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A new blow up criterion for the 3D magneto-micropolar fluid flows without magnetic diffusion

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Abstract

This note obtains a new regularity criterion for the three-dimensional magneto-micropolar fluid flows in terms of one velocity component and the gradient field of the magnetic field. The authors prove that the weak solution (u, ω, b) to the magneto-micropolar fluid flows can be extended beyond time $t = T$, provided if $u_3 \in L^{\beta}(0, T; L^{\alpha}(R^3))$ with $\frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}$, $\alpha > \frac{10}{3}$ and $\nabla b \in L^{\frac{4p}{3(p-2)}}(0, T; \dot{M}_{p,q}(R^3))$ with $1 < q \leq p < \infty$ and $p \geq 3$.

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1 Introduction

The aim of this paper is to understand the regularity criterion for the following three-dimensional magneto-micropolar fluid flows without magnetic diffusion:

$$\begin{cases} \partial_t u - (\mu + \chi)\Delta u + \nabla \pi = -u \cdot \nabla u + b \cdot \nabla b - \chi \nabla \times \omega, & (x, t) \in R^3 \times (0, T), \\ \partial_t \omega - \gamma \Delta \omega - \kappa \nabla \operatorname{div} \omega + 2\chi \omega = -u \cdot \nabla \omega + \chi \nabla \times u, & (x, t) \in R^3 \times (0, T), \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, & (x, t) \in R^3 \times (0, T), \\ \nabla \cdot u = \nabla \cdot b = 0, & (x, t) \in R^3 \times (0, T), \\ u|_{t=0} = u_0, \quad \omega|_{t=0} = \omega_0, \quad b|_{t=0} = b_0, & x \in R^3. \end{cases} \quad (1.1)$$

This system is a special case of the classical three-dimensional magneto-micropolar fluid flows

$$\begin{cases} \partial_t u - (\mu + \chi)\Delta u + \nabla \pi = -u \cdot \nabla u + b \cdot \nabla b - \chi \nabla \times \omega, & (x, t) \in R^3 \times (0, T), \\ \partial_t \omega - \gamma \Delta \omega - \kappa \nabla \operatorname{div} \omega + 2\chi \omega = -u \cdot \nabla \omega + \chi \nabla \times u, & (x, t) \in R^3 \times (0, T), \\ \partial_t b + u \cdot \nabla b - \nu \Delta b = b \cdot \nabla u, & (x, t) \in R^3 \times (0, T), \\ \operatorname{div} u = \operatorname{div} b = 0, & (x, t) \in R^3 \times (0, T), \\ u|_{t=0} = u_0, \quad \omega|_{t=0} = \omega_0, \quad b|_{t=0} = b_0, & x \in R^3, \end{cases} \quad (1.2)$$

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where $u = (u_1, u_2, u_3)$, $\omega = (\omega_1, \omega_2, \omega_3)$, $b = (b_1, b_2, b_3)$ and π denote the unknown velocity field, the micro-rotational velocity, the magnetic field and the unknown scalar pressure at the point $(x, t) \in \mathbb{R}^3 \times (0, T)$, respectively. While u_0, ω_0, b_0 are the prescribed initial data and $\operatorname{div} u = \operatorname{div} b = 0$ in the sense of distributions. The constants $\mu, \chi, \kappa, \gamma$ are positive numbers associated with the properties of the material, where μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities (more details see [11]).

The magneto-micropolar fluid system (1.2) was first proposed by Galdi and Rionero [7]. The existence of global-in-time weak solutions was then established by Rojas-Medar and Boldrini [14], while the local strong solutions and global strong solutions in bounded domain for the small initial data were considered, respectively, by Rojas-Medar [13] and Ortega-Torres and Rojas-Medar [12]. However, whether the local strong solutions can exist globally or the global weak solution is regular and unique is an outstanding open problem. Hence there are many regularity criteria to ensure the smoothness of solutions. Gala [2] proved that, if $u \in L^{\frac{2}{1-r}}(0, T; \dot{M}_{p, \frac{3}{r}}^{\frac{3}{r}}(\mathbb{R}^3))$ or $\nabla u \in L^{\frac{2}{2-r}}(0, T; \dot{M}_{p, \frac{3}{r}}^{\frac{3}{r}}(\mathbb{R}^3))$, then the local smooth solution (u, ω, b) can be extended beyond $t = T$. Zhang and Yao [17] demonstrated that, if $\nabla u \in L^p(0, T; \dot{F}_{q, \frac{2q}{3}}^0(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{3}{q} = 2$, $\frac{3}{2} < q \leq \infty$, then the weak solution (u, ω, b) is smooth on $[0, T]$.

When the micro-rotational velocity $\omega = 0$ and $\chi = 0$, Eq. (1.2) becomes the standard magneto-hydrodynamic equations. In recent years, the problem of regularity criteria involving one component has been investigated for the MHD equations (see e.g. [3–5, 8, 9]). In 2016, Yamazaki [15] proved that, if

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3))$$

with $\frac{2}{p} + \frac{3}{q} \leq \frac{1}{3} + \frac{1}{2q}$, $\frac{15}{2} < q \leq \infty$ and

$$j_3 \in L^{p'}(0, T; L^{q'}(\mathbb{R}^3))$$

with $\frac{2}{p'} + \frac{3}{q'} \leq 2$, $\frac{13}{2} < q' \leq \infty$, then the weak solution (u, b) is regular, where j_3 is the third component of $\nabla \times b = (j_1, j_2, j_3)$. Later Zhang [16] refined the result of Yamazaki's. He proved that, if

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3))$$

with $\frac{2}{p} + \frac{3}{q} \leq \frac{4}{9} + \frac{1}{3q}$, $\frac{15}{2} < q \leq \infty$ and

$$j_3 \in L^{p'}(0, T; L^{q'}(\mathbb{R}^3))$$

with $\frac{2}{p'} + \frac{3}{q'} \leq 2$, $\frac{3}{2} < q' \leq \infty$, then the weak solution (u, b) is regular.

We further assume that $\omega = \chi = 0$, the system (1.1) is usually named MHD equations without magnetic diffusion. In order to present our motivation, we list some information on regularity criteria for 2D-MHD equations without magnetic diffusion. In 2011, Zhou and Fan [18] proved if $\nabla b \in L^1(0, T; \operatorname{BMO}(\mathbb{R}^2))$, then the local strong solution (u, b) is regular. Gala, Ragusa and Ye [6] improved Zhou and Fan's result. They showed that, if $\nabla b \in L^{\frac{p}{p-1}}(0, T; \dot{M}_{p, q}^{\frac{p}{p-1}}(\mathbb{R}^2))$ with $p \geq q > 1$, the local strong solution (u, b) to the MHD equations with magnetic diffusion is regular. Motivated by [2, 15], and [6], we will investigate

the regularity criterion on the weak solution to the magneto-micropolar flows involving one velocity component and the gradient of magnetic field satisfying (1.3) and (1.4). Our result is stated as follows.

Theorem 1.1 *Let $(u_0, b_0) \in H^1(R^3)$ and $\omega_0 \in H^1(R^3)$, with the initial data $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, ω, b) be the weak solution to the equations (1.1) defined on $[0, T)$ for some $0 < T < \infty$. If (u, b) satisfies*

$$u_3 \in L^\beta(0, T; L^\alpha(R^3)), \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \quad \alpha > \frac{10}{3}, \quad (1.3)$$

and

$$\nabla b \in L^{\frac{4p}{3(p-2)}}(0, T; \dot{M}_{p,q}(R^3)), \quad 1 < q \leq p < \infty, p \geq 3, \quad (1.4)$$

then the solution (u, ω, b) to (1.1) can be smoothly extended beyond $t = T$.

Remark 1.1 To the best of our knowledge, this is the first regularity criterion result is concerned with weak solution to the 3D incompressible MHD equations without magnetic diffusion in Morrey Campanato space. The worst difficulty is to handle the nonlinear term $\int_{R^3} u \cdot \nabla u \cdot \Delta u \, dx$. For the two dimension case, due to $\int_{R^2} u \cdot \nabla u \cdot \Delta u = 0$, the condition $\nabla b \in L^{\frac{2p}{p-2}}(0, T; \dot{M}_{p,q}(R^2))$ is sufficient (see [6]). Compared with the result in [2], due to the magneto-micropolar fluid flows discussed in Theorem 1.1 there is lack of magnetic diffusion, it increases the difficulties of dealing with the nonlinear terms in H^1 -energy estimates, especially for the term $\int_{R^3} u \cdot \nabla w \cdot \Delta w \, dx$. Fortunately, the L^3 -energy estimate for ω helps us to overcome these problems.

When $\omega = 0$, $\chi = 0$, the magneto-micropolar equations (1.1) become the classical MHD equations without magnetic diffusion. Theorem 1.1 converts into the following corollary.

Corollary 1.1 *Let $(u_0, b_0) \in H^1(R^3)$ with the initial data $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Assume that (u, b) be the weak solution to the incompressible MHD equations defined on $[0, T)$ for some $0 < T < \infty$. If (u, b) satisfies*

$$u_3 \in L^\beta(0, T; L^\alpha(R^3)), \quad \frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3}{4} + \frac{1}{2\alpha}, \quad \alpha > \frac{10}{3},$$

and

$$\nabla b \in L^{\frac{4p}{3(p-2)}}(0, T; \dot{M}_{p,q}(R^3)), \quad 1 < q \leq p < \infty, p \geq 3,$$

then the local strong solution (u, b) can be smoothly extended beyond $t = T$.

Remark 1.2 Comparing with the results either in [15] or in [16], though the MHD equations discussed in Corollary 1.1 show lack of magnetic diffusion and the spatial space of magnetic field is enlightened, it is hard to say our results have refined the one in [15] or that in [16].

The difficulties and strategy are listed as follows:

- The first big tiger is to estimate the nonlinear term $\int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta u \, dx$. Since $\int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta u \, dx \neq 0$, it is impossible to handle it as in the second dimension. We have to present the horizontal energy estimate $\|\nabla_h u\|_{L^2}, \|\nabla_h b\|_{L^2}$.
- The second thorn is the nonlinear term $\int_{\mathbb{R}^3} u \cdot \nabla w \cdot \Delta w \, dx$. Integrating by part gives $\int_{\mathbb{R}^3} \Delta u \cdot \nabla w \cdot w \, dx$ and $\int_{\mathbb{R}^3} \nabla u \cdot \nabla^2 w \cdot w \, dx$. As usual, if we use the Hölder inequality and the Young inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^3} u \cdot \nabla w \cdot \Delta w \, dx \right| &\leq \|u\|_{L^6} \|\nabla w\|_{L^3} \|\Delta w\|_{L^2} \leq \|\nabla u\|_{L^2} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\Delta w\|_{L^2}^{\frac{3}{2}} \\ &\leq \frac{1}{4} \|\Delta w\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|\nabla w\|_{L^2}^2, \end{aligned}$$

which does not work. Thanks to the L^3 -norm estimate of w and some suitable interpolating inequality, which can be found in Sect. 3, we can overcome these difficulties. More precisely, these methods helps us to handle $\int_{\mathbb{R}^3} \Delta u \cdot \nabla w \cdot w \, dx$ and $\int_{\mathbb{R}^3} \nabla u \cdot \nabla^2 w \cdot w \, dx$ successfully.

The rest of this paper is organized as follows. The definition of some functional spaces and some useful lemmas are presented in Sect. 2. The L^3 -norm of w is given in Sect. 3. The proof of Theorem 1.1 is presented in Sect. 4.

2 Preliminaries and some basic facts

In this section, we will present some information on the Morrey space and introduce the definition of a weak solution to the magneto-micropolar equation (1.1).

Definition 2.1 (see [10]) For $1 < q \leq p \leq \infty$, the homogeneous Morrey space is presented as follows:

$$\dot{M}_{p,q} = \left\{ f \in L_{loc}^q(\mathbb{R}^3) : \|f\|_{\dot{M}_{p,q}(\mathbb{R}^3)} = \sup_{x \in \mathbb{R}^3} \sup_{R>0} R^{\frac{3}{p}-\frac{3}{q}} \|f\|_{L^q(B(x,R))} < \infty \right\},$$

where $B(x, R)$ denotes the closed ball in $\mathbb{R}^3$ with center x and radius R .

Definition 2.2 (see [10]) Let $1 < p' \leq q' < \infty$, we address the homogeneous space $Z_{p',q'}$ defined by

$$\begin{aligned} Z_{p',q'} = \left\{ f \in L^{p'} \mid f = \sum_{k \in \mathbb{N}} g_k, \quad \text{where } (g_k) \subset L_{\text{comp}}^{q'}(\mathbb{R}^3), \right. \\ \left. \sum_{k \in \mathbb{N}} d_k^{3(\frac{1}{p'}-\frac{1}{q'})} \|g_k\|_{L^{q'}} < \infty, \quad \text{where } \forall k, d_k = \text{diam}(\text{supp } g_k) < \infty \right\}. \end{aligned}$$

The following lemma plays a crucial role in proving the regularity criterion for the magneto-micropolar fluid flows (1.1).

Lemma 2.1 (see [10]) (1) Let $1 < p' \leq q' < \infty$ and p, q such that $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$, then $\dot{M}_{p,q}$ is the dual space of $Z_{p',q'}$.

(2) Let $1 < p' \leq q' < 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $r = \frac{3}{p}$, there exists $C > 0$ such that $\forall f \in L^2(R^3)$ and $\forall g \in \dot{H}^r(R^3), h \in \dot{M}_{p,q}(R^3)$ satisfies

$$\int_{R^3} |f(x)g(x)h(x)| dx \leq C \|h\|_{\dot{M}_{p,q}} \|fg\|_{Z_{p',q'}} \leq C \|h\|_{\dot{M}_{p,q}} \|f\|_{L^2} \|g\|_{\dot{H}^r}.$$

The Sobolev–Ladyzhenskaya inequality in the whole space R^3 reads as follows.

Lemma 2.2 (see [1]) *There exists a constant $C > 0$ such that*

$$\|\phi\|_{L^p} \leq C \|\phi\|_{L^2}^{\frac{6-p}{2p}} \|\partial_1 \phi\|_{L^2}^{\frac{p-2}{2p}} \|\partial_2 \phi\|_{L^2}^{\frac{p-2}{2p}} \|\partial_3 \phi\|_{L^2}^{\frac{p-2}{2p}}, \quad (2.1)$$

for every $\phi \in H^1(R^3)$ and every $p \in [2, 6]$, where C is a constant depending only on p .

The definition of a weak solution to the magnetic micropolar equation is provided in the following.

Definition 2.3 Let $(u_0, b_0) \in L^2_\sigma(R^3), \omega \in L^2(R^3)$ and $T > 0$. A measurable function (u, b, ω) is said to be a weak solution to (1.1) on $(0, T)$ if

- (i) $(u, b) \in L^\infty(0, T; L^2_\sigma(R^3)) \cap L^2(0, T; H^1(R^3))$, and $\omega \in L^\infty(0, T; L^2(R^3)) \cap L^2(0, T; H^1(R^3))$;
- (ii) for every $\phi, \varphi \in H^1(0, T; H^1_\sigma(R^3))$ and $\psi \in H^1(0, T; H^1(R^3))$ with $\phi(T) = \varphi(T) = \psi(T) = 0$,

$$\begin{aligned} & \int_0^T \langle -u, \partial_t \phi \rangle + \langle u \cdot \nabla u, \phi \rangle + (\mu + \chi) \langle \nabla u, \nabla \phi \rangle dt \\ & - \int_0^T \langle b \cdot \nabla b, \phi \rangle + \chi \langle \nabla \times \omega, \phi \rangle dt = -\langle u_0, \phi_0 \rangle, \\ & \int_0^T \langle -\omega, \partial_t \varphi \rangle + \gamma \langle \omega, \nabla \varphi \rangle + \kappa \langle \nabla \cdot u, \nabla \cdot \varphi \rangle dt \\ & + \int_0^T \langle u \cdot \nabla \omega, \varphi \rangle + 2\chi \langle \omega, \varphi \rangle - 2\chi \langle \nabla \times u, \varphi \rangle dt = -\langle \omega_0, \phi_0 \rangle, \end{aligned}$$

and

$$\int_0^T \langle -b, \partial_t \psi \rangle + \langle u \cdot \nabla b, \psi \rangle - \langle b \cdot \nabla u, \psi \rangle dt = -\langle b_0, \psi_0 \rangle,$$

where $L^2_\sigma = \{u | u \in L^2, \nabla \cdot u = 0\}$.

3 Some useful lemmas

The following L^3 -energy estimate is needed to prove our result.

Lemma 3.1 *Let (u, ω, b) be the weak solution to the magneto-micropolar equation (1.1). Then*

$$\|\omega\|_{L^3}^3 + \int_0^t \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^2 d\tau \leq C(\|\omega_0\|_{L^3}, \|\omega_0\|_{L^2}, \|u_0\|_{L^2}, \|b_0\|_{L^2}, T), \quad (3.1)$$

for any $t \in [0, T)$.

Proof Multiplying (1.1)₂ by $|\omega|\omega$ and integrating over R^3 , we have

$$\begin{aligned} & \frac{1}{3} \frac{d}{dt} \|\omega\|_{L^3}^3 + 2\|\omega\|_{L^3}^3 + \frac{4}{9} \|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2 + \frac{1}{2} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\omega|^{\frac{1}{2}} \nabla \cdot \omega\|_{L^2}^2 \\ & \leq \left| \int_{R^3} \nabla \times u \cdot \omega |\omega| dx \right|, \end{aligned} \quad (3.2)$$

where we have used

$$\begin{aligned} \int_{R^3} (\nabla \cdot \omega) \nabla \cdot (|\omega|\omega) dx &= \int_{R^3} \nabla \cdot \omega (|\omega| \nabla \cdot \omega + \omega \nabla |\omega|) dx \\ &= \|\omega|^{\frac{1}{2}} \nabla \cdot \omega\|_{L^2}^2 + \int_{R^3} \nabla \cdot \omega \cdot \omega \cdot \nabla |\omega| dx \\ &\geq \frac{1}{2} \|\omega|^{\frac{1}{2}} \nabla \cdot \omega\|_{L^2}^2 - \frac{1}{2} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2}^2. \end{aligned}$$

To estimate the nonlinear term on the right hand side, integrating by parts and using the Young inequality and (4.1), we obtain

$$\begin{aligned} \int_{R^3} \nabla \times u \cdot \omega |\omega| dx &\leq \int_{R^3} |u| |\omega| |\nabla \omega| dx + \int_{R^3} |u| |\omega| |\nabla \times \omega| dx \\ &\leq C \int_{R^3} |u| |\omega| |\nabla \omega| dx \\ &\leq C \|u\|_{L^3} \|\omega|^{\frac{1}{2}}\|_{L^6} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2} \\ &\leq \frac{1}{4} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\omega\|_{L^3}^3 + C \|\nabla u\|_{L^2}^{\frac{3}{2}} \\ &\leq \frac{1}{4} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\omega\|_{L^3}^3 + C (\|\nabla u\|_{L^2}^2 + 1). \end{aligned} \quad (3.3)$$

Substituting the above inequality into (3.2) gives

$$\begin{aligned} & \frac{d}{dt} \|\omega\|_{L^3}^3 + \|\omega\|_{L^3}^3 + \|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2 + \frac{3}{4} \|\omega|^{\frac{1}{2}} \nabla \omega\|_{L^2}^2 + \|\omega|^{\frac{1}{2}} \nabla \cdot \omega\|_{L^2}^2 \\ & \leq C (\|\nabla u\|_{L^2}^2 + 1). \end{aligned}$$

Integrating on $[0, t]$ and using (4.1) yield

$$\begin{aligned} \|\omega\|_{L^3}^3 + \int_0^t \|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2 d\tau &\leq \|\omega_0\|_{L^3}^3 + C \int_0^t (\|\nabla u\|_{L^2}^2 + 1) d\tau \\ &\leq C (\|\omega_0\|_{L^3}, \|\omega_0\|_{L^2}, \|u_0\|_{L^2}, \|b_0\|_{L^2}, T). \end{aligned} \quad \square$$

4 Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1. We will assume that $\mu = \chi = \gamma = \kappa = 1$ throughout this paper.

Let $[0, T^*)$ be the maximal time interval for the existence of the local smooth solution. If $T^* \geq T$, the conclusion is obviously valid, but for $T^* < T$, we would show that

$$\limsup_{t \rightarrow T^*} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \omega(\cdot, t)\|_{L^2}^2 + \|\nabla b(\cdot, t)\|_{L^2}^2) \leq C$$

under the assumption of (1.3) and (1.4). Hence, according to the definition of T^* , this leads to a contradiction.

Step 1: L^2 -energy estimate A standard energy method says

$$\begin{aligned} & \| (u(t), \omega(t), b(t)) \|_{L^2}^2 + 2 \int_0^t \| \nabla u \|_{L^2}^2 d\tau + 2 \int_0^t \| \nabla \omega \|_{L^2}^2 d\tau \\ & + 2 \int_0^t \| \nabla \cdot \omega \|_{L^2}^2 d\tau + 2 \int_0^t \| \omega \|_{L^2}^2 d\tau \leq \| (u_0, \omega_0, b_0) \|_{L^2}^2. \end{aligned} \quad (4.1)$$

Step 2: H^1 -Horizontal energy estimate

We first establish the horizontal gradient of the velocity u and magnetic field b . Taking ∇_h on both sides of Eqs. (1.1)₁ and (1.1)₃, multiplying by $\nabla_h u$ and $\nabla_h b$, respectively, and integrating over $\mathbf{R}^3$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\| \nabla_h u \|_{L^2}^2 + \| \nabla_h b \|_{L^2}^2) + 2 \| \nabla_h \nabla u \|_{L^2}^2 \\ & = - \int_{\mathbf{R}^3} \nabla_h (u \cdot \nabla u) \cdot \nabla_h u \, dx \\ & \quad + \int_{\mathbf{R}^3} (\nabla_h (b \cdot \nabla b) \cdot \nabla_h u + \nabla_h (b \cdot \nabla u) \cdot \nabla_h b - \nabla_h (u \cdot \nabla b) \cdot \nabla_h b) \, dx \\ & \quad + \int_{\mathbf{R}^3} \nabla_h (\nabla \times \omega) \cdot \nabla_h u \, dx \\ & := A_1 + A_2 + A_3. \end{aligned} \quad (4.2)$$

We start to estimate A_2 . From (1.1)₄ and Lemma 2.1 and the fact $\nabla \cdot u = \nabla \cdot b = 0$, we know that ($p \geq 3$)

$$\begin{aligned} A_2 &= \int_{\mathbf{R}^3} \sum_{k=1}^2 \partial_k (b \cdot \nabla b) \partial_k u \, dx + \int_{\mathbf{R}^3} \sum_{k=1}^2 \partial_k (b \cdot \nabla u) \partial_k b \, dx \\ & \quad - \int_{\mathbf{R}^3} \sum_{k=1}^2 \partial_k (u \cdot \nabla b) \partial_k b \, dx \\ &= \int_{\mathbf{R}^3} \sum_{k=1}^2 (\partial_k b \cdot \nabla b \partial_k u + \partial_k b \cdot \nabla u \partial_k b) \, dx - \int_{\mathbf{R}^3} \sum_{k=1}^2 \partial_k u \cdot \nabla b \partial_k b \, dx \\ &\leq C \int_{\mathbf{R}^3} |\nabla b| |\nabla u| |\nabla b| \, dx \leq C \| \nabla b \|_{\dot{M}_{p,q}} \| \nabla b \|_{L^2} \| \nabla u \|_{\dot{H}^{\frac{3}{p}}} \\ &\leq C \| \nabla b \|_{\dot{M}_{p,q}} \| \nabla b \|_{L^2} \| \nabla u \|_{L^2}^{1-\frac{3}{p}} \| \Delta u \|_{L^2}^{\frac{3}{p}}. \end{aligned} \quad (4.3)$$

Thanks to the Hölder and Young inequalities, one deduces

$$\begin{aligned} A_3 &= \int_{\mathbf{R}^3} \sum_{k=1}^2 \partial_k (\nabla \times \omega) \cdot \partial_k u \, dx - \int_{\mathbf{R}^3} \sum_{k=1}^2 \nabla \times \omega \cdot \partial_k \partial_k u \, dx \\ &\leq C \| \nabla \times \omega \|_{L^2} \| \nabla_h \nabla u \|_{L^2} \\ &\leq \frac{1}{2} \| \nabla_h \nabla u \|_{L^2}^2 + C \| \nabla \omega \|_{L^2}^2. \end{aligned} \quad (4.4)$$

Now it is time to deal with the first term A_1 . Integrating by parts, we get

$$\begin{aligned}
 A_1 &= \int_{R^3} u \cdot \nabla u \cdot \Delta_h u \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 u_i \partial_i u_j \partial_k \partial_k u_j \, dx + \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 u_i \partial_i u_3 \partial_k \partial_k u_3 \, dx \\
 &\quad + \int_{R^3} \sum_{k=1}^2 \sum_{j=1}^3 u_3 \partial_3 u_j \partial_k \partial_k u_j \, dx \\
 &:= A_{11} + A_{12} + A_{13}.
 \end{aligned} \tag{4.5}$$

Using integration by parts again and applying the fact that $\operatorname{div} u = 0$, it yields

$$\begin{aligned}
 A_{11} &= - \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 (\partial_k u_i \partial_i u_j \partial_k u_j + u_i \partial_i \partial_k u_j \partial_k u_j) \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 \partial_k u_i \partial_i u_j \partial_k u_j \, dx + \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 \partial_i u_i \partial_k u_j \partial_k u_j \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 \partial_k u_i \partial_i u_j \partial_k u_j \, dx - \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \sum_{j=1}^2 \partial_3 u_3 \partial_k u_j \partial_k u_j \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i,j=1}^2 \partial_k u_i \partial_i u_j \partial_k u_j \, dx - \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \sum_{j=1}^2 \partial_3 u_3 \partial_k u_j \partial_k u_j \, dx \\
 &= - \int_{R^3} \partial_3 u_3 ((\partial_1 u_2)^2 + \partial_2 u_1 \partial_1 u_2 + (\partial_2 u_1)^2 + (\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2) \, dx \\
 &\quad - \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \sum_{j=1}^2 \partial_3 u_3 \partial_k u_j \partial_k u_j \, dx
 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
 A_{12} &= - \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 (\partial_k u_i \partial_i u_3 \partial_k u_3 + u_i \partial_i \partial_k u_3 \partial_k u_3) \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) \, dx \\
 &\quad + \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 \partial_i u_i \partial_k u_3 \partial_k u_3 \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) \, dx \\
 &\quad - \frac{1}{2} \int_{R^3} \sum_{k=1}^2 \partial_3 u_3 \partial_k u_3 \partial_k u_3 \, dx \\
 &= \int_{R^3} \sum_{k=1}^2 \sum_{i=1}^2 (\partial_k \partial_i u_i u_3 \partial_k u_3 + \partial_k u_i u_3 \partial_k \partial_i u_3) \, dx
 \end{aligned} \tag{4.7}$$

$$+ \int_{\mathbb{R}^3} \sum_{k=1}^2 u_3 \partial_3 \partial_k u_3 \partial_k u_3 \, dx.$$

Substituting (4.6) and (4.7) into (4.5) yields

$$A_1 \leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| \, dx. \quad (4.8)$$

Thanks to the Hölder inequality, (2.1) and the Young inequality, we obtain for $\alpha > 3$

$$\begin{aligned} A_1 &\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}} \\ &\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^6}^{\frac{3}{\alpha}} \\ &\leq C \|u_3\|_{L^\alpha} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla_h \nabla u\|_{L^2}^{\frac{2}{\alpha}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{\alpha}} \\ &\leq \frac{1}{2} \|\nabla_h \nabla u\|_{L^2}^2 + C \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}}, \end{aligned} \quad (4.9)$$

which, along with (4.3), (4.4) and (4.2), gives

$$\begin{aligned} &\frac{d}{dt} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + 2 \|\nabla_h \nabla u\|_{L^2}^2 \\ &\leq C \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} + C \|\nabla \omega\|_{L^2}^2 \\ &\quad + C \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}}. \end{aligned} \quad (4.10)$$

Integrating over $[0, t)$ and using (4.1), one can verify

$$\begin{aligned} &\sup_{0 \leq \tau \leq t} (\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + 2 \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 \, d\tau \\ &\leq C (\|\nabla_h u_0\|_{L^2}^2 + \|\nabla_h b_0\|_{L^2}^2 + 1) + C \int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} \, d\tau \\ &\quad + C \int_0^t \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}} \, d\tau. \end{aligned} \quad (4.11)$$

Step 3: H^1 -full energy estimate

Multiplying (1.1)₁, (1.1)₂ and (1.1)₃ by $-\Delta u$, $-\Delta \omega$ and $-\Delta b$, respectively, and integrating over $\mathbb{R}^3$, then adding them we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + 2 \|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\nabla \nabla \cdot \omega\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta u \, dx - \int_{\mathbb{R}^3} b \cdot \nabla b \cdot \Delta u \, dx + \int_{\mathbb{R}^3} u \cdot \nabla \omega \cdot \Delta \omega \, dx \\ &\quad - 2 \int_{\mathbb{R}^3} \nabla \times u \cdot \Delta \omega \, dx + \int_{\mathbb{R}^3} u \cdot \nabla b \cdot \Delta b \, dx - \int_{\mathbb{R}^3} b \cdot \nabla u \cdot \Delta b \, dx \\ &:= B_1 + B_2 + B_3 + B_4 + B_5 + B_6, \end{aligned} \quad (4.12)$$

where we have used the inequalities

$$\begin{aligned} \int_{\mathbb{R}^3} (-\nabla \nabla \cdot \omega)(-\Delta \omega) dx &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_j \omega_j \partial_k \partial_k \omega_i dx \\ &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k \partial_j \omega_j \partial_k \partial_i \omega_i dx \\ &= \|\nabla \nabla \cdot \omega\|_{L^2}^2 \end{aligned}$$

and

$$\int_{\mathbb{R}^3} \nabla \times \omega \cdot \Delta u dx = \int_{\mathbb{R}^3} \Delta \omega \cdot \nabla \times u dx.$$

Now, we estimate B_2, B_5 and B_6 . After applying integration by parts, $\operatorname{div} u = \operatorname{div} b = 0$, the Hölder inequality, Lemma 2.1, the Sobolev interpolation inequality and the Young inequality, we have

$$\begin{aligned} B_2 + B_5 + B_6 &= \int_{\mathbb{R}^3} \nabla b \cdot \nabla b \cdot \nabla u dx - \int_{\mathbb{R}^3} \nabla u \cdot \nabla b \cdot \nabla b dx + \int_{\mathbb{R}^3} \nabla b \cdot \nabla u \cdot \nabla b dx \\ &\leq C \int_{\mathbb{R}^3} |\nabla b| |\nabla u| |\nabla b| dx \\ &\leq C \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{\dot{H}^{\frac{3}{p}}} \\ &\leq C \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}} \\ &\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} \|\nabla b\|_{L^2}^{\frac{2p}{2p-3}} \|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}} \\ &\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} (\|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \end{aligned} \quad (4.13)$$

We can infer from the Hölder and the Young inequalities that

$$B_4 \leq 2 \int_{\mathbb{R}^3} \nabla(\nabla \times u) \cdot \nabla \omega dx \leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\nabla \omega\|_{L^2}^2. \quad (4.14)$$

Applying integration by parts we obtain

$$\begin{aligned} B_3 &= - \int_{\mathbb{R}^3} \nabla u \cdot \nabla \omega \cdot \nabla \omega dx \\ &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla(\nabla \omega) \cdot \omega dx + \int_{\mathbb{R}^3} \nabla(\nabla u) \cdot \nabla \omega \cdot \omega dx \\ &:= B_{31} + B_{32}. \end{aligned} \quad (4.15)$$

Thanks to the Hölder inequality, the interpolation inequality with $3 \leq \alpha \leq 9$ and Lemma 3.1, we arrive at

$$\begin{aligned}
 B_{31} &\leq \int_{\mathbb{R}^3} |\nabla u| |\Delta \omega| |\omega| dx \leq \|\Delta \omega\|_{L^2} \|\omega\|_{L^\alpha} \|\nabla u\|_{L^{\frac{2\alpha}{\alpha-2}}} \\
 &\leq \|\Delta \omega\|_{L^2} \|\omega\|_{L^{\frac{9}{3}}}^{\frac{9-\alpha}{2\alpha}} \|\omega\|_{L^{\frac{9}{9}}}^{\frac{3\alpha-9}{2\alpha}} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^6}^{\frac{3}{\alpha}} \\
 &\leq \|\Delta \omega\|_{L^2} \|\omega\|_{L^6}^{\frac{3}{2}} \|\omega\|_{L^6}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\Delta u\|_{L^2}^{\frac{3}{\alpha}} \\
 &\leq \|\Delta \omega\|_{L^2} \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla u\|_{L^2}^{1-\frac{3}{\alpha}} \|\Delta u\|_{L^2}^{\frac{3}{\alpha}} \\
 &\leq \frac{1}{4} \|\Delta \omega\|_{L^2}^2 + C \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^{2(1-\frac{3}{\alpha})} \|\nabla u\|_{L^2}^{2(1-\frac{3}{\alpha})} \|\Delta u\|_{L^2}^{\frac{6}{\alpha}} \\
 &\leq \frac{1}{4} \|\Delta \omega\|_{L^2}^2 + \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^2 \|\nabla u\|_{L^2}^2.
 \end{aligned} \tag{4.16}$$

Similarly, the term B_{32} can be bounded as follows:

$$\begin{aligned}
 B_{32} &\leq \int_{\mathbb{R}^3} |\Delta u| |\nabla \omega| |\omega| dx \leq \|\Delta u\|_{L^2} \|\omega\|_{L^\alpha} \|\nabla \omega\|_{L^{\frac{2\alpha}{\alpha-2}}} \\
 &\leq \|\Delta u\|_{L^2} \|\omega\|_{L^{\frac{9}{3}}}^{\frac{9-\alpha}{2\alpha}} \|\omega\|_{L^{\frac{9}{9}}}^{\frac{3\alpha-9}{2\alpha}} \|\nabla \omega\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla \omega\|_{L^6}^{\frac{3}{\alpha}} \\
 &\leq \|\Delta u\|_{L^2} \|\omega\|_{L^6}^{\frac{3}{2}} \|\omega\|_{L^6}^{1-\frac{3}{\alpha}} \|\nabla \omega\|_{L^2}^{1-\frac{3}{\alpha}} \|\Delta \omega\|_{L^2}^{\frac{3}{\alpha}} \\
 &\leq \|\Delta u\|_{L^2} \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^{1-\frac{3}{\alpha}} \|\nabla \omega\|_{L^2}^{1-\frac{3}{\alpha}} \|\Delta \omega\|_{L^2}^{\frac{3}{\alpha}} \\
 &\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^{2(1-\frac{3}{\alpha})} \|\nabla \omega\|_{L^2}^{2(1-\frac{3}{\alpha})} \|\Delta \omega\|_{L^2}^{\frac{6}{\alpha}} \\
 &\leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + \frac{1}{4} \|\Delta \omega\|_{L^2}^2 + C \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^2 \|\nabla \omega\|_{L^2}^2.
 \end{aligned} \tag{4.17}$$

For the term B_1 , similar to A_1 , we find that

$$\begin{aligned}
 B_1 &= \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \Delta_h u dx + \int_{\mathbb{R}^3} u \cdot \nabla u \cdot \partial_3 \partial_3 u dx \\
 &= - \int_{\mathbb{R}^3} \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_i \partial_i u_j \partial_k u_j dx - \int_{\mathbb{R}^3} \sum_{k=1}^2 \sum_{i,j=1}^3 u_i \partial_i \partial_k u_j \partial_k u_j dx \\
 &\quad + \int_{\mathbb{R}^3} \sum_{i,j=1}^3 u_i \partial_i u_j \partial_3 \partial_3 u_j dx \\
 &= - \int_{\mathbb{R}^3} \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_i \partial_i u_j \partial_k u_j dx - \int_{\mathbb{R}^3} \sum_{i,j=1}^3 \partial_3 u_i \partial_i u_j \partial_3 u_j dx \\
 &\quad - \int_{\mathbb{R}^3} \sum_{i,j=1}^3 u_i \partial_i \partial_3 u_j \partial_3 u_j dx \\
 &= - \int_{\mathbb{R}^3} \sum_{k=1}^2 \sum_{i,j=1}^3 \partial_k u_i \partial_i u_j \partial_k u_j dx - \int_{\mathbb{R}^3} \sum_{j=1}^3 \sum_{i=1}^2 \partial_3 u_i \partial_i u_j \partial_3 u_j dx
 \end{aligned} \tag{4.18}$$

$$\begin{aligned}
& - \int_{\mathbb{R}^3} \sum_{j=1}^3 \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx \\
& \leq C \int_{\mathbb{R}^3} |\nabla_h u| |\nabla u|^2 dx \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^4}^2 \\
& \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^6}^{\frac{3}{2}} \\
& \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{2}} \\
& \leq C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Combining (4.13), (4.14), (4.16), (4.17) and (4.18), then (4.12) becomes

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \\
& \leq C (\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\
& \quad + C \|\nabla \omega\|_{L^2}^2 + C \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \nabla u\|_{L^2} \|\Delta u\|_{L^2}^{\frac{1}{2}}.
\end{aligned} \tag{4.19}$$

Integrating over $[0, t]$ yields

$$\begin{aligned}
& (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int_0^t \|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 d\tau \\
& \leq C \int_0^t (\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\nabla |\omega|^{\frac{3}{2}}\|_{L^2}^2) (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) d\tau \\
& \quad + C (\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + 1) + CG(t),
\end{aligned} \tag{4.20}$$

where

$$G(t) = \int_0^t \|\nabla_h u\|_{L^2} \|\nabla_h \nabla u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} d\tau.$$

We proceed to estimate $G(t)$. From (4.1) and (4.11), we deduce that

$$\begin{aligned}
G(t) & \leq \sup_{0 \leq \tau \leq t} \|\nabla_h u\|_{L^2} \left(\int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq C \left(\sup_{0 \leq \tau \leq t} \|\nabla_h u\|_{L^2}^2 + \int_0^t \|\nabla_h \nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq C \left((\|\nabla_h u_0\|_{L^2}^2 + \|\nabla_h b_0\|_{L^2}^2 + 1) + \int_0^t \|u_3\|_{L^{\frac{2\alpha}{\alpha-2}}}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} d\tau \right. \\
& \quad \left. + \int_0^t \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}} d\tau \right) \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{1}{4}} \\
& \leq C (\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + C \left(\int_0^t \|u_3\|_{L^{\frac{2\alpha}{\alpha-2}}}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_{L^2}^{\frac{2(\alpha-3)}{\alpha-2}} \|\Delta u\|_{L^2}^{\frac{2}{\alpha-2}} d\tau \right)^{\frac{4}{3}} \\
& \quad + \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2}^{1-\frac{3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}} d\tau \right)^{\frac{4}{3}} + \frac{1}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{1}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + \left(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{4(\alpha-3)}{3(\alpha-2)}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{4}{3(\alpha-2)}} \\
&\quad + C \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}^{\frac{2p}{2p-3}}}^{\frac{2p}{2p-3}} \|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}} d\tau \right)^{\frac{2(2p-3)}{3p}} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau \right)^{\frac{2}{p}} \\
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{3}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + C \left(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{4(\alpha-3)}{3\alpha-10}} \\
&\quad + C \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}^{\frac{2p}{2p-3}}}^{\frac{2p}{2p-3}} \|\nabla u\|_{L^2}^{\frac{2(p-3)}{2p-3}} d\tau \right)^{\frac{2(2p-3)}{3(p-2)}} \\
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{3}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + C \left(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{4(\alpha-3)}{3\alpha-10}} \\
&\quad + C \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}^{\frac{2p}{2p-3}}}^{\frac{2p}{2p-3}} (\|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) d\tau \right)^{\frac{2(2p-3)}{3(p-2)}} \\
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{3}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + C \left(\int_0^t \|u_3\|_{L^\alpha}^{\frac{2\alpha}{\alpha-3}} \|\nabla u\|_{L^2}^{\frac{3\alpha-10}{2(\alpha-3)}} \|\nabla u\|_{L^2}^{\frac{\alpha-2}{2(\alpha-3)}} d\tau \right)^{\frac{4(\alpha-3)}{3\alpha-10}} \\
&\quad + C \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}^{\frac{2p}{2p-3}}}^{\frac{2p}{2p-3}} (\|\nabla b\|_{L^2}^{\frac{3(p-2)}{2p-3}} + \|\nabla u\|_{L^2}^{\frac{3(p-2)}{2p-3}}) (\|\nabla b\|_{L^2}^{\frac{p}{2p-3}} + \|\nabla u\|_{L^2}^{\frac{p}{2p-3}}) d\tau \right)^{\frac{2(2p-3)}{3(p-2)}} \\
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{3}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + C \left(\int_0^t \|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} \|\nabla u\|_{L^2}^2 d\tau \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{\alpha-2}{3\alpha-10}} \\
&\quad + C \left(\int_0^t \|\nabla b\|_{\dot{M}_{p,q}^{\frac{4p}{3(p-2)}}}^{\frac{4p}{3(p-2)}} (\|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) d\tau \right) \left(\int_0^t \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 d\tau \right)^{\frac{p}{3(p-2)}} \\
&\leq C(\|\nabla_h u_0\|_{L^2}^{\frac{8}{3}} + \|\nabla_h b_0\|_{L^2}^{\frac{8}{3}} + 1) + \frac{3}{4} \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
&\quad + C \int_0^t (\|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} + \|\nabla b\|_{\dot{M}_{p,q}^{\frac{4p}{3(p-2)}}}^{\frac{4p}{3(p-2)}}) (\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) d\tau.
\end{aligned}$$

Inserting the above inequality into (4.20), we have

$$\begin{aligned}
&(\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int_0^t (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2) d\tau \\
&\leq C(\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + 1)
\end{aligned}$$

$$\begin{aligned}
& + C \int_0^t \left(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2 + \|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} + \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}} \right) \\
& \times \left(\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) d\tau,
\end{aligned}$$

Gronwall's inequality and Lemma 3.1 help to obtain

$$\begin{aligned}
& \left(\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) + \int_0^t \left(\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 \right) d\tau \\
& \leq C \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + 1 \right) \\
& \quad \times \exp \left\{ \int_0^T \left(\|\nabla b\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}} + \|\nabla|\omega|^{\frac{3}{2}}\|_{L^2}^2 + \|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} + \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}} \right) d\tau \right\} \\
& \leq C \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + 1 \right) \\
& \quad \times \exp \left\{ C \int_0^T \left(1 + \|u_3\|_{L^\alpha}^{\frac{8\alpha}{3\alpha-10}} + \|\nabla b\|_{\dot{M}_{p,q}}^{\frac{4p}{3(p-2)}} \right) d\tau \right\},
\end{aligned}$$

which completes the proof of Theorem 1.1.

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Cao, C., Wu, J.: Two regularity criteria for the 3D MHD equations. *J. Differ. Equ.* **248**, 2263–2274 (2010)
2. Gala, S.: Regularity criteria for the 3D magneto-microplar fluid equations in the Morrey–Campanato space. *Nonlinear Differ. Equ. Appl.* **17**, 181–194 (2010)
3. Gala, S., Ragusa, M.A.: A logarithmic regularity criterion for the two-dimensional MHD equations. *J. Math. Anal. Appl.* **444**, 1752–1758 (2016)
4. Gala, S., Ragusa, M.A.: On the regularity criterion of weak solutions for the 3D MHD equations. *Z. Angew. Math. Phys.* **68**, 140 (2017)
5. Gala, S., Ragusa, M.A.: A new regularity criterion for the 3D incompressible MHD equations via partial derivatives. *J. Math. Anal. Appl.* **481**, 123497 (2020)
6. Gala, S., Ragusa, M.A., Ye, Z.: An improved blow-up criterion for smooth solutions of the two-dimensional MHD equations. *Math. Methods Appl. Sci.* **40**, 279–285 (2017)
7. Galdi, G.P., Rionero, S.: A note on the existence and uniqueness of solutions of the microplar fluid equations. *Int. J. Eng. Sci.* **15**, 105–108 (1997)
8. Jia, X., Zhou, Y.: Regularity criteria for the 3D MHD equations involving partial components. *Nonlinear Anal., Real World Appl.* **13**, 410–418 (2012)
9. Jia, X., Zhou, Y.: Regularity criteria for the 3D MHD equations via partial derivatives. *Kinet. Relat. Models* **5**, 505–516 (2012)
10. Lemarié-Rieusset, P.G.: The Navier–Stokes equations in the critical Morrey–Campanato space. *Rev. Mat. Iberoam.* **23**, 897–930 (2007)
11. Lukaszewicz, G.: *Micropolar Fluids: Theory and Applications*. Birkhäuser, Berlin (1998)
12. Ortega-Torres, E.E., Rojas-Medar, M.A.: Magneto-microplar fluid motion: global existence of strong solutions. *Abstr. Appl. Anal.* **4**, 109–125 (1999)

13. Rojas-Medar, M.A.: Magneto-micropolar fluid motion: existence and uniqueness of strong solutions. *Math. Nachr.* **188**, 301–319 (1997)
14. Rojas-Medar, M.A., Boldrini, J.L.: Magneto-micropolar fluid motion: existence of weak solutions. *Internet Rev. Mat. Complut.* **11**, 443–460 (1998)
15. Yamazaki, K.: Regularity criteria of MHD system involving one velocity and one current density component. *J. Math. Fluid Mech.* **16**, 551–570 (2014)
16. Zhang, Z.: Regularity criteria for the 3D MHD equations involving one current density and the gradient of one velocity component. *Nonlinear Anal.* **115**, 41–49 (2015)
17. Zhang, Z., Yao, Z., Wang, X.: A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces. *Nonlinear Anal.* **74**, 2220–2225 (2011)
18. Zhou, Y., Fan, J.: A regularity criterion for the 2D MHD system with zero magnetic diffusivity. *J. Math. Anal. Appl.* **378**, 169–172 (2011)

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