Omega_1} H_{,i}H_{,i}\,dx + \int_{\Omega_2} I_{,i}I_{,i}\,dx + \frac{d^2}{2}\int_{\Omega_1} H_{,t}H_{,t}\,dx + \frac{d^2}{2}\int_{\Omega_2} I_{,t}I_{,t}\,dx.
\end{aligned}
\tag{2.23}
$$

Since

$$
H_{,i} = \frac{\partial H}{\partial n}n_i + s_i\nabla_s H, \qquad I_{,i} = \frac{\partial I}{\partial n}n_i + s_i\nabla_s I,
\tag{2.24}
$$

where n and s are the normal and tangential vectors to $\partial\Omega$, respectively, and $\nabla_s H$ and $\nabla_s I$ are the tangential derivatives, we have

$$
\begin{aligned}
&\int_{\Gamma_1} x_in_i\left(\frac{\partial H}{\partial n}\right)^2 dS + \int_{\Gamma_2} x_in_i\left(\frac{\partial I}{\partial n}\right)^2 dS \\
&\quad\le \int_{\Gamma_1} x_in_i|\nabla_s H|^2\,dS - 2\int_{\Gamma_1} x_is_i\nabla_s H\frac{\partial H}{\partial n}\,dS + \int_{\Gamma_2} x_in_i|\nabla_s I|^2\,dS \\
&\qquad - 2\int_{\Gamma_2} x_is_i\nabla_s I\frac{\partial I}{\partial n}\,dS + \int_{\Omega_1} H_{,i}H_{,i}\,dx + \int_{\Omega_2} I_{,i}I_{,i}\,dx \\
&\qquad + \frac{d^2}{2}\int_{\Omega_1} H_{,t}H_{,t}\,dx + \frac{d^2}{2}\int_{\Omega_2} I_{,t}I_{,t}\,dx.
\end{aligned}
\tag{2.25}
$$

We know Ω is star-shaped with respect to the region and, setting $m = \min_{\partial\Omega} x_in_i > 0$, we then obtain

$$
\begin{aligned}
&m\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS + m\int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS \\
&\quad\le \left(d + \frac{2d^2}{m}\right)\int_{\Gamma_1} |\nabla_s H|^2\,dS + \frac{m}{2}\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS \\
&\qquad + \left(d + \frac{2d^2}{m}\right)\int_{\Gamma_2} |\nabla_s I|^2\,dS + \frac{m}{2}\int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS + \int_{\Omega_1} H_{,i}H_{,i}\,dx \\
&\qquad + \int_{\Omega_2} I_{,i}I_{,i}\,dx + \frac{d^2}{2}\int_{\Omega_1} H_{,t}H_{,t}\,dx + \frac{d^2}{2}\int_{\Omega_2} I_{,t}I_{,t}\,dx.
\end{aligned}
\tag{2.26}
$$

Multiplying (2.1)$_1$ by $2H_{,t}$ and integrating over Ω_1, we find

$$\begin{aligned}
&2\int_{\Omega_1} H_{,t}H_{,t}\,dx \\
&\quad = 2\int_{\Omega_1} H_{,t}\Delta H\,dx = 2\int_{\Gamma_1} T_{U,t}\frac{\partial H}{\partial n}\,dS + 2\int_L H_{,t}H_{,3}n_3^{(1)}\,dS - 2\int_{\Omega_1} H_{,it}H_{,i}\,dx \\
&\quad \le \frac{d^2}{2m}\int_{\Gamma_1}(T_{U,t})^2\,dS + \frac{2m}{d^2}\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS - 2\int_L I_{,t}I_{,3}n_3^{(2)}\,dS \\
&\qquad - \frac{d}{dt}\int_{\Omega_1} H_{,i}H_{,i}\,dx.
\end{aligned} \tag{2.27}$$

Similarly, we get

$$\begin{aligned}
2\int_{\Omega_2} I_{,t}I_{,t}\,dx \le{}& \frac{d^2}{2m}\int_{\Gamma_2}(S_{U,t})^2\,dS + \frac{2m}{d^2}\int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS \\
&+ 2\int_L I_{,t}I_{,3}n_3^{(2)}\,dS - \frac{d}{dt}\int_{\Omega_2} I_{,i}I_{,i}\,dx.
\end{aligned} \tag{2.28}$$

Combining (2.26)–(2.28), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{d}{dt}\int_{\Omega_2} I_{,i}I_{,i}\,dx \\
&\quad \le \frac{4}{d^2}\left(\int_{\Omega_1} H_{,i}H_{,i}\,dx + \int_{\Omega_2} I_{,i}I_{,i}\,dx\right) + \left(\frac{4}{d} + \frac{8}{m}\right)\left(\int_{\Gamma_1}|\nabla_s H|^2\,dS + \int_{\Gamma_2}|\nabla_s I|^2\,dS\right) \\
&\qquad + \frac{d^2}{4m}\left(\int_{\Gamma_1}(T_{U,t})^2\,dS + \int_{\Gamma_2}(S_{U,t})^2\,dS\right).
\end{aligned} \tag{2.29}$$

Therefore, integrating (2.29) yields

$$F_1(t) \le D_1(t) + \frac{4}{d^2}\int_0^t F_1(\eta)\,d\eta. \tag{2.30}$$

Gronwall inequality now implies (2.16). □

Lemma 3 *For the functions H and I, we have the following estimates*:

$$\int_0^t\int_{\Omega_1} H_{,\eta}H_{,\eta}\,dx\,d\eta + \int_0^t\int_{\Omega_2} I_{,\eta}I_{,\eta}\,dx\,d\eta \le m_3(t), \tag{2.31}$$

where $m_3(t) = \frac{d^2 m_2(t)}{2m} + \frac{1}{2}(\int_{\Omega_1} H_{0,i}H_{0,i}\,dx + \int_{\Omega_2} I_{0,i}I_{0,i}\,dx) + \frac{m}{2d^2}(\int_0^t\int_{\Gamma_1}(\frac{\partial H}{\partial n})^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2}(\frac{\partial I}{\partial n})^2\,dS\,d\eta)$.

Proof Multiplying (2.1)$_1$ by $2H_{,t}$ and integrating over Ω_1, we find

$$\begin{aligned}
2\int_{\Omega_1} H_{,t}H_{,t}\,dx &= 2\int_{\Omega_1} H_{,t}\Delta H\,dx \\
&= 2\int_{\Gamma_1} T_{U,t}\frac{\partial H}{\partial n}\,dS + 2\int_L H_{,t}H_{,3}n_3^{(1)}\,dS - 2\int_{\Omega_1} H_{,it}H_{,i}\,dx \\
&\le \frac{d^2}{m}\int_{\Gamma_1}(T_{U,t})^2\,dS + \frac{m}{d^2}\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS \\
&\quad - 2\int_L I_{,t}I_{,3}n_3^{(2)}\,dS - \frac{d}{dt}\int_{\Omega_1} H_{,i}H_{,i}\,dx.
\end{aligned} \tag{2.32}$$

Similarly, we get

$$\begin{aligned}
&2\int_{\Omega_2} I_{,t}I_{,t}\,dx \\
&\quad \le \frac{d^2}{m}\int_{\Gamma_2}(S_{U,t})^2\,dS + \frac{m}{d^2}\int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS + 2\int_L I_{,t}I_{,3}n_3^{(2)}\,dS - \frac{d}{dt}\int_{\Omega_2} I_{,i}I_{,i}\,dx.
\end{aligned} \tag{2.33}$$

Combining (2.26), (2.32), and (2.33), we obtain

$$\begin{aligned}
&\left(\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS + \int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS\right) + \frac{d^2}{m}\left(\frac{d}{dt}\int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{d}{dt}\int_{\Omega_2} I_{,i}I_{,i}\,dx\right) \\
&\quad \le \frac{4}{m}\left(\int_{\Omega_1} H_{,i}H_{,i}\,dx + \int_{\Omega_2} I_{,i}I_{,i}\,dx\right) + \frac{d^4}{m^2}\left(\int_{\Gamma_1}(T_{U,t})^2\,dS + \int_{\Gamma_2}(S_{U,t})^2\,dS\right) \\
&\qquad + \left(\frac{4d}{m} + \frac{8d^2}{m^2}\right)\left(\int_{\Gamma_1}|\nabla_s H|^2\,dS + \int_{\Gamma_2}|\nabla_s I|^2\,dS\right).
\end{aligned} \tag{2.34}$$

Therefore, integrating (2.34) yields

$$\begin{aligned}
&\left(\int_0^t\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS\,d\eta + \int_0^t\int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS\,d\eta\right) \\
&\quad \le \frac{4}{m}\int_0^t m_1(\eta)\,d\eta + \left(\frac{4d}{m} + \frac{8d^2}{m^2}\right)\left(\int_0^t\int_{\Gamma_1}|\nabla_s H|^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2}|\nabla_s I|^2\,dS\,d\eta\right) \\
&\qquad + \frac{d^4}{m^2}\left(\int_0^t\int_{\Gamma_1}(T_{U,t})^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2}(S_{U,t})^2 dS d\eta\right) \\
&\qquad + \frac{d^2}{m}\left(\int_{\Omega_1} T_{0,i}T_{0,i}\,dx + \int_{\Omega_2} S_{0,i}S_{0,i}\,dx\right) = m_2(t).
\end{aligned} \tag{2.35}$$

Combining (2.32) and (2.33), we obtain

$$\begin{aligned}
&\int_{\Omega_1} H_{,t}H_{,t}\,dx + \int_{\Omega_2} I_{,t}I_{,t}\,dx + \frac{1}{2}\left(\frac{d}{dt}\int_{\Omega_1} H_{,i}H_{,i}\,dx + \frac{d}{dt}\int_{\Omega_2} I_{,i}I_{,i}\,dx\right) \\
&\quad \le \frac{d^2}{2m}\left(\int_{\Gamma_1}(T_{U,t})^2\,dS + \int_{\Gamma_2}(S_{U,t})^2\,dS\right) \\
&\qquad + \frac{m}{2d^2}\left(\int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS + \int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS\right).
\end{aligned} \tag{2.36}$$

Therefore, integrating (2.36) yields the desired result (2.31). □

Lemma 4 *For the functions H and I, we have the following estimates*:

$$\int_{\Omega_1} H^2\,dx + \int_{\Omega_2} I^2\,dx \le m_4(t), \tag{2.37}$$

with $m_4(t) = \int_{\Omega_1} T_0^2\,dx + \int_{\Omega_2} S_0^2\,dx + \int_0^t\int_{\Gamma_1} T_U^2\,dS\,d\eta + \int_0^t\int_{\Gamma_2} S_U^2\,dS\,d\eta + m_2(t)$.

Proof Multiplying $(2.1)_1$ by $2H$ and integrating over Ω_1, we find

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_1} H^2\,dx &= 2\int_{\Omega_1} HH_{,t}\,dx = 2\int_{\Omega_1} H\Delta H\,dx\\ &= 2\int_{\Gamma_1} T_U\frac{\partial H}{\partial n}\,dS + 2\int_L HH_{,3}n_3^{(1)}\,dS - 2\int_{\Omega_1} H_{,i}H_{,i}\,dx\\ &\le \int_{\Gamma_1} T_U^2\,dS + \int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS - 2\int_L II_{,3}n_3^{(2)}\,dS.\end{aligned} \tag{2.38}$$

Similarly, we get

$$\frac{d}{dt}\int_{\Omega_2} I^2\,dx \le \int_{\Gamma_2} S_U^2\,dS + \int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS + 2\int_L II_{,3}n_3^{(2)}\,dS. \tag{2.39}$$

Combining (2.38) and (2.39), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_1} H^2\,dx + \frac{d}{dt}\int_{\Omega_2} I^2\,dx\\ &\quad\le \int_{\Gamma_1} T_U^2\,dS + \int_{\Gamma_2} S_U^2\,dS + \int_{\Gamma_1}\left(\frac{\partial H}{\partial n}\right)^2 dS + \int_{\Gamma_2}\left(\frac{\partial I}{\partial n}\right)^2 dS.\end{aligned} \tag{2.40}$$

Therefore, integrating (2.40) yields the desired result (2.37). □

Lemma 5 *For the temperatures T and S, we have the following estimates*:

$$\begin{aligned}&\int_{\Omega_1} T^2\,dx + \int_{\Omega_2} S^2\,dx + \kappa\int_0^t\int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta + \kappa\int_0^t\int_{\Omega_2} S_{,i}S_{,i}\,dx\,d\eta\\ &\quad\le 4\int_0^t\int_{\Omega_1} T^2\,dx\,d\eta + 4\int_0^t\int_{\Omega_2} S^2\,dx\,d\eta + \frac{4F_M^2}{\kappa}\int_0^t\int_{\Omega_1} u_iu_i\,dx\,d\eta\\ &\qquad + \frac{4F_M^2}{\kappa}\int_0^t\int_{\Omega_2} v_iv_i\,dx\,d\eta + m_4(t) + 2\int_0^t m_4(\eta)\,d\eta\\ &\qquad + 2\kappa\int_0^t m_1(\eta)\,d\eta + 2m_3(t) + 4\int_0^t\int_{\Omega_1} Q^2\,dx\,d\eta + 4\int_0^t\int_{\Omega_2} Q_s^2\,dx\,d\eta.\end{aligned} \tag{2.41}$$

Proof A combination of (2.7), (2.16), (2.31), and (2.37) leads to the desired result (2.41). □

Lemma 6 *For the solutions (u_i, T) and (v_i, S) of equations* (1.1) *and* (1.2), *if we let* $F_2(t) = \int_{\Omega_1} T^2\,dx + \int_{\Omega_2} S^2\,dx + \int_{\Omega_1} u_iu_i\,dx$, $m_5 = \max\{4 + G_1^2 + \frac{4}{\kappa}F_M^2G_1^2, 2 + \frac{8}{\kappa}F_M^2\}$, $D_2(t) = (1 + \frac{4}{\kappa}F_M^2)\int_{\Omega_1} f_if_i\,dx + m_4(t) + 2\int_0^t m_4(\eta)\,d\eta + 2\kappa\int_0^t m_1(\eta)\,d\eta + 2m_3(t) + 4\int_0^t\int_{\Omega_1} Q^2\,dx\,d\eta +$

$4\int_0^t\int_{\Omega_2} Q_s^2\,dx\,d\eta$, *we get*

$$F_2(t) \le D_2(t) + m_5 e^{m_5 t}\int_0^t D_2(\eta)e^{-m_5\eta}\,d\eta = m_6(t), \tag{2.42}$$

$$\int_0^t\int_{\Omega_1} |u|^3\,dx\,d\eta \le \frac{G_1^2+1+|G_1^2-1|}{4\lambda}m_6(t) + \frac{1}{2\lambda}\int_{\Omega_1} f_i f_i\,dx = \frac{m_7(t)}{\lambda}, \tag{2.43}$$

$$\int_0^t\int_{\Omega_1} T_{,i}T_{,i}\,dx\,d\eta + \int_0^t\int_{\Omega_2} S_{,i}S_{,i}\,dx\,d\eta \le \frac{1}{\kappa}D_2(t) + \frac{m_5}{\kappa}\int_0^t m_6(\eta)\,d\eta = m_8(t), \tag{2.44}$$

and

$$\int_0^t\int_{\Omega_2} v_i v_i\,dx\,d\eta \le \frac{G_1^2+1+|G_1^2-1|}{2}\int_0^t m_6(\eta)\,d\eta + \int_{\Omega_1} f_i f_i\,dx = m_9(t), \tag{2.45}$$

where $m_7(t) = \frac{G_1^2+1+|G_1^2-1|}{4}m_6(t) + \frac{1}{2}\int_{\Omega_1} f_i f_i\,dx$.

Proof Multiplying (1.1)$_1$ by $2u_i$ and integrating over Ω_1, we find

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_1} u_i u_i\,dx &= 2\int_{\Omega_1} u_i u_{i,t}\,dx\\ &= -2\lambda\int_{\Omega_1} |u|u_i u_i\,dx - 2\int_{\Omega_1} p_{,i}u_i\,dx + 2\int_{\Omega_1} g_i T u_i\,dx.\end{aligned} \tag{2.46}$$

For the second function on the right-hand side of (2.46), using the divergence theorem and equations (1.3), (1.5), we get

$$-2\int_{\Omega_1} p_{,i}u_i\,dx = -2\int_L p u_3 n_3^{(1)}\,dS = 2\int_L q v_3 n_3^{(2)}\,dS = 2\int_{\Omega_2} q_{,i}v_i\,dx. \tag{2.47}$$

If we insert (1.2)$_1$ and (2.47) into (2.46), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega_1} u_i u_i\,dx + 2\lambda\int_{\Omega_1} |u|u_i u_i\,dx\\ &\le 2\int_{\Omega_2} q_{,i}v_i\,dx + 2\int_{\Omega_1} g_i T u_i\,dx\\ &\le 2\int_{\Omega_2} (g_i S - v_i)v_i\,dx + \int_{\Omega_1} g_i g_i T^2\,dx + \int_{\Omega_1} u_i u_i\,dx\\ &\le \frac{1}{2}\int_{\Omega_2} g_i g_i S^2\,dx + G_1^2\int_{\Omega_1} T^2\,dx + \int_{\Omega_1} u_i u_i\,dx\\ &\le \frac{1}{2}G_1^2\int_{\Omega_2} S^2\,dx + G_1^2\int_{\Omega_1} T^2\,dx + \int_{\Omega_1} u_i u_i\,dx.\end{aligned} \tag{2.48}$$

Therefore, integrating (2.48) yields

$$\begin{aligned}\int_{\Omega_1} u_i u_i\,dx \le{}& G_1^2\int_0^t\int_{\Omega_1} T^2\,dx\,d\eta + \frac{1}{2}G_1^2\int_0^t\int_{\Omega_2} S^2\,dx\,d\eta\\ &+ \int_0^t\int_{\Omega_1} u_i u_i\,dx\,d\eta + \int_{\Omega_1} f_i f_i\,dx.\end{aligned} \tag{2.49}$$

Similarly, we get

$$\int_0^t \int_{\Omega_2} v_i v_i \,dx\,d\eta \le G_1^2 \int_0^t \int_{\Omega_1} T^2 \,dx\,d\eta + G_1^2 \int_0^t \int_{\Omega_2} S^2 \,dx\,d\eta + \int_0^t \int_{\Omega_1} u_i u_i \,dx\,d\eta + \int_{\Omega_1} f_i f_i \,dx. \tag{2.50}$$

Combining (2.41), (2.49), and (2.50), we obtain

$$\begin{aligned}
&\int_{\Omega_1} T^2 \,dx + \int_{\Omega_2} S^2 \,dx + \int_{\Omega_1} u_i u_i \,dx + \kappa \int_0^t \int_{\Omega_1} T_{,i} T_{,i} \,dx\,d\eta + \kappa \int_0^t \int_{\Omega_2} S_{,i} S_{,i} \,dx\,d\eta \\
&\quad \le \left(4 + G_1^2 + \frac{4}{\kappa} F_M^2 G_1^2\right) \int_0^t \int_{\Omega_1} T^2 \,dx\,d\eta + \left(4 + \frac{1}{2} G_1^2 + \frac{4}{\kappa} F_M^2 G_1^2\right) \int_0^t \int_{\Omega_2} S^2 \,dx\,d\eta \\
&\qquad + \left(2 + \frac{8}{\kappa} F_M^2\right) \int_0^t \int_{\Omega_1} u_i u_i \,dx\,d\eta + \left(1 + \frac{4}{\kappa} F_M^2\right) \int_{\Omega_1} f_i f_i \,dx + m_4(t) \\
&\qquad + 2 \int_0^t m_4(\eta)\,d\eta + 2\kappa \int_0^t m_1(\eta)\,d\eta + 2m_3(t) + 4 \int_0^t \int_{\Omega_1} Q^2 \,dx\,d\eta \\
&\qquad + 4 \int_0^t \int_{\Omega_2} Q_s^2 \,dx\,d\eta.
\end{aligned} \tag{2.51}$$

We can get

$$F_2(t) \le D_2(t) + m_5 \int_0^t F_2(\eta)\,d\eta. \tag{2.52}$$

Gronwall inequality now implies the desired result (2.42). □

Similarly, we can also get the desired result (2.43).
Combining (2.51) and (2.42), we obtain the desired result (2.44).
Combining (2.50) and (2.42), we obtain the desired result (2.45).

Lemma 7 *For the temperatures T and S, we have the following estimates*:

$$\max\left\{\sup\,[0,\tau]\|T\|_\infty, \sup_{[0,\tau]} \|S\|_\infty\right\} \le e^{2\tau} \max\left\{\sup_{[0,\tau]} \|Q\|_\infty, \sup_{[0,\tau]} \|Q_s\|_\infty, F_M\right\} = N_M. \tag{2.53}$$

Proof Multiplying $(1.1)_3$ by $2r(T^{2r-1} - H^{2r-1})$ and integrating over $\Omega_1 \times [0,t]$, (where $r > 2$), we find

$$\begin{aligned}
&2r \int_0^t \int_{\Omega_1} u_i T^{2r-1} T_{,i} \,dx\,d\eta - 2r \int_0^t \int_{\Omega_1} u_i H^{2r-1} T_{,i} \,dx\,d\eta \\
&\quad - 2r \int_0^t \int_{\Omega_1} \left(T^{2r-1} - H^{2r-1}\right) Q \,dx\,d\eta \\
&\quad = 2r\kappa \int_0^t \int_{\Omega_1} \left(T^{2r-1} - H^{2r-1}\right) \Delta T \,dx\,d\eta \\
&\quad - 2r \int_0^t \int_{\Omega_1} \left(T^{2r-1} - H^{2r-1}\right) T_{,\eta} \,dx\,d\eta.
\end{aligned} \tag{2.54}$$

For the first function on the right-hand side of (2.54), using the divergence theorem and equations (1.3), (1.5), we get

$$\begin{aligned}
&2r\kappa\int_0^t\int_{\Omega_1}\left(T^{2r-1}-H^{2r-1}\right)\Delta T\,dx\,d\eta\\
&=2r\kappa\int_0^t\int_L T^{2r-1}T_{,3}n_3^{(1)}\,dS\,d\eta-2r\kappa\int_0^t\int_L H^{2r-1}T_{,3}n_3^{(1)}\,dS\,d\eta\\
&\quad-\frac{2\kappa(2r-1)}{r}\int_0^t\int_{\Omega_1}\left(T^r\right)_{,i}\left(T^r\right)_{,i}\,dx\,d\eta+2r\kappa(2r-1)\int_0^t\int_{\Omega_1}H^{2r-2}H_{,i}T_{,i}\,dx\,d\eta\\
&\le-2r\kappa\int_0^t\int_L S^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta+2r\kappa\int_0^t\int_L I^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta\\
&\quad+r\kappa(2r-1)F_M^{2r-2}\int_0^t m_1(\eta)\,d\eta+r\kappa(2r-1)F_M^{2r-2}m_8(t).
\end{aligned}\tag{2.55}$$

For the first function on the left-hand side of (2.54), using the divergence theorem and equations (1.4), (2.1), we find

$$\begin{aligned}
2r\int_0^t\int_{\Omega_1}u_iT^{2r-1}T_{,i}\,dx\,d\eta&=\int_0^t\int_{\Omega_1}u_i\left(T^{2r}\right)_{,i}\,dx\,d\eta=\int_0^t\int_L T^{2r}u_3n_3^{(1)}\,dS\,d\eta\\
&=-\int_0^t\int_L S^{2r}v_3n_3^{(2)}\,dS\,d\eta.
\end{aligned}\tag{2.56}$$

For the second function on the left-hand side of (2.54), we get

$$\begin{aligned}
&2\left|\int_0^t\int_{\Omega_1}u_iH^{2r-1}T_{,i}\,dx\,d\eta\right|\\
&\le 2F_M^{2r-1}\int_0^t\int_{\Omega_1}u_iT_{,i}\,dx\,d\eta\\
&\le F_M^{2r-1}\int_0^t\int_{\Omega_1}u_iu_i\,dx\,d\eta+F_M^{2r-1}\int_0^t\int_{\Omega_1}T_{,i}T_{,i}\,dx\,d\eta\\
&\le F_M^{2r-1}\int_0^t m_6(\eta)\,d\eta+F_M^{2r-1}m_8(t).
\end{aligned}\tag{2.57}$$

For the third function on the left-hand side of (2.54), using Young inequality, we get

$$\begin{aligned}
2r\left|\int_0^t\int_{\Omega_1}\left(T^{2r-1}-H^{2r-1}\right)Q\,dx\,d\eta\right|&\le 2\int_0^t\int_{\Omega_1}Q^{2r}\,dx\,d\eta\\
&\quad+(2r-1)\int_0^t\int_{\Omega_1}T^{2r}\,dx\,d\eta\\
&\quad+(2r-1)\int_0^t\int_{\Omega_1}H^{2r}\,dx\,d\eta.
\end{aligned}\tag{2.58}$$

For the second function on the right-hand side of (2.54), using Young inequality and equations (1.4), (2.1), we find

$$\begin{aligned}
&-2r\int_0^t\int_{\Omega_1}\left(T^{2r-1}-H^{2r-1}\right)T_{,\eta}\,dx\,d\eta\\
&\quad=-\int_{\Omega_1}T^{2r}\,dx+2r\int_{\Omega_1}H^{2r-1}T\,dx-(2r-1)\int_{\Omega_1}T_0^{2r}\,dx\\
&\qquad-2r(2r-1)\int_0^t\int_{\Omega_1}H^{2r-2}H_{,\eta}T\,dx\,d\eta\\
&\quad\le-\frac{1}{2}\int_{\Omega_1}T^{2r}\,dx+(2r-1)2^{\frac{1}{2r-1}}\int_{\Omega_1}H^{2r}\,dx+r(2r-1)F_M^{2r-2}m_3(t)\\
&\qquad+r(2r-1)F_M^{2r-2}\int_0^t m_6(\eta)\,d\eta.
\end{aligned}\tag{2.59}$$

Combining (2.54)–(2.59), we obtain

$$\begin{aligned}
&\int_{\Omega_1}T^{2r}\,dx-2\int_0^t\int_L S^{2r}v_3n_3^{(2)}\,dS\,d\eta\\
&\qquad+4r\kappa\int_0^t\int_L S^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta-4r\kappa\int_0^t\int_L I^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta\\
&\quad\le(4r-2)\int_0^t\int_{\Omega_1}T^{2r}\,dx\,d\eta+4\int_0^t\int_{\Omega_1}Q^{2r}\,dx\,d\eta+(4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_1}H^{2r}\,dx\\
&\qquad+(4r-2)\int_0^t\int_{\Omega_1}H^{2r}\,dx\,d\eta+2r\kappa(2r-1)F_M^{2r-2}\int_0^t m_1(\eta)\,d\eta\\
&\qquad+2r(2r-1)F_M^{2r-2}m_3(t)+\left[2F_M^{2r-1}+2r(2r-1)F_M^{2r-2}\right]\int_0^t m_6(\eta)\,d\eta\\
&\qquad+\left[2F_M^{2r-1}+2r\kappa(2r-1)F_M^{2r-2}\right]m_8(t).
\end{aligned}\tag{2.60}$$

Similarly, we get

$$\begin{aligned}
&\int_{\Omega_2}S^{2r}\,dx+2\int_0^t\int_L S^{2r}v_3n_3^{(2)}\,dS\,d\eta-4r\kappa\int_0^t\int_L S^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta\\
&\qquad+4r\kappa\int_0^t\int_L I^{2r-1}S_{,3}n_3^{(2)}\,dS\,d\eta\\
&\quad\le(4r-2)\int_0^t\int_{\Omega_2}S^{2r}\,dx\,d\eta+(4r-2)\int_0^t\int_{\Omega_2}I^{2r}\,dx\,d\eta+4\int_0^t\int_{\Omega_2}Q_s^{2r}\,dx\,d\eta\\
&\qquad+(4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_2}I^{2r}\,dx+2r\kappa(2r-1)F_M^{2r-2}\int_0^t m_1(\eta)\,d\eta\\
&\qquad+2r(2r-1)F_M^{2r-2}m_3(t)+2r(2r-1)F_M^{2r-2}\int_0^t m_6(\eta)\,d\eta\\
&\qquad+\left[2F_M^{2r-1}+2r\kappa(2r-1)F_M^{2r-2}\right]m_8(t)+2F_M^{2r-1}m_9(t).
\end{aligned}\tag{2.61}$$

Combining (2.60) and (2.61), we get

$$\begin{aligned}
&\int_{\Omega_1} T^{2r}\,dx + \int_{\Omega_2} S^{2r}\,dx \\
&\quad\le (4r-2)\int_0^t\int_{\Omega_1} T^{2r}\,dx\,d\eta + 4\int_0^t\int_{\Omega_1} Q^{2r}\,dx\,d\eta \\
&\qquad + (4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_1} H^{2r}\,dx + (4r-2)\int_0^t\int_{\Omega_1} H^{2r}\,dx\,d\eta \\
&\qquad + (4r-2)\int_0^t\int_{\Omega_2} S^{2r}\,dx\,d\eta + 4\int_0^t\int_{\Omega_2} Q_s^{2r}\,dx\,d\eta + (4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_2} I^{2r}\,dx \\
&\qquad + (4r-2)\int_0^t\int_{\Omega_2} I^{2r}\,dx\,d\eta + 4r\kappa(2r-1)F_M^{2r-2}\int_0^t m_1(\eta)\,d\eta \\
&\qquad + 4r(2r-1)F_M^{2r-2}m_3(t) + 2F_M^{2r-1}m_9(t) \\
&\qquad + \bigl[2F_M^{2r-1} + 4r(2r-1)F_M^{2r-2}\bigr]\int_0^t m_6(\eta)\,d\eta \\
&\qquad + \bigl[4F_M^{2r-1} + 4r\kappa(2r-1)F_M^{2r-2}\bigr]m_8(t).
\end{aligned} \tag{2.62}$$

Letting

$$\begin{aligned}
F_3(t) &= \int_{\Omega_1} T^{2r}\,dx + \int_{\Omega_2} S^{2r}\,dx, \\
D_3(t) &= 4\int_0^t\int_{\Omega_1} Q^{2r}\,dx\,d\eta + (4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_1} H^{2r}\,dx + (4r-2)\int_0^t\int_{\Omega_1} H^{2r}\,dx\,d\eta \\
&\quad + 4\int_0^t\int_{\Omega_2} Q_s^{2r}\,dx\,d\eta + (4r-2)2^{\frac{1}{2r-1}}\int_{\Omega_2} I^{2r}\,dx + (4r-2)\int_0^t\int_{\Omega_2} I^{2r}\,dx\,d\eta \\
&\quad + 4r\kappa(2r-1)F_M^{2r-2}\int_0^t m_1(\eta)\,d\eta + 4r(2r-1)F_M^{2r-2}m_3(t) + 2F_M^{2r-1}m_9(t) \\
&\quad + [2F_M^{2r-1} + 4r(2r-1)F_M^{2r-2}]\int_0^t m_6(\eta)\,d\eta + [4F_M^{2r-1} + 4r\kappa(2r-1)F_M^{2r-2}]m_8(t),
\end{aligned}$$

we get

$$F_3(t) \le D_3(t) + (4r-2)\int_0^t F_3(\eta)\,d\eta. \tag{2.63}$$

Gronwall inequality now implies

$$\int_0^t F_3(\eta)\,d\eta \le \int_0^t D_3(\eta)e^{(4r-2)(t-\eta)}\,d\eta \le e^{(4r-2)t}\int_0^t D_3(\eta)\,d\eta. \tag{2.64}$$

Raising to the power of $\frac{1}{2r}$ both sides of (2.64), we have

$$\left[\int_0^t F_3(\eta)\,d\eta\right]^{\frac{1}{2r}} \le \bigl[e^{(4r-2)t}\bigr]^{\frac{1}{2r}}\left[\int_0^t D_3(\eta)\,d\eta\right]^{\frac{1}{2r}}. \tag{2.65}$$

From the definition of $F_3(t)$, we have

$$\begin{aligned}
&\max\left\{\left(\int_0^t\int_\Omega T^{2r}\,dx\,d\eta\right)^{\frac{1}{2r}},\left(\int_0^t\int_\Omega S^{2r}\,dx\,d\eta\right)^{\frac{1}{2r}}\right\}\\
&\quad\leq\left[\int_0^t F_3(\eta)\,d\eta\right]^{\frac{1}{2r}}\leq\left[e^{(4r-2)t}\right]^{\frac{1}{2r}}\left[\int_0^t D_3(\eta)\,d\eta\right]^{\frac{1}{2r}}.
\end{aligned}\tag{2.66}$$

Using the facts

$$\lim_{r\to\infty}\left(\int_0^t\int_\Omega T^{2r}\,dx\,d\eta\right)^{\frac{1}{2r}}=\sup_{[0,\tau]}\|T\|_\infty,$$
$$\lim_{r\to\infty}\left(\int_0^t\int_\Omega S^{2r}\,dx\,d\eta\right)^{\frac{1}{2r}}=\sup_{[0,\tau]}\|S\|_\infty,$$

and the equality

$$\lim_{n\to\infty}\left(a_1^n+a_2^n+\cdots+a_p^n\right)^{\frac{1}{n}}=\max\{a_1,a_2,a_3,\dots,a_p\},$$

with $a_1,a_2,\dots,a_p$ all nonnegative constants, we can get the desired result (2.53). □

3 Continuous dependence results for the Forchheimer coefficient λ

In this section,we will discuss the continuous dependence on the Forchheimer coefficient λ. Let (u_i,T,p) and (v_i,S,q) be the solutions of (1.1)–(1.5) with $\lambda=\lambda_1$ Similarly, we set (u_i^*,T^*,p^*) and (v_i^*,S^*,q^*) to be the solutions of (1.1)–(1.5) with $\lambda=\lambda_2$.

We define $\omega_i=u_i-u_i^*$, $\theta=T-T^*$, $\pi=p-p^*$, $\hat{\lambda}=\lambda_1-\lambda_2$, and $\omega_i^m=v_i-v_i^*$, $\theta^m=S-S^*$, $\pi^m=q-q^*$.

We find that (ω_i,θ,π) satisfy the following equations:

$$\begin{cases}
\frac{\partial\omega_i}{\partial t}=-(\lambda_1|u|u_i-\lambda_2|u^*|u_i^*)-\pi_{,i}+g_i\theta,\\
\frac{\partial\omega_i}{\partial x_i}=0,\\
\frac{\partial\theta}{\partial t}+u_i\frac{\partial\theta}{\partial x_i}+\omega_i\frac{\partial T^*}{\partial x_i}=\kappa\Delta\theta,
\end{cases}\tag{3.1}$$

and $(\omega_i^m,\theta^m,\pi^m)$ satisfy

$$\begin{cases}
\omega_i^m=-\pi_{,i}^m+g_i\theta^m,\\
\frac{\partial\omega_i^m}{\partial x_i}=0,\\
\frac{\partial\theta^m}{\partial t}+v_i\frac{\partial\theta^m}{\partial x_i}+\omega_i^m\frac{\partial S^*}{\partial x_i}=\kappa\Delta\theta^m.
\end{cases}\tag{3.2}$$

The boundary conditions are

$$\begin{cases}
\omega_i=0,\qquad\theta=0,\qquad(x,t)\in\Gamma_1\times[0,\tau],\\
\omega_i^m n_i=0,\qquad\theta^m=0,\qquad(x,t)\in\Gamma_2\times[0,\tau],
\end{cases}\tag{3.3}$$

and additionally the initial conditions are given at $t = 0$, i.e.,

$$\begin{cases} \omega_i(x,0) = 0, \qquad \theta(x,0) = 0, \quad x \in \Omega_1, \\ \theta^m(x,0) = 0, \qquad\qquad\qquad x \in \Omega_2. \end{cases} \tag{3.4}$$

The conditions on interface L ar

$$\begin{cases} \omega_3 = \omega_3^m, \qquad \theta = \theta^m, \qquad \theta_{,3} = \theta_{,3}^m, \\ \pi = \pi^m. \end{cases} \tag{3.5}$$

Theorem *Let (u_i, T, p) and (v_i, S, q) be the classical solutions to the initial-boundary value problem (1.1)–(1.5) with $\lambda = \hat{\lambda}_1$, while (u_i^*, T^*, p^*) and (v_i^*, S^*, q^*) are the classical solutions to the initial-boundary value problem (1.1)–(1.5) with $\lambda = \hat{\lambda}_2$. We define (ω_i, θ, π) and $(\omega_i^m, \theta^m, \pi^m)$ to be the differences of these two solutions, respectively. Then the solutions (u_i, T, p) and (v_i, S, q) converge to the solutions (u_i^*, T^*, p^*) and (v_i^*, S^*, q^*) as the Forchheimer coefficient $\hat{\lambda}$ tends to 0. The differences of solutions satisfy*

$$\int_{\Omega_1} \theta^2\,dx + \int_{\Omega_2} \left(\theta^{\mathrm{m}}\right)^2 dx + \frac{N_M^2}{2\kappa}\int_{\Omega_1} \omega_i\omega_i\,dx \le \hat{\lambda}^2 m_{11}(t), \tag{3.6}$$

where $m_{10} = \max\{\frac{N_M^2}{\kappa}, \frac{N_M^2 G_1^2}{2\kappa}\}$, $m_{11}(t) = \frac{N_M^2}{2\kappa\lambda_1\lambda_2} m_7(t) + \frac{N_M^2 m_{10}}{2\kappa\lambda_1\lambda_2} e^{m_{10}t}\int_0^t m_7(\eta)e^{-m_{10}\eta}\,d\eta$.

Moreover, the differences of velocities satisfy the following estimates:

$$\int_0^t \int_{\Omega_2} \omega_i^m\omega_i^m\,dx\,d\eta \le \hat{\lambda}^2\left(\frac{m_7(t)}{\lambda_1\lambda_2} + m_{12}\int_0^t m_{11}(\eta)\,d\eta\right), \tag{3.7}$$

where $m_{12} = \max\{G_1^2, \frac{2\kappa}{N_M^2}\}$.

Proof Multiplying $(3.1)_1$ by $2\omega_i$ and integrating over Ω_i, we see

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_1} \omega_i\omega_i\,dx \\ &\quad = -2\hat{\lambda}\int_{\Omega_1} |u|u_i\omega_i\,dx - 2\lambda_2\int_{\Omega_1} \left(|u|u_i - \left|u^*\right|u_i^*\right)\omega_i\,dx - 2\int_{\Omega_1} \pi_{,i}\omega_i\,dx \\ &\qquad + 2\int_{\Omega_1} g_i\theta\omega_i\,dx. \end{aligned} \tag{3.8}$$

For the third function on the right-hand side of (3.8), using the divergence theorem and Eqs. (3.3), (3.5), we get

$$-2\int_{\Omega_1} \pi_{,i}\omega_i\,dx = -2\int_L \pi\omega_3 n_3^{(1)}\,dS = 2\int_L \pi^m\omega_3^m n_3^{(2)}\,dS = 2\int_{\Omega_2} \pi_{,i}^m\omega_i^m\,dx. \tag{3.9}$$

For the second function on the right-hand side of (3.8), we have

$$\begin{aligned} &2\left(|u|u_i - \left|u^*\right|u_i^*\right)\omega_i \\ &\quad = 2\left(|u|^3 + \left|u^*\right|^3\right) - 2u_iu_i^*\left(|u| + \left|u^*\right|\right) \\ &\quad = \left(|u| + \left|u^*\right|\right)\left[\left(|u|^2 + \left|u^*\right|^2 - 2|u|\left|u^*\right|\right) + \left(|u|^2 + \left|u^*\right|^2 - 2u_iu_i^*\right)\right] \\ &\quad = \left(|u| + \left|u^*\right|\right)\left[\left(|u| + \left|u^*\right|\right)^2 + \omega_i\omega_i\right] \\ &\quad \geq |u|\omega_i\omega_i. \end{aligned} \tag{3.10}$$

For the first function on the right-hand side of (3.8), we have

$$-2\hat{\lambda}\int_{\Omega_1}|u|u_i\omega_i\,dx \leq \frac{\hat{\lambda}^2}{\lambda_2}\int_{\Omega_1}|u|^3\,dx + \lambda_2\int_{\Omega_1}|u|\omega_i\omega_i\,dx. \tag{3.11}$$

Combining (3.8)–(3.11), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega_1}\omega_i\omega_i\,dx \\ &\quad \leq \frac{\hat{\lambda}^2}{\lambda_2}\int_{\Omega_1}|u|^3\,dx + 2\int_{\Omega_2}\pi_{,i}^m\omega_i^m\,dx + 2\int_{\Omega_1}g_i\theta\omega_i\,dx \\ &\quad \leq \frac{\hat{\lambda}^2}{\lambda_2}\int_{\Omega_1}|u|^3\,dx + 2\int_{\Omega_2}\left(g_i\theta^m - \omega_i^m\right)\omega_i^m\,dx + \int_{\Omega_1}\omega_i\omega_i\,dx + G_1^2\int_{\Omega_1}\theta^2\,dx \\ &\quad \leq \frac{\hat{\lambda}^2}{\lambda_2}\int_{\Omega_1}|u|^3\,dx - \int_{\Omega_2}\omega_i^m\omega_i^m\,dx + \int_{\Omega_1}\omega_i\omega_i\,dx \\ &\qquad + G_1^2\int_{\Omega_1}\theta^2\,dx + G_1^2\int_{\Omega_1}\left(\theta^m\right)^2\,dx. \end{aligned} \tag{3.12}$$

In order to estimate $\int_{\Omega_1}\theta\theta\,dx + \int_{\Omega_2}\theta^{\mathrm{m}}\theta^m\,dx$, we multiply $(3.1)_3$ by 2θ and get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_1}\theta^2\,dx &= 2\int_{\Omega_1}\theta\theta_{,t}\,dx \\ &= 2\int_{\Omega_1}\theta\left(\kappa\Delta\theta - u_i\theta_{,i} - \omega_iT_{,i}^*\right)dx \\ &= 2\kappa\int_{\Omega_1}\theta\Delta\theta\,dx - 2\int_{\Omega_1}\theta u_i\theta_{,i}\,dx - 2\int_{\Omega_1}\theta\omega_iT_{,i}^*\,dx. \end{aligned} \tag{3.13}$$

For the first function on the right-hand side of (3.13), using the divergence theorem and Eqs. (3.3), (3.5), we get

$$\begin{aligned} 2\kappa\int_{\Omega_1}\theta\Delta\theta\,dx &= 2\int_L\theta\kappa\theta_{,3}n_3^{(1)}\,dS - 2\kappa\int_{\Omega_1}\theta_{,i}\theta_{,i}\,dx \\ &\leq -2\int_L\theta^m\kappa\theta_{,3}^m n_3^{(2)}\,dS - 2\kappa\int_{\Omega_1}\theta_{,i}\theta_{,i}\,dx. \end{aligned} \tag{3.14}$$

For the second function on the right-hand side of (3.13), using the divergence theorem and Eqs. (1.3), (3.5), we get

$$\begin{aligned} -2\int_{\Omega_1}\theta u_i\theta_{,i}\,dx &= -\int_{\Omega_1}u_i(\theta)^2_{,i}\,dx = -\int_L u_3\theta^2 n_3^{(1)}\,dS \\ &= \int_L v_3\big(\theta^m\big)^2 n_3^{(2)}\,dS = 2\int_{\Omega_2}\theta^m v_i\theta^m_{,i}\,dx. \end{aligned} \tag{3.15}$$

For the third function on the right-hand side of (3.13), using the divergence theorem and Eqs. (1.5), (3.3), and (3.5), we get

$$\begin{aligned} -2\int_{\Omega_1}\theta\omega_i T^*_{,i}\,dx &= -2\int_L \theta\omega_3 T^* n_3^{(1)}\,dS + 2\int_{\Omega_1}\theta_{,i}\omega_i T^*\,dx \\ &= 2\int_L \theta^m\omega_3^m S^* n_3^{(2)}\,dS + 2\int_{\Omega_1}\theta_{,i}\omega_i T^*\,dx. \end{aligned} \tag{3.16}$$

Combining (3.13)–(3.16), we get

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_1}\theta^2\,dx &\le -2\kappa\int_{\Omega_1}\theta_{,i}\theta_{,i}\,dx + 2\int_{\Omega_1}\theta_{,i}\omega_i T^*\,dx - 2\int_L\theta^m\kappa\theta^m_{,3}n_3^{(2)}\,dS \\ &\quad + 2\int_L\theta^m\omega_3^m S^* n_3^{(2)}\,dS + 2\int_{\Omega_2}\theta^m v_i\theta^m_{,i}\,dx \\ &\le \frac{N_M^2}{2\kappa}\int_{\Omega_1}\omega_i\omega_i\,dx - 2\int_L\theta^m\kappa\theta^m_{,3}n_3^{(2)}\,dS \\ &\quad + 2\int_L\theta^m\omega_3^m S^* n_3^{(2)}\,dS + 2\int_{\Omega_2}\theta^m v_i\theta^m_{,i}\,dx. \end{aligned} \tag{3.17}$$

Similarly, we multiply $(3.2)_3$ by $2\theta^m$, we have

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_2}\big(\theta^m\big)^2\,dx &\le \frac{N_M^2}{2\kappa}\int_{\Omega_2}\omega_i^m\omega_i^m\,dx + 2\int_L\theta^m\kappa\theta^m_{,3}n_3^{(2)}\,dS \\ &\quad - 2\int_L\theta^m\omega_3^m S^* n_3^{(2)}\,dS - 2\int_{\Omega_2}\theta^m v_i\theta^m_{,i}\,dx. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we have

$$\frac{d}{dt}\left(\int_{\Omega_1}\theta^2\,dx + \int_{\Omega_2}\big(\theta^{\mathrm{m}}\big)^2\,dx\right) \le \frac{N_M^2}{2\kappa}\int_{\Omega_1}\omega_i\omega_i\,dx + \frac{N_M^2}{2\kappa}\int_{\Omega_2}\omega_i^m\omega_i^m\,dx. \tag{3.19}$$

Combining (3.12) and (3.19), we have

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\Omega_1}\theta^2\,dx + \int_{\Omega_2}\big(\theta^{\mathrm{m}}\big)^2\,dx + \frac{N_M^2}{2\kappa}\int_{\Omega_1}\omega_i\omega_i\,dx\right) \\ &\quad\le \frac{\hat{\lambda}^2}{\lambda_2}\frac{N_M^2}{2\kappa}\int_{\Omega_1}|u|^3\,dx + \frac{N_M^2}{\kappa}\int_{\Omega_1}\omega_i\omega_i\,dx + \frac{N_M^2G_1^2}{2\kappa}\int_{\Omega_1}\theta^2\,dx \\ &\quad\quad + \frac{N_M^2G_1^2}{2\kappa}\int_{\Omega_1}\big(\theta^m\big)^2\,dx. \end{aligned} \tag{3.20}$$

If we let $F_4(t) = \int_{\Omega_1} \theta^2\,dx + \int_{\Omega_2} (\theta^{\mathrm{m}})^2\,dx + \frac{N_M^2}{2\kappa}\int_{\Omega_1} \omega_i\omega_i\,dx$, $m_{10} = \max\{\frac{N_M^2}{\kappa}, \frac{N_M^2 G_1^2}{2\kappa}\}$. Therefore, integrating (3.20) yields

$$F_4(t) \le \hat{\lambda}^2 \frac{N_M^2}{2\kappa\lambda_1\lambda_2} m_7(t) + m_{10}\int_0^t F_4(\eta)\,d\eta. \tag{3.21}$$

Gronwall inequality implies

$$F_4(t) \le \hat{\lambda}^2 \frac{N_M^2}{2\kappa\lambda_1\lambda_2} m_7(t) + \hat{\lambda}^2 \frac{N_M^2 m_{10}}{2\kappa\lambda_1\lambda_2} e^{m_{10}t}\int_0^t m_7(\eta)e^{-m_{10}\eta}\,d\eta = \hat{\lambda}^2 m_{11}(t), \tag{3.22}$$

where $m_{11}(t) = \frac{N_M^2}{2\kappa\lambda_1\lambda_2} m_7(t) + \frac{N_M^2 m_{10}}{2\kappa\lambda_1\lambda_2} e^{m_{10}t}\int_0^t m_7(\eta)e^{-m_{10}\eta}\,d\eta$.

Inserting (3.22) into (3.12), we have

$$\int_0^t \int_{\Omega_2} \omega_i^m\omega_i^m\,dx\,d\eta \le \hat{\lambda}^2\left(\frac{m_7(t)}{\lambda_1\lambda_2} + m_{12}\int_0^t m_{11}(\eta)\,d\eta\right), \tag{3.23}$$

where $m_{12} = \max\{G_1^2, \frac{2\kappa}{N_M^2}\}$. □

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