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On instability of Rayleigh–Taylor problem for incompressible liquid crystals under L^1-norm

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Abstract

We investigate the nonlinear Rayleigh–Taylor (RT) instability of a nonhomogeneous incompressible nematic liquid crystal in the presence of a uniform gravitational field. We first analyze the linearized equations around the steady state solution. Thus we construct solutions of the linearized problem that grow in time in the Sobolev space H^4, then we show that the RT equilibrium state is linearly unstable. With the help of the established unstable solutions of the linearized problem and error estimates between the linear and nonlinear solutions, we establish the nonlinear instability of the density, the horizontal and vertical velocities under L^1-norm.

Keywords: Rayleigh–Taylor instability; Liquid crystals; Steady state solution

1 Introduction

The instability arises when steady states of two fluid layers with different densities are accelerated in the direction toward the denser fluid [1]. This phenomenon was first studied by Rayleigh [2] and then Taylor [3], thus is called the Rayleigh–Taylor (RT) instability. In the last decades, this phenomenon has been extensively investigated from both physical and numerical aspects, see [4, 5] for examples. It has been also widely investigated how the RT instability evolves under the effects of other physical factors, such as elasticity [6–8], rotation [9], internal surface tension [10–12], magnetic fields [13–17], and so on. In particular, to the best of our knowledge, the linear Rayleigh–Taylor instability is well understood, see [5, 9], for instance; however, there are only few mathematical analysis results on nonlinear problems in the literature.

In this paper, we further mathematically prove the RT instability in incompressible liquid crystal materials in the presence of a uniform gravitational field in a bounded domain

$\Omega \subset \mathbb{R}^3$:

$$
\begin{cases}
\rho_t + u \cdot \nabla \rho = 0, \\
\rho u_t + \rho u \cdot \nabla u - \mu \Delta u + \nabla p = -\Delta d \cdot \nabla d - g\rho e_3, \\
d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \\
\operatorname{div} u = 0, \\
u(0,x) = u^0(x), \qquad d(0,x) = d^0(x), \\
(u,d)|_{\partial\Omega} = (0, e_3).
\end{cases}
\tag{1.1}
$$

Here the unknown function ρ is the density of the nematic liquid crystals, u the velocity, and p the pressure, d represents the macroscopic average of the nematic liquid crystal orientation field. Also $\mu > 0$ is the coefficient of viscosity, $g > 0$ is the gravitational constant, $e_3 = (0,0,1)^T$ is the vertical unit vector, and $-\rho g e_3$ is the gravitational force.

In this paper we study the Rayleigh–Taylor (RT) instability of system (1.1). To this purpose, we consider a density profile $\bar{\rho} := \bar{\rho}(x_3) \in C^5(\bar{\Omega})$, which satisfies

$$
\inf_{x\in\Omega}\{\bar{\rho}\} > 0, \tag{1.2}
$$

and an RT condition

$$
\bar{\rho}'\left(x_3^0\right) > 0 \quad \text{for some } x^0 \in \Omega, \tag{1.3}
$$

where x_3^0 denotes the third component of x^0. Then we further define a pressure $\bar{p}$ (unique up to a constant) by the relation

$$
\nabla \bar{p} = -g\bar{\rho} e_3.
$$

The condition (1.3) means that there is a region in which the RT density profile has a larger density with increasing x_3 (height), thus leading to RT instability.

Obviously, $R_C := (\bar{\rho}, 0, \bar{p}, e_3)$ is an RT equilibrium-state solution of the system (1.1). Now, we denote the perturbation by

$$
\varrho = \rho - \bar{\rho}, \qquad u = u - 0, \qquad q = p - \bar{p}, \qquad \sigma = d - e_3,
$$

then, (ϱ, u, q, σ) satisfies the following perturbation equations:

$$
\begin{cases}
\varrho_t + u \cdot \nabla(\varrho + \bar{\rho}) = 0, \\
(\varrho + \bar{\rho}) u_t + (\varrho + \bar{\rho}) u \cdot \nabla u + \nabla q - \mu \Delta u = -\Delta\sigma \cdot \nabla\sigma - g\varrho e_3, \\
\sigma_t + u \cdot \nabla \sigma = \Delta \sigma + |\nabla \sigma|^2 \sigma + |\nabla \sigma|^2 e_3, \\
\operatorname{div} u = 0,
\end{cases}
\tag{1.4}
$$

with the initial-boundary value conditions:

$$
(\varrho, u, \sigma)|_{t=0} = \left(\varrho^0, u^0, \sigma^0\right), \qquad (u, \sigma)|_{\partial\Omega} = (0, 0), \tag{1.5}
$$

where Ω is a general bounded domain. In this article, the initial-boundary value problem (1.4)–(1.5) is called the LCRT problem.

If the perturbation is small, we omit the nonlinear terms, and thus get the linearized LCRT equations

$$\begin{cases} \varrho_t + \bar{\rho}' u_3 = 0, \\ \bar{\rho} u_t + \nabla q + g\varrho e_3 = \mu \Delta u, \\ \sigma_t = \Delta \sigma, \\ \operatorname{div} u = 0. \end{cases} \tag{1.6}$$

The linearized equations (1.6) and the initial-boundary values (1.5) constitute the linearized LCRT problem. It is well-known that the linearized LCRT problem is convenient in mathematical analysis in order to have an insight into the physical and mathematical mechanisms of the instability.

1.1 Main results

Before stating our main result, we shall introduce some mathematical notations of Sobolev spaces:

$$\int := \int_\Omega, \qquad L^p := L^p(\Omega) = W^{0,p}(\Omega), \qquad \|\cdot\|_k := \|\cdot\|_{H^k}, \qquad I_T := (0,T),$$
$$H_0^k := \{\eta \in H^k(\Omega) | \eta|_{\partial\Omega} = 0\}, \qquad H_\sigma^k := \{\eta \in H_0^k(\Omega) | \operatorname{div} \eta = 0\},$$
$$\|(u,q,\sigma)\|_{S,k} = \sqrt{\|(u,\sigma)\|_{k+2}^2 + \|q\|_{k+1}^2}, \qquad \underline{H}^i := \left\{\eta \in H^i \middle| \int \eta \, dy = 0\right\},$$
$$\mathcal{E}(t) := \sum_{i=0}^{2} \|\partial_t^i \varrho\|_{4-i}^2 + \sum_{i=0}^{1} \|\partial_t^i (u,q,\sigma)\|_{S,2-2i}^2 + \|(u_{tt},\sigma_{tt})\|_0^2,$$
$$\mathcal{D}(t) := \sum_{i=0}^{1} \|\partial_t^i (u,q,\sigma)\|_{S,3-2i}^2 + \|(u_{tt},\sigma_{tt})\|_1^2,$$
$$a \lesssim b \quad \text{means that } a \le cb \text{ for some positive constant } c,$$

where $1 < p \le \infty$, k is a nonnegative integer, and the positive constant c may depend on the domain occupied by the fluids and other known physical parameters such as g, ϖ, α, and μ, and vary from line to line.

Next we state the instability result in the LCRT problem

Theorem 1 *Let Ω be a C^5-bounded domain, and let the density profile $\bar{\rho} \in C^5(\bar{\Omega})$ satisfy* (1.2)–(1.3)*. Then, the equilibrium $(\bar{\rho}, 0, \bar{q}, e_3)$ to LCRT problem* (1.4)–(1.5) *is unstable in Hadamard sense, that is, there are positive constants Λ, m_0, ε, δ_0, and*

$$(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0, u^r, q^r) \in H^4(\Omega) \times H_\sigma^4(\Omega) \times \underline{H}^3(\Omega) \times H_\sigma^4(\Omega) \times \underline{H}^3(\Omega),$$

such that, for any given $\delta \in (0, \delta_0)$, there is a unique classical solution $(\varrho, u, \sigma) \in C^0(\bar{I}_T, H^4(\Omega) \times H_\sigma^4(\Omega) \times H^4(\Omega))$, with a unique associated (perturbation) pressure $q \in C^0(\bar{I}_T, \underline{H}^3)$, *to the LCRT problem with the initial data*

$$\left(\varrho^0, u^0, q^0\right) := \delta\left(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0\right) + \delta^2\left(0, u^r, q^r\right),$$

but the solution satisfies

$$\left\|\varrho\left(T^\delta\right)\right\|_{L^1(\Omega)}, \left\|u_h\left(T^\delta\right)\right\|_{L^1(\Omega)}, \left\|u_3\left(T^\delta\right)\right\|_{L^1(\Omega)} \geq \varepsilon, \tag{1.7}$$

for some escape time $T^\delta := \frac{1}{\Lambda} \ln \frac{2\varepsilon}{m_0 \delta} \in I_T$. *In addition, the initial data* ϱ^0, u^0, q^0, *and* σ^0 *satisfy the compatibility conditions*:

$$\begin{aligned} &\operatorname{div}\left(\left(u^0 \cdot \nabla u^0 + \left(\nabla q^0 - \mu \Delta u^0\right.\right.\right. \\ &\quad \left.\left.\left. + g\varrho^0 e_3 + \Delta \sigma^0 \cdot \nabla \sigma^0\right)/\left(\varrho^0 + \bar{\rho}\right)\right) = 0 \quad \textit{in } \Omega, \end{aligned} \tag{1.8}$$

$$\nabla q^0 - \mu \Delta u^0 + g\varrho^0 e_3 + \Delta \sigma^0 \cdot \nabla \sigma^0 = 0 \quad \textit{on } \partial\Omega, \tag{1.9}$$

$$\Delta \sigma^0 + \left|\nabla \sigma^0\right|^2 e_3 = 0 \quad \textit{on } \partial\Omega. \tag{1.10}$$

The proof of Theorem 1 is based on a bootstrap instability method, which has its origin in [18, 19]. We mention that many authors have established various versions of the bootstrap methods for mathematical proofs of various flow instabilities, see [20–22], for example. We complete the proof of Theorem 1 in four steps. Firstly, we introduce unstable solutions to the linearized LCRT problem, in view of the linearized LCRT problem, we can obtain a growing mode ansatz of solutions, i.e., for some $\Lambda > 0$, $(\varrho, u, q, \sigma) := e^{\Lambda t}(-\bar{\rho}'\tilde{u}_3/\Lambda, \tilde{u}, \tilde{q}, 0)$, see Proposition 1. Secondly, by using the standard energy method, we establish a Gronwall-type energy inequality of the local-in-time solution of the LCRT problem, see Proposition 2. Thirdly, we use initial data of solutions of the linearized LCRT problem to construct initial data for solutions of the LCRT problem, so that the modified initial data $(\varrho_0^\delta, u_0^\delta, q_0^\delta) := \delta(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0) + \delta^2(0, u^r, q^r)$ belongs to $H^4 \times H_\sigma^4 \times \underline{H}^3$ and satisfies necessary compatibility condition, see Proposition 4. Finally, we introduce the error estimates between the solutions of the linearized and nonlinear LCRT problems, and then prove the nonlinear solution is unstable under L^1-norm.

Now, we will introduce some well-known mathematical results, which will be used in the proof of Theorem 1.

Lemma 1 (1) *Embedding inequalities* (*see* [23, *Theorem* 4.12]):

$$\|f\|_{L^p} \lesssim \|f\|_1 \quad \textit{for } 2 \leq p \leq 6, \tag{1.11}$$

$$\|f\|_{C^0(\bar{\Omega})} = \|f\|_{L^\infty} \lesssim \|f\|_2. \tag{1.12}$$

(2) *Estimates of the product of functions in Sobolev spaces* (*denoted as product estimates*):

$$\|fg\|_j \lesssim \begin{cases} \|f\|_1 \|g\|_1 & \textit{for } j = 0; \\ \|f\|_j \|g\|_2 & \textit{for } 0 \leq j \leq 2; \\ \|f\|_2 \|g\|_j + \|f\|_j \|g\|_2 & \textit{for } 3 \leq j \leq 5, \end{cases} \tag{1.13}$$

which can be easily verified by Hölder's inequality and the embedding inequality (1.11)–(1.12).

(3) *Interpolation inequality in* H^j (*see* [23, *Theorem* 5.2]):

$$\|f\|_j \lesssim \|f\|_0^{1-\frac{j}{i}} \|f\|_i^{\frac{j}{i}} \leq C(\varepsilon)\|f\|_0 + \varepsilon\|f\|_i \quad \textit{for any } 0 \leq j < i, \varepsilon > 0. \tag{1.14}$$

2 Linear instability

Proposition 1 *Under the assumptions of Theorem* 1, *the LCRT equilibrium state* R_C *is linearly unstable, that is, there is an unstable solution in the form*

$$(\varrho, u, q, \sigma) := e^{\Lambda t}\left(-\bar{\rho}'\tilde{u}_3/\Lambda, \tilde{u}, \tilde{q}, 0\right) \tag{2.1}$$

to (1.5)–(1.6), *where* $(\tilde{u}, \tilde{q}) \in H_\sigma^4 \times \underline{H}^3$ *solves the following boundary problem*:

$$\begin{cases} \Lambda^2\bar{\rho}\tilde{u} + \Lambda\nabla\tilde{q} = \Lambda\mu\Delta\tilde{u} + g\bar{\rho}'\tilde{u}_3 e_3, \\ \operatorname{div}\tilde{u} = 0, \qquad \tilde{u}|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

with the constant growth rate Λ *defined by*

$$\Lambda^2 = \sup_{w \in H_\sigma^1} \frac{g\int \bar{\rho}' w_3^2\,dx - \Lambda\mu\int |\nabla w|^2\,dx}{\int \bar{\rho}|w|^2\,dx}. \tag{2.3}$$

Moreover, $\tilde{u}$ *satisfies*

$$\bar{\rho}'\tilde{u} \neq 0, \qquad \tilde{u}_{\mathrm{h}} \neq 0, \quad \textit{and} \quad \tilde{u}_3 \neq 0. \tag{2.4}$$

Proof Please refer to the proof of [24, Theorem 1.1]. □

3 Gronwall-type energy inequality of nonlinear solutions

We derive that any small solution of the LCRT problem enjoys a Gronwall-type energy inequality. We will derive such an inequality by the *a priori* estimate method for simplicity. Let (ϱ, u, σ) be a solution of LCRT problem such that

$$\sup_{0 \leq t < T} \sqrt{\left\|\left(\varrho(t), u(t), \sigma(t)\right)\right\|_4^2} \leq \delta \in (0,1) \quad \text{for some } T > 0. \tag{3.1}$$

Moreover, the solution enjoys fine regularity, which makes valid the procedure of formal deduction. In addition, we rewrite (1.4) with the boundary-value condition in (1.5) as a nonhomogeneous form:

$$\begin{cases} \varrho_t + \bar{\rho}'u_3 = \mathcal{N}_1 := -u\cdot\nabla\varrho, \\ \bar{\rho}u_t + \nabla q - \mu\Delta u + g\varrho e_3 = \mathcal{N}_2 := -\Delta\sigma\cdot\nabla\sigma - (\varrho + \bar{\rho})u\cdot\nabla u - \varrho u_t, \\ \sigma_t - \Delta\sigma = \mathcal{N}_3 := -u\cdot\nabla\sigma + |\nabla\sigma|^2\sigma + |\nabla\sigma|^2 e_3, \\ \operatorname{div} u = 0, \\ u(0,x) = u^0(x), \qquad \sigma(0,x) = \sigma^0(x), \\ (u,\sigma)|_{\partial\Omega} = (0,0). \end{cases} \tag{3.2}$$

Lemma 2 *Under the assumption* (3.1) *with sufficiently small* δ, *it holds that*

$$\frac{d}{dt}\|\sqrt{\bar{\rho}}u\|_0^2 + c\|u\|_1^2 \lesssim \|(\varrho, u_3)\|_0^2 + \sqrt{\mathcal{E}}\mathcal{D}. \tag{3.3}$$

Proof Multiplying $(3.2)_2$ by u in L^2 and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\bar{\rho}}u\|_0^2 + \mu\|\nabla u\|_0^2 = -g\int \varrho u_3\,dx + \int \mathcal{N}_2 \cdot u\,dx. \tag{3.4}$$

By (3.1) and product estimate, it holds that

$$\int \mathcal{N}_2 \cdot u\,dx \lesssim \sqrt{\mathcal{E}}\mathcal{D}. \tag{3.5}$$

Thus we immediately derive (3.3) from (3.4) and (3.5) by using the Young's and Friedrichs's inequalities. □

Lemma 3 *Under the assumption* (3.1) *with sufficiently small* δ, *it holds that*

$$\frac{d}{dt}\|\varrho\|_4^2 \lesssim \|\varrho\|_4\|u\|_4. \tag{3.6}$$

Proof Let $\alpha :=(\alpha_1,\alpha_2,\alpha_3)$ be a multiindex of order $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 4$, and $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$.

Using integration by parts, we can get

$$\int u \cdot \nabla\partial^\alpha\varrho\partial^\alpha\varrho\,dx = 0,$$

thus, applying ∂^α to $(3.2)_1$, and then multiplying the resulting identity by $\partial^\alpha\varrho$ in L^2, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\partial^\alpha\varrho\right\|_0^2 = -\int \partial^\alpha\left(\bar{\rho}'u_3\right)\partial^\alpha\varrho\,dx + \int\left(u\cdot\nabla\partial^\alpha\varrho - \partial^\alpha(u\cdot\nabla\varrho)\right)\cdot\partial^\alpha\varrho\,dx := I_{1,\alpha}. \tag{3.7}$$

By (3.1) and product estimate, it holds that

$$I_{1,\alpha} \lesssim \|\varrho\|_4\|u\|_4 \quad \text{for any } 0 \leq |\alpha| \leq 4. \tag{3.8}$$

Thus putting (3.8) into (3.7), we immediately derive (3.6). □

Lemma 4 *Under the assumption* (3.1) *with sufficiently small* δ, *it holds that*

$$\frac{d}{dt}\|\sqrt{\varrho+\bar{\rho}}u_t\|_0^2 + c\|u_t\|_1^2 \lesssim \|u_3\|_0^2 + \sqrt{\mathcal{E}}\mathcal{D}, \tag{3.9}$$

$$\frac{d}{dt}\|\sqrt{\varrho+\bar{\rho}}u_{tt}\|_0^2 + c\|u_{tt}\|_1^2 \lesssim \|\partial_t u_3\|_0^2 + \sqrt{\mathcal{E}}\mathcal{D}. \tag{3.10}$$

Proof Let $1 \le i \le 2$. Applying ∂_t^i to $(1.4)_2$, $(1.4)_4$, and $(3.2)_6$, we get

$$\begin{cases} \partial_t^i((\varrho+\bar{\rho})u_t) + \partial_t^i((\varrho+\bar{\rho})u\cdot\nabla u) + \nabla\partial_t^i q \\ \quad = \mu\Delta\partial_t^i u - \partial_t^i(\Delta\sigma\cdot\nabla\sigma) - g\partial_t^i\varrho e_3, \\ \operatorname{div}\partial_t^i u = 0, \\ \partial_t^i u|_{\partial\Omega} = 0. \end{cases} \tag{3.11}$$

Multiplying $(3.11)_1$ with $i = 2$ by u_{tt} in L^2, and using the integration by parts and $(3.2)_1$, we can get that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{\varrho+\bar{\rho}}u_{tt}\|_0^2\big) + \mu\|\nabla u_{tt}\|_0^2 \\ &\quad = \int \big(g\partial_t\big(\bar{\rho}'u_3 - \mathcal{N}_1\big)\big)e_3\cdot u_{tt}\,dx - \int \partial_t^2\big((\varrho+\bar{\rho})u\cdot\nabla u\big)\cdot u_{tt}\,dx \\ &\qquad + \frac{1}{2}\int \varrho_t u_{tt}^2\,dx + \int \big((\varrho+\bar{\rho})u_{ttt} - \partial_t^2\big((\varrho+\bar{\rho})u_t\big)\big)\cdot u_{tt}\,dx \\ &\qquad + \int \partial_t^2(\Delta\sigma\cdot\nabla\sigma)\cdot u_{tt}\,dx \\ &\quad =: \sum_{i=1}^{5} I_i. \end{aligned} \tag{3.12}$$

By (3.1) and product estimate, it holds that

$$\sum_{i=1}^{4}|I_i| \lesssim \|u_{tt}\|_0\big(\|\partial_t u_3\|_0 + \sqrt{\mathcal{E}\mathcal{D}}\big).$$

Moreover, by using (3.1), integration by parts, and product estimate, we can obtain

$$|I_5| \lesssim \big(\|u_{tt}\|_0 + \|u_{tt}\|_1\big)\sqrt{\mathcal{E}\mathcal{D}}. \tag{3.13}$$

Putting the above two estimates into (3.12), and then using Friedrichs's and Young's inequalities, we get (3.10). Similarly, we can easily derive (3.9) from (3.11) with $i = 1$. □

Lemma 5 *Under the assumption* (3.1) *with sufficiently small* δ, *it holds that*

$$\frac{d}{dt}\|\sigma\|_0^2 + c\|\sigma\|_1^2 \lesssim \sqrt{\mathcal{E}}\mathcal{D}, \tag{3.14}$$

$$\frac{d}{dt}\|\sigma_t\|_0^2 + c\|\sigma_t\|_1^2 \lesssim \sqrt{\mathcal{E}}\mathcal{D}, \tag{3.15}$$

$$\frac{d}{dt}\|\sigma_{tt}\|_0^2 + c\|\sigma_{tt}\|_1^2 \lesssim \sqrt{\mathcal{E}}\mathcal{D}, \tag{3.16}$$

Proof Let $1 \le i \le 2$, applying ∂_t^i to $(3.2)_3$, we get

$$\partial_t^{i+1}\sigma - \Delta\partial_t^i\sigma = \partial_t^i\mathcal{N}_3, \tag{3.17}$$

multiplying (3.17) by σ in L^2 with $i = 0$, then using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|_0^2 + \|\nabla\sigma\|_0^2 = \int \mathcal{N}_3 \cdot \sigma \, dx. \tag{3.18}$$

By (3.1), integration by parts and product estimate, it holds that

$$\int \mathcal{N}_3\sigma \lesssim \sqrt{\mathcal{E}}\mathcal{D}.$$

Putting the above estimate into (3.18), and then using Young's inequality, we get (3.14). Similarly, we can easily derive (3.15) and (3.16) from (3.17) with $i = 1$ and $i = 2$, respectively. □

Lemma 6 *Under the assumption* (3.1) *with sufficiently* δ, *it holds that*

$$\|(u,q)\|_{s,2} \lesssim \|\varrho\|_2 + \|(u_{tt},u)\|_0 + \|\sigma\|_4\|\sigma\|_3, \tag{3.19}$$

$$\|\partial_t(u,q)\|_{s,0} \lesssim \|u\|_0 + \|u_{tt}\|_0 + (\|\varrho\|_2 + \|u\|_0)\|u\|_1 + \|\sigma_t\|_2\|\sigma\|_3, \tag{3.20}$$

$$\|(u,q)\|_{s,3} \lesssim \|\varrho\|_3 + \|u_{tt}\|_1 + \sqrt{\mathcal{E}\mathcal{D}}, \tag{3.21}$$

$$\|\partial_t(u,q)\|_{s,1} \lesssim \|(u_3,u_{tt})\|_1 + \sqrt{\mathcal{E}\mathcal{D}}. \tag{3.22}$$

Proof Applying ∂_t^i to $(3.2)_2$, we have

$$\bar{\rho}\partial_t^{i+1}u + \nabla\partial_t^i q - \mu\Delta\partial_t^i u = \partial_t^i\mathcal{N}_2 - g\partial_t^i\varrho e_3. \tag{3.23}$$

By $(3.11)_2$, $(3.11)_3$, and (3.23), for $i = 0$ and $i = 1$, we get the following Stokes problem:

$$\begin{cases} \nabla\partial_t^i q - \mu\Delta\partial_t^i u = \mathcal{M}_1, \\ \operatorname{div}\partial_t^i u = 0, \\ \partial_t^i u|_{\partial\Omega} = 0, \end{cases} \tag{3.24}$$

where we have defined

$$\mathcal{M}_1 = \begin{cases} -g\varrho e_3 - \bar{\rho}u_t + \mathcal{N}_2 & \text{for } i = 0, \\ g(\bar{\rho}'u_3 - \mathcal{N}_1)e_3 - \bar{\rho}u_{tt} + \partial_t\mathcal{N}_2 & \text{for } i = 1. \end{cases} \tag{3.25}$$

Applying the classical Stokes estimate to (3.24) yields

$$\|\partial_t^i(u,q)\|_{S,2-2i} \lesssim \begin{cases} \|(\varrho,u_t,\mathcal{N}_2)\|_2 & \text{for } i = 0, \\ \|(u_3,u_{tt},\mathcal{N}_1,\partial_t\mathcal{N}_2)\|_0 & \text{for } i = 1, \end{cases} \tag{3.26}$$

and

$$\|\partial_t^i(u,q)\|_{S,3-2i} \lesssim \begin{cases} \|(\varrho,u_t,\mathcal{N}_2)\|_3 & \text{for } i = 0, \\ \|(u_3,u_{tt},\mathcal{N}_1,\partial_t\mathcal{N}_2)\|_1 & \text{for } i = 1. \end{cases} \tag{3.27}$$

By (3.1), we can estimate that

$$\begin{aligned}\left\|(\mathcal{N}_1,\partial_t\mathcal{N}_2)\right\|_0 &\lesssim \left(\|\varrho\|_2+\|\varrho_t\|_0\right)\|u\|_1+\|u_t\|_2\left(\|\varrho_t\|_0+\|u\|_1\right)\\ &\quad+\|\varrho\|_2\|u_{tt}\|_0+\|\sigma_t\|_2\|\sigma\|_3, \qquad (3.28)\\ \|\mathcal{N}_2\|_2 &\lesssim \|u\|_2\|u\|_3+\|\varrho\|_2\|u_t\|_2+\|\sigma\|_4\|\sigma\|_3, \qquad (3.29)\\ \|\mathcal{N}_2\|_3+\left\|(\mathcal{N}_1,\partial_t\mathcal{N}_2)\right\|_1 &\lesssim \sqrt{\mathcal{E}\mathcal{D}}. \qquad (3.30)\end{aligned}$$

In addition, using (3.1) and $(3.2)_1$, we have

$$\|\varrho_t\|_0\lesssim\|u\|_0. \qquad (3.31)$$

By using Young's inequality and the four estimates above, we get (3.19)–(3.22) from (3.26) and (3.27). □

Lemma 7 *Under the assumption* (3.1) *with sufficiently* δ, *it holds that*

$$\begin{aligned}&\|\sigma\|_4\lesssim\|\sigma_t\|_2+\|\sigma\|_3\|u\|_2, \qquad (3.32)\\ &\|\sigma\|_5\lesssim\|\sigma_t\|_3+\sqrt{\mathcal{E}\mathcal{D}}, \qquad (3.33)\\ &\|\sigma_t\|_2\lesssim\|u_t\|_0\|\sigma\|_3+\|\sigma_{tt}\|_0, \qquad (3.34)\\ &\|\sigma_t\|_3\lesssim\|\sigma_{tt}\|_1+\sqrt{\mathcal{E}\mathcal{D}}. \qquad (3.35)\end{aligned}$$

Proof Applying ∂_t^i to $(3.2)_3$ and $(3.2)_6$, we get the following elliptic equations:

$$\begin{cases}-\Delta\partial_t^i\sigma=\partial_t^i(\mathcal{N}_3-\sigma_t),\\ \partial_t^i\sigma|_{\partial\Omega}=0.\end{cases} \qquad (3.36)$$

Applying the classical regularity theory for an elliptic problem (3.36) with $i=0,1,2,3$, we have

$$\left\|\partial_t^k\sigma\right\|_{i-2k+2}\lesssim\left\|\partial_t^k(\mathcal{N}_3,\sigma_t)\right\|_{i-2k}. \qquad (3.37)$$

(1) Taking $(i,k)=(2,0)$ in (3.37), we immediately get

$$\|\sigma\|_4\lesssim\|\mathcal{N}_3\|_2+\|\sigma_t\|_2. \qquad (3.38)$$

Now we estimate $\|\mathcal{N}_3\|_2$ and, by the definition of $\mathcal{N}_3$, get

$$\|\mathcal{N}_3\|_2\lesssim\|u\cdot\nabla\sigma\|_2+\left\||\nabla\sigma|^2\sigma\right\|_2+\left\||\nabla\sigma|^2\right\|_2\lesssim\|\sigma\|_3\|u\|_2+\|\sigma\|_0\|\sigma\|_3^2+\|\sigma\|_3^2, \quad (3.39)$$

thus, by using (3.1), we get (3.32) from (3.38) and (3.39).

(2) Taking $(i,k)=(3,0)$ in (3.37), we immediately get

$$\|\sigma\|_5\lesssim\|\mathcal{N}_3\|_3+\|\sigma_t\|_3, \qquad (3.40)$$

and similarly to (3.39), we get

$$\|\mathcal{N}_3\|_3 \lesssim \|u\|_3\|\sigma\|_4 + \|\sigma\|_4^2\|\sigma\|_3 + \|\sigma\|_4^2, \tag{3.41}$$

thus (3.33) follows by putting (3.41) into (3.40).

(3) Taking $(i,k) = (2,1)$ in (3.37), we can obtain that

$$\|\sigma_t\|_2 \lesssim \|\partial_t\mathcal{N}_3\|_0 + \|\sigma_{tt}\|_0. \tag{3.42}$$

By a simple calculation, we get

$$\|\partial_t\mathcal{N}_3\|_0 \lesssim \|u_t\|_0\|\sigma\|_3 + \|\sigma_t\|_2, \tag{3.43}$$

thus by putting (3.43) into (3.42), we obtain (3.34).

(4) Taking $(i,k) = (3,1)$ in (3.37), we obtain that

$$\|\sigma_t\|_3 \lesssim \|\partial_t\mathcal{N}_3\|_1 + \|\sigma_{tt}\|_1, \tag{3.44}$$

similarly, we get that

$$\|\partial_t\mathcal{N}_3\|_1 \lesssim \|u_t\|_1\|\sigma\|_3 + \|u\|_2\|\sigma_t\|_1 + \|\sigma_t\|_1\|\sigma\|_3, \tag{3.45}$$

thus, (3.35) follows form the above two estimates. □

Lemma 8 *Under the assumption* (3.1) *with sufficiently* δ, *we have that*

$$\mathcal{E} \textit{ is equivalent to } \left\|(\varrho,u,\sigma)\right\|_4^2. \tag{3.46}$$

Proof By (3.19), (3.20), (3.32), and (3.34), to get (3.46), it suffices to derive, for sufficiently small δ,

$$\|\varrho_t\|_3 + \|\varrho_{tt}\|_2 + \|u_t\|_0 + \|u_{tt}\|_0 + \|\sigma_{tt}\|_0 \lesssim \left\|(\varrho,u,\sigma)\right\|_4. \tag{3.47}$$

Next, we verify (3.47).

Multiplying (3.23) with $i = 1$ by u_{tt} in L^2, we infer that

$$\|\sqrt{\bar{\rho}}u_{tt}\|_0^2 = \int \left(g\left(\bar{\rho}'u_3 - \mathcal{N}^1\right)e_3 + \mu\Delta u_t + \partial_t\mathcal{N}_2\right)\cdot u_{tt}\,dy.$$

Using (1.2), (3.1), (3.28), (3.31), and Young's inequality, we can derive from the above identity that

$$\|u_{tt}\|_0^2 \lesssim \|u\|_1 + \|u_t\|_2^2 + \|\sigma_t\|_2\|\sigma\|_3. \tag{3.48}$$

Next we shall estimate for $\|u_t\|_2$ and $\|\sigma_t\|_2$.

By using (3.29), we can obtain from $(3.2)_2$ that

$$\|u_t\|_2 \lesssim \|\varrho\|_2 + \|q\|_3 + \|u\|_4 + \|\sigma\|_4 \tag{3.49}$$

and

$$\begin{cases} -\Delta q = \mathcal{M}_2 := \bar{\rho}'\partial_t u_3 + g\partial_3\varrho + \operatorname{div}((\varrho+\bar{\rho})u\cdot\nabla u + \Delta\sigma\cdot\nabla\sigma) + u_t\cdot\nabla\varrho, \\ \nabla q\cdot\vec{n}|_{\partial\Omega} = \mathcal{M}_3 := (\mu\Delta u - g\varrho e_3 - \Delta\sigma\cdot\nabla\sigma)\cdot\vec{n}, \end{cases} \tag{3.50}$$

where $\vec{n}$ denotes the unit outer normal vector on $\partial\Omega$. Applying the classical elliptic regularity theory to (3.50) yields that

$$\|q\|_3 \lesssim \|\mathcal{M}_2\|_1 + \|\mathcal{M}_3\|_{H^{3/2}(\partial\Omega)} \lesssim \|(\varrho,u,\sigma)\|_4 + \|u_t\|_1.$$

Inserting the above estimate into (3.49), and then using interpolation inequality, we arrive at

$$\|u_t\|_2 \lesssim \|(\varrho,u,\sigma)\|_4 + \|u_t\|_0. \tag{3.51}$$

Next we shall estimate u_t.

We multiply $(3.2)_2$ by u_t in L^2, and then use the integration by parts to obtain

$$\|\sqrt{\bar{\rho}}u_t\|_0^2 = \int(\mu\Delta u + \mathcal{N}_2 - g\varrho e_3)\cdot u_t\,dx.$$

Using (3.1) and Young's inequality, we can derive from the above identity that

$$\|u_t\|_0^2 \lesssim \|(\varrho,u)\|_2^2 + \|\sigma\|_3^2. \tag{3.52}$$

Thus, we derive from (3.48), (3.51), and (3.52) that, for sufficiently small δ,

$$\|u_t\|_2 + \|u_{tt}\|_0 \lesssim \|(\varrho,u,\sigma)\|_4^2 + \|\sigma_t\|_2^2. \tag{3.53}$$

Next we estimate $\|\sigma_t\|_2$.

By (3.39), we can derive from $(3.2)_3$ that

$$\|\sigma_t\|_2 \lesssim \|(u,\sigma)\|_4. \tag{3.54}$$

Now, we estimate σ_{tt}. Multiplying $(3.36)_1$ with $i=1$ by σ_{tt} in L^2, we infer that

$$\|\sigma_{tt}\|_0^2 = \int(\Delta\sigma + \partial_t\mathcal{N}_3)\cdot\sigma_{tt}\,dy. \tag{3.55}$$

Using (3.1), (3.43), and Young's inequality, we derive from the above identity

$$\|\sigma_{tt}\|_0^2 \lesssim \|u_t\|_0^2\|\sigma\|_3^2 + \|\sigma_t\|_2^2 + \|\sigma\|_2^2. \tag{3.56}$$

Thus, by (3.52), (3.54), and (3.56), we can get

$$\|\sigma_t\|_2 + \|\sigma_{tt}\|_0 \lesssim \|(\varrho,u,\sigma)\|_4. \tag{3.57}$$

Finally, by (3.2)$_1$, we have

$$\|\varrho_t\|_3 \lesssim \|u\|_3, \tag{3.58}$$

$$\|\varrho_{tt}\|_2 \lesssim \|u_t\|_2 + \|u_2\|\|\varrho_t\|_3. \tag{3.59}$$

Thus, we get (3.46) from (3.52) and (3.53), (3.57), and above two estimates. □

Proposition 2 *There exist a constant $\delta_1 \in (0,1)$ and $C > 0$ such that, for any $\delta \leq \delta_1$, if the solution (ϱ, u, σ) of LCRT problem satisfies (3.1), then for any $t \in (0,T)$, the solution (ϱ, v, σ) satisfies the Gronwall-type energy inequality*

$$\mathcal{E} + \int_0^t \mathcal{D}\,d\tau \leq C_1\left(\int_0^t \left\|(\varrho, v, \sigma)\right\|_0^2 d\tau + \left\|\left(\varrho^0, v^0, \sigma^0\right)\right\|_4^2\right) + \Lambda \int_0^t \mathcal{E}\,d\tau. \tag{3.60}$$

Proof We derive from Lemmas 2–5 that, for sufficiently large constant c_1,

$$\frac{d}{dt}\mathcal{E}_1 + c\mathcal{D}_1 \leq c_1\left(\left\|(\varrho, u_3)\right\|_0^2 + \|\varrho\|_4\|u\|_4 + \sqrt{\mathcal{E}}\mathcal{D}\right)/c, \tag{3.61}$$

where we have defined

$$\mathcal{E}_1 := \|\sqrt{\bar{\rho}}u\|_0^2 + \left\|\sqrt{\varrho + \bar{\rho}}(c_1 u_t, u_{tt})\right\|_0^2 + \left\|(\sigma, \sigma_t, \sigma_{tt})\right\|_0^2 + \|\varrho\|_4^2,$$

$$\mathcal{D}_1 := \left\|(c_1 u_t, u_{tt})\right\|_1^2 + \|u\|_1^2 + \left\|(\sigma, \sigma_t, \sigma_{tt})\right\|_1^2.$$

Integrating (3.61) over $(0,t)$, and then using interpolation and Young's inequalities, we get, for any given $\varepsilon > 0$,

$$\begin{aligned}
&\mathcal{E}_1 + c\int_0^t \left(\mathcal{D}_1 + \left\|\varrho(\tau)\right\|_3^2\right) d\tau \\
&\quad \leq \mathcal{E}_1(0) + \varepsilon \int_0^t \|\varrho\|_4^2\,d\tau + \frac{1}{c}\int_0^t \left(\left\|(\varrho, u)(\tau)\right\|_0^2 + \sqrt{\mathcal{E}}\mathcal{D}\right) d\tau,
\end{aligned} \tag{3.62}$$

where the positive constant c depends on ε.

Noting that, by (3.31), (3.58), (3.58), (3.59), and Lemmas 6–8, we easily derive that there exists a constant c such that, for sufficiently small δ,

$$\mathcal{E}_1, \mathcal{E}, \text{ and } \left\|(\varrho, u, \sigma)\right\| \text{ are equivalent for any } t \geq 0, \tag{3.63}$$

$$\mathcal{D} \leq c\left(\mathcal{D}_1 + \|\varrho\|_3^2\right). \tag{3.64}$$

Consequently, we immediately derive (3.60) from (3.62)–(3.64). □

Proposition 3 (1) *Let $\bar{\rho} \in C^5(\bar{\Omega})$. Then there are a sufficiently small $\delta_2 \in (0,1)$ and $K_1 > 0$ such that if $(\varrho^0, u^0, \sigma^0)$ satisfies*

$$\sqrt{\left\|\left(\varrho^0, u^0, \sigma^0\right)\right\|_4^2} < \delta_2, \tag{3.65}$$

$$\inf_{x \in \Omega}\left\{\left(\varrho^0 + \bar{\rho}\right)(x)\right\} \geq K_1 > 0, \tag{3.66}$$

and the compatibility conditions

$$\operatorname{div} u^0 = 0 \quad \textit{in } \Omega, \tag{3.67}$$

$$(\sigma_t, u_t)|_{t=0} = 0 \quad \textit{on } \partial\Omega, \tag{3.68}$$

then there exist a local existence time $T^{\max} > 0$ *(depending on* δ_2*, the domain and the known parameters) and a unique local-in-time classical solution* $(\varrho, u, \sigma, q) \in C^0(H^4 \times H^4_\sigma \times H^4_0 \times \underline{H}^3)$ *to the LCRT problem.*

(2) *In addition, if the solution* (ϱ, u, σ) *further satisfies*

$$\sup_{t\in[0,T)} \sqrt{\|(\varrho, u, \sigma)\|_4^2} \le \delta_1 \quad \textit{for some } T < T^{\max},$$

then (ϱ, u, σ) *enjoys the equivalent estimate* (3.46) *and the Gronwall-type energy inequality* (3.60).

Remark 1 For any given initial data $(\varrho^0, u^0, \sigma^0) \in H^4 \times H^4_\sigma \times H^4_0$ satisfying (3.65)–(3.67) with sufficiently small δ_2, there exists a unique local-in-time strong solution $(\varrho, u, \sigma, q) \in C^0([0,T), H^2 \times H^2_\sigma \times H^2_0 \times H^1)$. Moreover, the initial date of q is a weak solution to

$$\begin{cases} \operatorname{div}(\nabla q^0/(\varrho^0 + \bar\rho)) \\ \quad = \operatorname{div}((\mu\Delta u^0 - \Delta\sigma^0 \cdot \nabla\sigma^0 - g\varrho^0 e_3)/(\varrho^0 + \bar\rho) - u^0 \cdot \nabla u^0) \quad \text{in } \Omega, \\ \nabla q^0 \cdot \vec{n} = (\mu\Delta u^0 - g\varrho^0 e_3 - \Delta\sigma^0 \cdot \nabla\sigma^0) \cdot \vec{n} \quad \text{on } \partial\Omega. \end{cases} \tag{3.69}$$

If the condition (3.68) is further satisfied, i.e., $(\varrho^0, u^0, \sigma^0)$ satisfies

$$\begin{cases} \nabla q^0 - \mu\Delta u^0 + g\varrho^0 e_3 + \Delta\sigma^0 \cdot \nabla\sigma^0 = 0 \quad \text{on } \partial\Omega, \\ \Delta\sigma^0 + |\nabla\sigma^0|^2 e_3 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.70}$$

then we can improve the regularity of (ϱ, u, σ) so that it is a classical solution for sufficiently small δ_2.

Remark 2 For any classical solution (ϱ, u, σ) constructed by Proposition 3, and for any given $t_0 \in (0, T^{\max})$, we take $(\varrho, u, \sigma)|_{t=t_0}$ as a new initial datum. Then the new initial data can define a unique local-in-time classical solution $(\tilde\varrho, \tilde u, \tilde\sigma, \tilde q)$ constructed by Proposition 3, moreover, the initial data of $\tilde q$ is equal to $q|_{t=t_0}$ by unique solvability of (3.69).

4 Construction of initial data for the nonlinear problem

For any given $\delta > 0$, let

$$(\varrho^{\mathrm{a}}, u^{\mathrm{a}}, q^{\mathrm{a}}) = \delta e^{\Lambda t}(\tilde\varrho^0, \tilde u^0, \tilde q^0), \tag{4.1}$$

where $(\tilde\varrho^0, \tilde u^0, \tilde q^0) := (-\bar\rho' \tilde u_3/\Lambda, \tilde u, \tilde q)$, and $(\tilde u, \tilde q) \in H^4_\sigma \times \underline{H}^3$ comes from Proposition 1. Then $(\varrho^{\mathrm{a}}, u^{\mathrm{a}}, q^{\mathrm{a}})$ is a solution to the linearized RTLC equations, and enjoys the estimate, for any $i \ge 0$,

$$\|\partial_t^i(u^{\mathrm{a}}, q^{\mathrm{a}})\|_{\mathrm{S},2} \le c(i)\delta e^{\Lambda t}. \tag{4.2}$$

Moreover, by (2.4),

$$\|\tilde{\varrho}^0\|_{L^1}\|\tilde{u}_h^0\|_{L^1}\|\tilde{u}_3^0\|_{L^1} > 0.$$

Next we shall modify the initial data of the linear solutions.

Proposition 4 *Let $(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0)$ be the same as in (4.1), then is a constant δ_3, such that for any $\delta \in (0, \delta_3)$, there exists $(u^r, q^r) \in H^4_\sigma \times \underline{H}^3$ enjoying the following properties:*

(1) *The modified initial data*

$$\left(\varrho_0^\delta, u_0^\delta, q_0^\delta\right) := \delta\left(\tilde{\varrho}^0, \tilde{u}^0, \tilde{q}^0\right) + \delta^2\left(0, u^r, q^r\right) \tag{4.3}$$

belongs to $\underline{H}^4_0 \times H^4_\sigma \times \underline{H}^3$, and satisfies the compatibility conditions $(3.69)_1$ and (3.70) with (u_0^δ, q_0^δ) in place of (u^0, q^0).

(2) *The uniform estimate holds:*

$$\sqrt{\|u^r\|_4^2 + \|q^r\|_3^2} \le C_2, \tag{4.4}$$

where the constant $C_2 \ge 1$ depends on the domain, the density profile, and the known parameters, but is independent of δ.

Proof Recalling the construction of $(\tilde{u}^0, \tilde{q}^0)$, we can see that $(\tilde{u}^0, \tilde{q}^0)$ satisfies

$$\begin{cases} \operatorname{div}\tilde{u}^0 = 0 \quad \text{in } \Omega, \\ \Lambda^2\bar{\rho}\tilde{u}^0 + \Lambda\nabla\tilde{q}^0 - \Lambda\mu\Delta\tilde{u}^0 = g\bar{\rho}'\tilde{u}_3^0 \quad \text{in } \Omega, \\ \tilde{u}^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

If $(u^r, \sigma^r, q^r) \in H^4_\sigma \times H^4_0 \times \underline{H}^3$ satisfies, for any given δ,

$$\begin{cases} \operatorname{div} u^r = 0 \quad \text{in } \Omega, \\ \nabla q^r - \mu\Delta u^r := g\bar{\rho}' u_3^r e_3 - \Delta\sigma_0^\delta \cdot \nabla\sigma_0^\delta \\ \qquad\qquad + (\varrho_0^\delta + \bar{\rho})(\Upsilon - (\delta\tilde{u}^0 + \delta^2 u^r)\cdot\nabla(\delta\tilde{u}^0 + \delta^2 u^r))/\delta^2 \quad \text{in } \Omega, \\ \operatorname{div}\Upsilon = -\delta\Lambda\operatorname{div}(\varrho_0^\delta\tilde{u}^0/(\varrho_0^\delta + \bar{\rho})) \quad \text{in } \Omega, \\ (\Upsilon, u^r) = 0 \quad \text{on } \Omega, \end{cases} \tag{4.6}$$

where $(\varrho_0^\delta, u_0^\delta, \sigma_0^\delta, q_0^\delta)$ is given in the mode (4.3), then, by (4.5), it is easy to check that $(\varrho_0^\delta, u_0^\delta, \sigma_0^\delta, q_0^\delta)$ $H_4 \times H^4_\sigma \times H^4_0 \times \underline{H}^3$, and it satisfies the compatibility conditions $(3.69)_1$ and (3.70) with (u_0^δ, q_0^δ) in place of (u^0, q^0). Next we construct such (u^r, σ^r, q^r) which satisfy (4.5) for sufficiently small δ.

Since $H^2 \hookrightarrow L^\infty$, there exists a constant $\delta_4 > 0$ such that

$$\inf_{x\in\bar{\Omega}}\{w + \bar{\rho}\} \ge \inf_{x\in\bar{\Omega}}\{\bar{\rho}\}/2 > 0 \quad \text{for any } w \text{ satisfying } \|w\|_2 \le \delta_4. \tag{4.7}$$

Moreover,

(1) There exists $(\Upsilon, q) \in H_0^2 \times \underline{H}^1$ such that

$$\begin{cases} \nabla q - \Delta \Upsilon = 0, \qquad \operatorname{div} \Upsilon = -\delta \Lambda \operatorname{div}(\varrho_0^\delta \tilde{u}^0/(\varrho_0^\delta + \bar{\rho})) \quad \text{in } \Omega, \\ \Upsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

and

$$\|(\Upsilon, q)\|_{S,0} \le \delta^2. \tag{4.9}$$

(2) There exists $(u^r, q^r) \in H_0^4 \times \underline{H}^3$ such that

$$\begin{cases} \operatorname{div} u^r = 0 \quad \text{in } \Omega \\ \nabla q^r - \mu \Delta u^r := g\bar{\rho}' u_3^r e_3 - \Delta \sigma_0^\delta \cdot \nabla \sigma_0^\delta \\ \qquad\qquad + (\varrho_0^\delta + \bar{\rho})(\Upsilon - (\delta \tilde{u}^0 + \delta^2 u^r) \cdot \nabla(\delta \tilde{u}^0 + \delta^2 u^r))/\delta^2 \quad \text{in } \Omega, \\ u^r = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

and $\|(u^r, q^r)\|_{S,2} \le c$, where Υ is constructed in (4.8). We mention that the constant c above is independent of δ. Thus we can get Proposition 4 from the above arguments. □

5 Error estimates and existence of escape times

Let

$$C_3 := \sqrt{\|(\tilde{\varrho}^0, \tilde{u}^0)\|_4^2 + C_2} \ge 1, \tag{5.1}$$
$$\delta < \delta_0 := \min\{\delta_1, \delta_2, 2C_3\delta_3, \delta_4\}/2C_3 < 1,$$

and (u_0^δ, q_0^δ) be constructed by Proposition 4.

Noting that

$$\sqrt{\|(\varrho_0^\delta, u_0^\delta)\|_4^2} \le C_3 \delta < 2C_3\delta_0 < \delta_4, \tag{5.2}$$

then

$$\inf_{x \in \bar{\Omega}} \{\varrho_0^\delta + \bar{\rho}\} > 0.$$

Thus, by the first assertion in Proposition 3, one sees that there is a (nonlinear) solution (u, q) of the problem defined in some time interval $I_{T^{\max}}$ with the initial value $(\varrho_0^\delta, u_0^\delta)$. Moreover, we have $\int q \, dx = \int q_0^\delta \, dx = 0$.

Let $\varepsilon_0 \in (0,1)$ be a constant, which will be defined in (5.9). We define

$$T^\delta := (\Lambda)^{-1} \ln(\varepsilon_0/\delta) > 0, \quad \text{i.e.,}\ \delta e^{\Lambda T^\delta} = \varepsilon_0,$$
$$T^* := \sup\{t \in (0, T^{\max}) \mid \|(\varrho, u)(\tau)\|_4 \le 2C_3\delta_0 \text{ for any } \tau \in [0, t)\},$$
$$T^{**} := \sup\{t \in (0, T^{\max}) \mid \|(\varrho, u)(\tau)\|_0 \le 2C_3\delta e^{\Lambda t} \text{ for any } \tau \in [0, t)\}, \tag{5.3}$$

where $T^{\max}$ denotes the maximal time of existence of the solution $(\varrho, u) \in C([0, T^{\max}), H^4 \times H^4_\sigma)$. Obviously, $T^* T^{**} > 0$ and

$$\|(\varrho, u)(T^*)\|_4 = 2C_3\delta_0, \quad \text{if } T^* < \infty, \tag{5.4}$$

$$\|(\varrho, u)(T^*)\|_0 = 2C_3\delta e^{\Lambda T^{**}}, \quad \text{if } T^{**} < T^{\max}. \tag{5.5}$$

We denote $T^{\min} := \min\{T^\delta, T^*, T^{**}\}$. By the definition of T^{**}, we can deduce from the estimate (3.60) that, for all $t < T^{\min}$,

$$\mathcal{E} \le c\delta^2 e^{2\Lambda t} + \Lambda \int_0^t \mathcal{E}\, \mathrm{d}\tau. \tag{5.6}$$

Applying Gronwall's inequality to the above estimate, we arrive at, for some constant C_4,

$$\mathcal{E} \le C_4\delta^2 e^{2\Lambda t} \quad \text{for all } t < T^{\min}. \tag{5.7}$$

In addition, we have the following error estimate between the nonlinear solution (ϱ, u) and the linear solution (ϱ^a, u^a).

Lemma 9 *Let $\varepsilon_0 \le \delta_4/C_4$. There exists a constant C_5 such that, for any $\delta \in (0,1)$ and any $t \in I_{T^{min}}$,*

$$\|(\varrho^d, u^d)\|_\aleph \le C_5\sqrt{\delta^3 e^{3\Lambda t}}, \tag{5.8}$$

where $(\varrho^d, u^d) := (\varrho, u) - (\varrho^a, u^a)$, $\aleph = L^1$ or L^2, and C_5 is independent of $T^{\min}$.

Proof Please refer to [25, Lemma 3.1]. We mention that the condition $\varepsilon_0 \le \delta_4/C_4$ makes sure that $\inf_{x\in\bar\Omega}\{\varrho + \bar\rho\} \ge \inf_{x\in\bar\Omega}\{\bar\rho\}/2 > 0$. □

Now we define that

$$\varepsilon_0 := \min\left\{\frac{C_3\delta_0}{C_4}, \frac{C_3^2}{4C_5^2}, \frac{m_0^2}{4C_5^2}\right\} > 0, \tag{5.9}$$

$$m_0 := \min\{\|\tilde\varrho^0\|_{L^1}, \|\tilde u_3^0\|_{L^1}, \|(\tilde u_1^0, \tilde u_2^0)\|_{L^1}\}. \tag{5.10}$$

It is easy to see that $m_0 > 0$ by (5.10). Now, we assert that

$$T^\delta = T^{\min} \ne T^* \text{ or } T^{**}, \tag{5.11}$$

which can be proved by contradiction as follows:

(1) If $T^{\min} = T^* < T^\delta$, then $T^* < \infty$. Moreover, $T^* \le T^{\max}$ by Proposition 3. Note that we can deduce from (5.3), (5.7), and (5.9) that

$$\|(\varrho, u)(T^*)\|_4 \le C_4\delta e^{\Lambda T^\delta} = C_4\varepsilon_0 < C_3\delta_0,$$

which contracts (5.4). Hence $T^{\min} \ne T^*$.

(2) If $T^{\min} = T^{**} < T^{\delta}$, then $T^{**} < T^{*} \le T^{\max}$. Moreover, making use of (4.1), (5.1), (5.3), (5.8), and (5.9), we see that

$$\begin{aligned}\|(\varrho,u)(T^{**})\|_0 &\le \|(\varrho^{\mathrm{a}},u^{\mathrm{a}})(T^{**})\|_0 + \|(\varrho^d,u^d)(T^{**})\|_0\\ &\le \delta C_3 e^{\Lambda T^{**}} + C_5\delta^{3/2}e^{3\Lambda T^{**}/2} \le \delta e^{\Lambda T^{**}}(C_3 + C_5\sqrt{\varepsilon_0})\\ &< 2\delta C_3 e^{\Lambda T^{**}},\end{aligned} \tag{5.12}$$

which also contradicts (5.5). Therefore, $T^{\min} \neq T^{**}$. We immediately see that (5.11) holds. This completes the proof of claim (5.11).

Since $T^{\delta} < T^{*} \le T^{\max}$, we can use (4.1), (5.8), and (5.9) to deduce that

$$\begin{aligned}\|\varrho(T^{\delta})\|_{L^1} &\ge \|\varrho^{\mathrm{a}}(T^{\delta})\|_{L^1} - \|\varrho^d(T^{\delta})\|_{L^1}\\ &\ge \delta e^{\Lambda T^{\delta}}\|\tilde{\varrho}_0\|_{L^1} - C_5\delta^{3/2}e^{3\Lambda T^{\delta}/2}\\ &\ge \varepsilon_0\|\tilde{\varrho}_0\|_{L^1} - C_5\varepsilon_0^{3/2} \ge m_0\varepsilon_0 - C_5\varepsilon_0^{3/2} \ge m_0\varepsilon_0/2.\end{aligned} \tag{5.13}$$

Similarly, we also have

$$\|u_3(T^{\delta})\|_{L^1} \ge m_0\varepsilon_0 - C_5\varepsilon_0^{3/2} \ge m_0\varepsilon_0/2,$$

and

$$\|(u_1,u_2)(T^{\delta})\|_{L^1} \ge m_0\varepsilon_0 - C_5\varepsilon_0^{3/2} \ge m_0\varepsilon_0/2,$$

where u_i denote the ith component of $u(T^{\delta})$ for $1 \le i \le 3$. This completes the proof of Theorem 1 by defining $\varepsilon := m_0\varepsilon_0/2$.

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