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Well posedness of a nonlinear mixed problem for a parabolic equation with integral condition

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Abstract

The aim of this work is to prove the well posedness of some posed linear and nonlinear mixed problems with integral conditions. First, an a priori estimate is established for the associated linear problem and the density of the operator range generated by the considered problem is proved by using the functional analysis method. Subsequently, by applying an iterative process based on the obtained results for the linear problem, the existence, uniqueness of the weak solution of the nonlinear problems is established.

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1 Introduction and statement of the problem

Some problems related to physical and technical issues can be effectively described in terms of nonlocal problems with integral conditions in partial differential equations. These nonlocal conditions arise mainly when the values on the boundary cannot be measured directly, while their average values are known. The problem of parabolic equation with integral condition is stated as follows: Let us consider the rectangular domain $Q =]0, 1[\times]0, T[$, then the problem is to find a solution $\sigma(x, t)$ of the following non-classical boundary value problem:

$$\mathcal{L}\sigma = \frac{\partial \sigma}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial \sigma}{\partial x} \right) = g \left(x, t, \sigma, \frac{\partial \sigma}{\partial x} \right), \quad \text{for } (x, t) \in]0, 1[\times]0, T[, \quad (1.1)$$

with the initial condition

$$l\sigma = \sigma(x, 0) = \varphi(x), \quad \text{for } x \in [0, 1], \quad (1.2)$$

and the Dirichlet boundary condition

$$\sigma(0, t) = 0, \quad \text{for } t \in [0, T], \quad (1.3)$$

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and the nonlocal condition

$$\int_0^\alpha \sigma(x, t) dx + \int_\beta^1 \sigma(x, t) dx = 0, \quad 0 \leq \alpha \leq \beta < 1 \quad \forall t \in [0, T]. \tag{1.4}$$

In addition, we assume that the function $a(x, t)$ and its derivatives satisfy the conditions

$$\begin{cases} 0 < a_0 \leq a(x, t) \leq a_1 & \forall (x, t) \in Q, \\ c_2 \leq \frac{\partial a}{\partial t}(x, t) \leq c_1, & \forall (x, t) \in Q, \\ \left| \frac{\partial a}{\partial x}(x, t) \right| \leq b, & \forall (x, t) \in Q, \end{cases} \tag{1.5}$$

where the functions $g(x, t, \sigma, \frac{\partial \sigma}{\partial x})$, $\varphi(x)$ are given, and we assume that the following matching conditions are satisfied:

$$\begin{cases} \varphi(0) = 0, \\ \int_0^\alpha \varphi(x) dx + \int_\beta^1 \varphi(x) dx = 0. \end{cases}$$

We also assume that there exists a positive constant d such that

$$\left| g\left(x, t, \sigma_1, \frac{\partial \sigma_1}{\partial x}\right) - g\left(x, t, \sigma_2, \frac{\partial \sigma_2}{\partial x}\right) \right| \leq d \left(|\sigma_1 - \sigma_2| + \left| \frac{\partial \sigma_1}{\partial x} - \frac{\partial \sigma_2}{\partial x} \right| \right),$$

for all $(x, t) \in Q$.

This type of problem can be found in various physic problems such as heat conduction [1–4], plasma physics [5], thermoelasticity [6], electrochemistry [7], chemical diffusion [8] and underground water flow [9–11]. Several research papers such as found in [1–4, 7, 12–18] have studied and solved the parabolic equation by combining the integral condition with Dirichlet condition or Newmann condition, or with purely integral conditions, using various methods. For hyperbolic equations, the unicity and existence of the solution have been studies in [13, 19–22] and the mixed-type equations in [23–27]. The elliptic equations were considered in [28, 29] and [30].

The linear problem associated to the problem stated in (1.1)–(1.4), for $\alpha = \beta = 0$, has been studied in [18] and for $\beta = 1$ in [16]. Meanwhile in [31] the solved problem is for the case $\alpha + \beta = 1$. It is worth mentioning that in [32] the author studied the same case where $\frac{\partial}{\partial x}(a \frac{\partial \sigma}{\partial x})$ was replaced by the Bessel operator.

In the present paper the motivation is to study and find a solution to the stated problem without imposing any conditions on the constants α and β in the interval $[0, 1]$. In addition, the nonlinear problem of the parabolic equation with integral condition defined on two parts of the boundary is solved.

First, an a priori estimate is established for the associated linear problem and the density of the operator range generated by the considered problem is proved by using the functional analysis method. Subsequently, by applying an iterative process based on the obtained results for the linear problem, the existence and uniqueness of the weak solution of the nonlinear problems is established.

The rest of the paper is organized as follows. In Sect. 2, the associated linear problem is stated. Section 3 deals with the proof of the uniqueness of the solution using an a priori

estimate. Section 4 gives the solvability of the considered linear problem. Finally, in Sect. 5, on the basis of the obtained results in Sects. 3 and 4, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the nonlinear problem.

2 Statement of the associated linear problem

In this section we introduce the linear problem and the different function spaces needed to investigate the mixed nonlocal problem given by the equation.

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) = f(x, t), \tag{2.1}$$

and the conditions given by (1.2)–(1.4).

The given problem (2.1), (1.2)–(1.4) can be considered as finding a solution of the operator equation $Lu = (\mathcal{L}u, lu) = \mathcal{F} = (f, \varphi)$, where the operator L has as a domain of definition $D(L)$ consisting of functions $u \in L^2(Q)$ such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t}(x, t) \in L^2(Q)$ and satisfying the conditions (1.3) and (1.4).

The operator L is an operator acting on E into \mathbb{F} , where E is the Banach space of functions $u \in L^2(Q)$, with a finite norm

$$\|u\|_E^2 = \int_Q (1-x)^2 \left[\left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx dt + \sup_t \int_0^1 \left[(1-x)^2 \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] dx. \tag{2.2}$$

\mathbb{F} is the Hilbert space of functions $F = (f, \varphi), f \in L^2(Q), \varphi \in H^1(0, 1)$ with the finite norm

$$\|F\|_{\mathbb{F}}^2 = \int_Q (1-x)^2 |f(x, t)|^2 dx dt + \int_0^1 \left[(1-x)^2 \left| \frac{d\varphi}{dx} \right|^2 + |\varphi|^2 \right] dx. \tag{2.3}$$

Then we show that the operator L has a closure \bar{L} and later on, in Sect. 3, we establish an energy inequality of the following type (see Theorem 3.1):

$$\|u\|_E \leq k \|Lu\|_{\mathbb{F}} \quad \forall u \in D(L). \tag{2.4}$$

Definition 2.1 A solution of the operator equation $\bar{L}u = F = (f, \varphi)$ is called a strong solution of problem (2.1)–(1.4).

Since the points of the graph of the operator \bar{L} are limits of sequences of points of the graph of L , we can extend the a priori estimate (2.4) to be applied to strong solutions by taking the limits, that is, we have the inequality

$$\|u\|_E \leq c \|\bar{L}u\|_{\mathbb{F}}, \quad \forall u \in D(\bar{L}). \tag{2.5}$$

From this inequality, we deduce the uniqueness of a strong solution, if it exists, and that the range of the operator \bar{L} coincides with the closure of the range of L .

Proposition 2.1 *The operator $L : E \rightarrow F$ admits a closure \bar{L} .*

Proof Let $u_n \in D(L)$ be a sequence such that

$$\lim_{n \rightarrow \infty} u_n = 0 \quad \text{in } E \tag{2.6}$$

and

$$\lim_{n \rightarrow \infty} Lu_n = F = (f, \varphi) \quad \text{in the space } F, \tag{2.7}$$

we then must show that $f = 0, \varphi = 0$.

Since (2.6) holds, we have

$$\lim_{n \rightarrow \infty} u_n = 0 \quad \text{in } D'(Q), \tag{2.8}$$

where $D'(Q)$ is the space of distribution on Q . By virtue of the continuity of derivation of $D'(Q)$ in $D'(Q)$, (2.8) implies that

$$\lim_{n \rightarrow \infty} \mathcal{E}u_n = 0 \quad \text{in } D'(Q),$$

According to (2.7), we have

$$\lim_{n \rightarrow \infty} \mathcal{E}u_n = f \quad \text{in } L^2_\rho(Q), \tag{2.9}$$

where $L^2_\rho(Q)$ is a Banach space with norm $\|u\|^2_{L^2_\rho(Q)} = \int_Q \frac{(1-x)^2}{2} |u|^2 dx dt$. Then

$$\lim_{n \rightarrow \infty} \mathcal{E}u_n = f \quad \text{in } D'(Q).$$

By virtue of the uniqueness of the limit in $D'(Q)$, we conclude that $f = 0$.

According to (2.7), we also conclude that

$$\lim_{n \rightarrow \infty} lu_n = \varphi \quad \text{in } H^1_\rho(0, 1),$$

where $H^1_\rho(0, 1)$ is a Banach space with norm $\|u\|^2_{H^1_\rho(0,1)} = \int_0^1 (\frac{(1-x)^2}{2} |\frac{\partial u}{\partial x}|^2 + |u|^2) dx$. By the fact that the canonical injection from $H^1_\rho(0, 1)$ into $D'(0, 1)$ is continuous, we deduce that

$$\lim_{n \rightarrow \infty} lu_n = \varphi \quad \text{in } D'(0, 1), \tag{2.10}$$

Moreover, since (2.6) holds and

$$\|lu_n\|_{H^1_\rho(0,1)} \leq \|u_n\|_E$$

we have

$$\lim_{n \rightarrow \infty} lu_n = 0 \quad \text{in } H^1_\rho(0, 1),$$

Hence

$$\lim_{n \rightarrow \infty} lu_n = 0 \quad \text{in } D'(0, 1), \tag{2.11}$$

By virtue of the uniqueness of the limit in $D'(0, 1)$, we conclude, from (2.10) and (2.11), that $\varphi = 0$. This proves Proposition 2.1 □

The following a priori estimate gives the uniqueness of the solution of the posed linear problem.

3 An energy inequality and its application

In this section, the uniqueness of the solution will be proved using an energy inequality method.

Theorem 3.1 *There exists a positive constant K , such that for each function $u \in D(L)$ we have*

$$\|u\|_E \leq K \|Lu\|_{\mathbb{F}}. \tag{3.1}$$

Proof Let

$$Mu = \lambda \frac{(1-x)^2}{2} \frac{\partial u}{\partial t} + \lambda(1-x) \int_0^x \frac{\partial u}{\partial t} d\zeta - xe^{\delta(x-1)} \int_x^1 g(\zeta, t) d\zeta,$$

where

$$g(\zeta, t) = k \int_0^\zeta \frac{\partial u}{\partial t} d\mu - (k-\lambda) \int_\alpha^\zeta \frac{\partial u}{\partial t} d\mu + (k-\lambda) \int_\beta^\zeta \frac{\partial u}{\partial t} d\mu,$$

and λ, k and δ are a positives scalar parameters such that

$$3 < \frac{\lambda}{k} < 4e^{-\delta} \quad \text{with } \delta > \ln\left(\frac{4}{3}\right). \tag{3.2}$$

Taking the scalar product in $L^2(Q^s)$, where $Q^s = [0, 1] \times [0, s]$ of Eq. (2.1) and the operator $e^{-ct}\overline{Mu}$, with $0 \leq s \leq T, c > 0$, we have

$$\begin{aligned} \Phi(u, u) &= \operatorname{Re} \int_{Q^s} e^{-ct} f(x, t) \overline{Mu} dx dt \\ &= \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial u}{\partial t} \overline{Mu} dx dt - \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u}{\partial x} \right) \overline{Mu} dx dt. \end{aligned} \tag{3.3}$$

Substituting Mu by its expression in the first term in the right-hand side of (3.3), we obtain

$$\begin{aligned} \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial u}{\partial t} \overline{Mu} dx dt &= \lambda \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\ &\quad - \operatorname{Re} \int_{Q^s} xe^{\delta(x-1)} e^{-ct} \frac{\partial u}{\partial t} \int_x^1 \overline{g(\zeta, t)} d\zeta dx dt \\ &\quad + \operatorname{Re} \int_{Q^s} \lambda(1-x) e^{-ct} \frac{\partial u}{\partial t} \int_0^x \overline{\frac{\partial u}{\partial t}} d\zeta dx dt. \end{aligned} \tag{3.4}$$

Integrating by parts the second term in the right-hand side of the last equality of (3.4) with respect to x , using the fact that $\frac{\partial u}{\partial t} = \frac{1}{k} xe^{\delta(x-1)} \frac{\partial g}{\partial x}$, then

$$\operatorname{Re} \int_0^1 xe^{\delta(x-1)} \frac{\partial u}{\partial t} \int_x^1 \overline{g(\zeta, t)} d\zeta dx = \frac{1}{k} \operatorname{Re} \int_0^1 xe^{\delta(x-1)} \frac{\partial g}{\partial x} \int_x^1 \overline{g(\zeta, t)} d\zeta dx$$

integrating by parts with respect to x , we obtain

$$\begin{aligned} & \operatorname{Re} \int_0^1 x e^{\delta(x-1)} \frac{\partial u}{\partial t} \int_x^1 \overline{g(\zeta, t)} d\zeta dx \\ &= \frac{1}{k} \operatorname{Re} \int_0^1 x e^{\delta(x-1)} \frac{\partial g}{\partial x} \int_x^1 \overline{g(\zeta, t)} d\zeta dx \\ &= \frac{1}{k} x e^{\delta(x-1)} g \int_x^1 \overline{g(\zeta, t)} d\zeta dx \Big|_{x=0}^{x=1} + \frac{1}{k} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} |g(x, t)|^2 dx \\ &\quad - \frac{1}{k} \operatorname{Re} \int_0^1 (1 + \delta x) e^{\delta(x-1)} g \int_x^1 \overline{g(\zeta, t)} d\zeta dx, \end{aligned}$$

using this equality

$$\begin{aligned} \frac{d}{dx} |h(x)|^2 &= \frac{d}{dx} h(x) \overline{h(x)} = \frac{dh(x)}{dx} \overline{h(x)} + h(x) \frac{d\overline{h(x)}}{dx} \\ &= \overline{\frac{dh(x)}{dx} h(x)} + h(x) \frac{d\overline{h(x)}}{dx} = 2 \operatorname{Re} \left(h(x) \frac{d\overline{h(x)}}{dx} \right). \end{aligned} \tag{3.5}$$

The last term in the previous equality becomes

$$\begin{aligned} & - \frac{1}{k} \operatorname{Re} \int_0^1 (1 + \delta x) e^{\delta(x-1)} g \int_x^1 \overline{g(\zeta, t)} d\zeta dx \\ &= \frac{1}{2k} \int_0^1 (1 + \delta x) e^{\delta(x-1)} \frac{d}{dx} \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 dx \\ &\quad + \frac{1}{2k} ((1 + \delta x) e^{\delta(x-1)}) \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 \Big|_{x=0}^{x=1} \\ &\quad - \frac{1}{2k} \int_0^1 (2\delta + \delta^2 x) e^{\delta(x-1)} \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 dx \\ &= - \frac{e^{-\delta}}{2k} \int_0^s e^{-ct} \left| \int_0^1 g(\zeta, t) d\zeta \right|^2 dt - \frac{1}{2k} \int_0^1 (2\delta + \delta^2 x) e^{\delta(x-1)} \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 dx. \end{aligned}$$

Then

$$\begin{aligned} & \operatorname{Re} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} \frac{\partial u}{\partial t} \int_x^1 \overline{g(\zeta, t)} d\zeta dx dt \\ &= \frac{1}{k} \operatorname{Re} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} \frac{\partial g}{\partial x} \int_x^1 \overline{g(\zeta, t)} d\zeta dx dt \\ &= \frac{1}{k} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} |g(x, t)|^2 dx dt - \frac{1}{2k} \int_{Q^s} (2\delta + \delta^2 x) e^{\delta(x-1)} e^{-ct} \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 dx dt \\ &\quad - \frac{e^{-\delta}}{2k} \int_0^s e^{-ct} \left| \int_0^1 g(\zeta, t) d\zeta \right|^2 dt. \end{aligned} \tag{3.6}$$

Similarly integrating by parts the last term of (3.4) with respect to x , we obtain

$$\begin{aligned}
 & \operatorname{Re} \int_{Q^s} \lambda(1-x)e^{-ct} \frac{\partial u}{\partial t} \overline{\int_0^x \frac{\partial u}{\partial t} d\zeta} dx dt \\
 &= \frac{\lambda}{2k^2} \int_{Q^s} e^{-ct} |g(x,t)|^2 dx dt \\
 & \quad + \frac{\lambda}{2k^2} \int_0^s e^{-ct} |g(0,t)|^2 dt - \frac{\lambda}{k^2} \operatorname{Re} \int_{Q^s} e^{-ct} g(0,t) \overline{g(x,t)} dx dt,
 \end{aligned} \tag{3.7}$$

From (3.7) and (3.6), equality (3.4) becomes

$$\begin{aligned}
 & \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial u}{\partial t} \overline{Mu} dx dt \\
 &= \lambda \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \frac{\lambda}{2k^2} \int_{Q^s} e^{-ct} |g(x,t)|^2 dx dt \\
 & \quad + \frac{\lambda}{2k^2} \int_{Q^s} e^{-ct} |g(0,t)|^2 dx dt + \frac{e^{-\delta}}{2k} \int_0^s e^{-ct} \left| \int_0^1 g(\zeta,t) d\zeta \right|^2 dt \\
 & \quad - \frac{1}{k} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} |g(x,t)|^2 dx dt \\
 & \quad + \frac{1}{2k} \int_{Q^s} (2\delta + \delta^2 x) e^{\delta(x-1)} e^{-ct} \left| \int_x^1 g(\zeta,t) d\zeta \right|^2 dx dt \\
 & \quad - \frac{\lambda}{k^2} \operatorname{Re} \int_{Q^s} e^{-ct} g(0,t) \overline{g(x,t)} dx dt.
 \end{aligned} \tag{3.8}$$

Similarly, substituting Mu by its expression in the last term in the right-hand side of (3.3), integrating by parts with respect to x , using the Dirichlet condition (1.3) and the integral condition (1.4) we get

$$\begin{aligned}
 & - \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial u}{\partial x} \right) \overline{Mu} dx dt \\
 &= \lambda \operatorname{Re} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} a(x,t) \frac{\partial u}{\partial x} \overline{\frac{\partial^2 u}{\partial x \partial t}} dx dt \\
 & \quad + \operatorname{Re} \int_{Q^s} (\lambda - kxe^{\delta(x-1)}) a(x,t) e^{-ct} u \overline{\frac{\partial u}{\partial t}} dx dt \\
 & \quad + \lambda \operatorname{Re} \int_{Q^s} \frac{\partial a(x,t)}{\partial x} e^{-ct} u \overline{\int_0^x \frac{\partial u}{\partial t} d\zeta} dx dt \\
 & \quad - \operatorname{Re} \int_{Q^s} \left(x \frac{\partial a}{\partial x} + 2a(1 + \delta x) \right) e^{\delta(x-1)} e^{-ct} u \overline{g(x,t)} dx dt \\
 & \quad + \operatorname{Re} \int_{Q^s} \left((1 + \delta x) \frac{\partial a}{\partial x} + (2\delta + \delta^2 x) a \right) e^{\delta(x-1)} e^{-ct} u \overline{\int_x^1 g(\zeta,t) d\zeta} dx dt.
 \end{aligned} \tag{3.9}$$

Integrating by parts the first two terms with respect to t in (3.9), using the condition (1.2) we have

$$\begin{aligned} & \lambda \operatorname{Re} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} a \frac{\partial u}{\partial x} \overline{\frac{\partial^2 u}{\partial x \partial t}} dx dt \\ &= \frac{\lambda}{2} \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \quad + \frac{\lambda}{2} \int_0^1 \frac{(1-x)^2}{2} e^{-cs} a(x,s) \left| \frac{\partial u}{\partial x} \right|^2 \Big|_{t=s} dx - \frac{\lambda}{2} \int_0^1 \frac{(1-x)^2}{2} a(x,0) \left| \frac{d\varphi}{dx} \right|^2 dx, \\ & \operatorname{Re} \int_{Q^s} (\lambda - kxe^{\delta(x-1)}) e^{-ct} a(x,t) u \overline{\frac{\partial u}{\partial t}} dx dt \\ &= \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) (\lambda - kxe^{\delta(x-1)}) e^{-ct} |u|^2 dx dt \\ & \quad + \int_0^1 (\lambda - kxe^{\delta(x-1)}) e^{-cs} a(x,s) |u|^2 \Big|_{t=s} dx - \int_0^1 (\lambda - kxe^{\delta(x-1)}) a(x,0) |\varphi|^2 dx, \end{aligned}$$

then from the above equalities and equalities (3.8) and (3.9), (3.3) becomes

$$\begin{aligned} & \frac{\lambda}{2} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \frac{\lambda}{2} \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \quad + \frac{1}{2k} \int_{Q^s} \left(\frac{\lambda}{k} - 2xe^{\delta(x-1)} \right) e^{-ct} |g(x,t)|^2 dx dt + \frac{\lambda}{2k^2} \int_{Q^s} e^{-ct} |g(0,t)|^2 dx dt \\ & \quad + \frac{1}{2k} e^{-\delta} \int_0^s e^{-ct} \left| \int_0^1 g(x,t) \right|^2 dx dt + \frac{1}{2} \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) (\lambda - kxe^{\delta(x-1)}) e^{-ct} |u|^2 dx dt \\ & \quad + \frac{1}{2k} \int_{Q^s} (2\delta + \delta^2 x) e^{\delta(x-1)} e^{-ct} \left| \int_x^1 g(\zeta,t) d\zeta \right|^2 dx dt \\ & \quad + \frac{\lambda}{2} \int_0^1 \frac{(1-x)^2}{2} e^{-cs} a(x,s) \left| \frac{\partial u}{\partial x} \right|^2 \Big|_{t=s} dx \\ & \quad + \frac{1}{2} \int_0^1 (\lambda - kxe^{\delta(x-1)}) e^{-cs} a(x,s) |u|^2 \Big|_{t=s} dx \\ & \quad + \lambda \operatorname{Re} \int_{Q^s} \frac{\partial a(x,t)}{\partial x} e^{-ct} u \int_0^x \overline{\frac{\partial u}{\partial t}} d\zeta dx dt \\ & \quad - \operatorname{Re} \int_{Q^s} \left(x \frac{\partial a}{\partial x} + 2a(1 + \delta x) \right) e^{\delta(x-1)} e^{-ct} \overline{ug(x,t)} dx dt \\ & \quad - \frac{\lambda}{2k^2} \operatorname{Re} \int_0^s e^{-ct} g(0,t) \int_0^1 \overline{g(x,t)} dx dt \\ & \quad + \operatorname{Re} \int_{Q^s} \left((1 + \delta x) \frac{\partial a}{\partial x} + (2\delta + \delta^2 x) a \right) e^{\delta(x-1)} e^{-ct} u \int_x^1 g(\zeta,t) d\zeta dx dt \\ &= \operatorname{Re} \int_{Q^s} e^{-ct} f \overline{Mu} dx dt + \frac{\lambda}{2} \int_0^1 \frac{(1-x)^2}{2} a(x,0) \left| \frac{d\varphi}{dx} \right|^2 dx \\ & \quad + \frac{1}{2} \int_0^1 (\lambda - kxe^{\delta(x-1)}) a(x,0) |\varphi|^2 dx. \tag{3.10} \end{aligned}$$

Using the Young inequality in the last four terms in the left-hand side of (3.10), and using the facts that

$$\int_{Q^s} e^{-ct} |u|^2 dx dt \leq 8 \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt$$

and

$$\int_{Q^s} e^{-ct} \left| \int_0^x \frac{\partial u}{\partial t} d\zeta \right|^2 dx dt \leq 8 \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt,$$

we get

$$\begin{aligned} & -\lambda \operatorname{Re} \int_{Q^s} e^{-ct} \frac{\partial a(x,t)}{\partial x} u \int_0^x \overline{\frac{\partial u}{\partial t} d\zeta} dx dt \\ & \leq 8\lambda \epsilon_1 b^2 \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt + \frac{2\lambda}{\epsilon_1} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt, \\ & \operatorname{Re} \int_{Q^s} \left(x \frac{\partial a}{\partial x} + 2a(1+\delta x) \right) e^{-ct} \overline{u g(x,t)} dx dt \\ & \leq \frac{8(2a_1(1+\delta) + b)^2}{2\epsilon_2} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt + \frac{\epsilon_2}{2} \int_{Q^s} e^{-ct} |g(x,t)|^2 dx dt, \\ & \operatorname{Re} \int_{Q^s} \left((1+\delta x) \frac{\partial a}{\partial x} + (2\delta + \delta^2 x) a \right) e^{\delta(x-1)} e^{-ct} u \int_x^1 \overline{g(\zeta,t)} d\zeta dx dt \\ & \leq \frac{8((2\delta + \delta^2)a_1 + (1+\delta)b)^2}{2\epsilon_3} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \quad + \frac{\epsilon_3}{2} \int_{Q^s} e^{\delta(x-1)} e^{-ct} \left| \int_x^1 \overline{g(\zeta,t)} d\zeta \right|^2 dx dt, \\ & \frac{\lambda}{2k^2} \operatorname{Re} \int_0^s g(0,t) \int_0^1 \overline{g(x,t)} dx dt \leq \frac{\lambda \epsilon_4}{4k^2} \int_0^s |g(0,t)|^2 dt + \frac{\lambda}{4\epsilon_4 k^2} \int_0^s \left| \int_0^1 g(x,t) dx \right|^2 dt. \end{aligned}$$

We choose $\epsilon_1 = 8$, $\epsilon_2 = 2$, $\epsilon_3 = \frac{\delta}{k}$, and $\epsilon_4 = 2$ and $c > 0$ such that

$$c > \frac{256b^2}{a_0} + 8 \frac{k}{\lambda a_0} (2a_1(1+\delta) + b)^2 + 8 \frac{k}{\lambda \delta a_0} [(2\delta + \delta^2)a_1 + (1+\delta)b]^2 + c_2, \tag{3.11}$$

therefore by combining the previous inequalities with (3.10), we get the following expression:

$$\begin{aligned} & \frac{\lambda}{4} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + M \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \quad + \frac{\lambda - k}{2} (ca_0 - c_2) \int_{Q^s} e^{-ct} |u|^2 dx dt + \left(\frac{\lambda}{2k^2} - \frac{3}{2k} \right) \int_{Q^s} e^{-ct} |g|^2 dx dt \\ & \quad + \frac{\lambda a_0}{2} \int_0^1 \frac{(1-x)^2}{2} e^{-cs} \left| \frac{\partial u}{\partial x} \right|_{t=s} dx + \lambda a_0 \int_0^1 e^{-cs} |u|^2_{t=s} dx \\ & \leq \operatorname{Re} \int_{Q^s} e^{-ct} f \overline{Mu} dx dt + \frac{\lambda a_1}{2} \int_0^1 \frac{(1-x)^2}{2} \left| \frac{d\varphi}{dx} \right|^2 dx + a_1 \int_0^1 |\varphi|^2 dx, \end{aligned} \tag{3.12}$$

where

$$M = \frac{\lambda}{2}(ca_0 - c_2) - 64\lambda b^2 - 4k(2a_1(1 + \delta) + b)^2 - 4\frac{k}{\delta}[(2\delta + \delta^2)a_1 + (1 + \delta)b]^2.$$

Substituting Mu by its expression in the first term in the right-hand side of (3.12), we obtain

$$\begin{aligned} & \operatorname{Re} \int_{Q^s} e^{-ct} f \overline{Mu} \, dx \, dt \\ &= \lambda \operatorname{Re} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} f \frac{\partial \overline{u}}{\partial t} \, dx \, dt \\ & \quad + \lambda \operatorname{Re} \int_{Q^s} (1-x) e^{-ct} f \int_0^x \frac{\partial \overline{u}}{\partial t} \, dx \, dt + \operatorname{Re} \int_{Q^s} e^{-ct} x f \int_x^1 \overline{g(x, \zeta)} \, d\zeta \, dx \, dt, \end{aligned} \tag{3.13}$$

each term in the right-hand side of (3.13), can be, respectively, controlled by

$$\begin{aligned} & \lambda \operatorname{Re} \int_{Q^s} e^{-ct} \frac{(1-x)^2}{2} f \frac{\partial \overline{u}}{\partial t} \, dx \, dt \\ & \leq 2\lambda \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} |f|^2 \, dx \, dt + \frac{\lambda}{8} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt, \\ & \lambda \operatorname{Re} \int_{Q^s} (1-x) e^{-ct} f \int_0^x \frac{\partial \overline{u}}{\partial t} \, dx \, dt \\ & \leq \frac{\lambda}{16} \int_{Q^s} e^{-ct} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt + 64\lambda \int_{Q^s} e^{-ct} \frac{(1-x)^2}{2} |f|^2 \, dx \, dt, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} & \operatorname{Re} \int_{Q^s} x e^{\delta(x-1)} e^{-ct} f \int_x^1 \overline{g(x, \zeta)} \, d\zeta \, dx \, dt \\ & \leq \frac{\lambda}{32} \int_{Q^s} \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt \left(\frac{\lambda^2 - 3k}{4k^2} \right) \int_{Q^s} \exp(-ct) |g(x, t)|^2 \, dx \, dt \\ & \quad + \left(\frac{2k^2}{\lambda^2 - 3k} + \frac{k^2}{128\lambda} \right) \int_{Q^s} \exp(-ct) \frac{(1-x)^2}{2} |f|^2 \, dx \, dt. \end{aligned}$$

The combination of the previous inequalities with (3.12) yields

$$\begin{aligned} & \int_{Q^s} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial t} \right|^2 \, dx \, dt + \int_{\Omega^s} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt \\ & \quad + \int_0^1 \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 \Big|_{t=s} \, dx + \int_0^1 |u|^2 \Big|_{t=s} \, dx \\ & \leq \sigma \left(\int_Q \frac{(1-x)^2}{2} |f|^2 \, dx \, dt + \int_0^1 \left(\frac{(1-x)^2}{2} \left| \frac{\partial \varphi}{\partial x} \right|^2 + |\varphi|^2 \right) \, dx \right), \end{aligned} \tag{3.15}$$

where

$$\sigma = \frac{\max\left(\frac{2k^2}{\lambda^2 - 3k} + \frac{k^2}{128\lambda} + 66\lambda, \frac{\lambda a_1}{2}\right)}{\min\left(M, \frac{\lambda - k}{2} e^{-\delta}(ca_0 - c_2), \frac{\lambda}{32}, \frac{\lambda - k}{2} e^{-\delta} a_0, \frac{\lambda a_0}{2}\right)} \exp(cT).$$

From Eqs. (1.1) and (3.15), we deduce that

$$\begin{aligned} & \int_{Q^s} \frac{(1-x)^2}{2} \left[\left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] dx dt + \int_{\Omega^s} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \quad + \int_0^1 \left(\frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) dx \Big|_{t=s} \\ & \leq K^2 \left[\int_Q \frac{(1-x)^2}{2} |f|^2 dx dt + \int_0^1 \left(\frac{(1-x)^2}{2} \left| \frac{\partial \varphi}{\partial x} \right|^2 + |\varphi|^2 \right) dx \right]. \end{aligned} \tag{3.16}$$

If we drop the second term in the last inequality and by taking the least upper bound of the left side with respect to s from 0 to T , we get the desired estimate (3.1) with $K^2 = \sigma + \frac{4+2\sigma+4b^2\sigma}{a_0^2}$.

Then the uniqueness of the strong solution results from the desired estimate (3.1) and (2.5) holds. □

This last inequality implies the following corollaries.

Corollary 3.1 *If a strong solution of (2.1)–(1.4) exists, it is unique and continuously depends on $F = (f, \varphi)$.*

Proof First, If u_1 and u_2 are two solutions of (2.1)–(1.4), then $u = u_1 - u_2$ is a solution of the problem

$$\begin{cases} \mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) = 0 & \text{for } (x, t) \in]0, 1[\times]0, T[, \\ lu = u(x, 0) = 0, & \text{for } x \in [0, 1], \\ u(0, t) = 0, & \text{for } t \in [0, T], \\ \int_0^\alpha \sigma(x, t) dx + \int_\beta^1 \sigma(x, t) dx = 0, & 0 \leq \alpha \leq \beta < 1, \forall t \in [0, T], \end{cases}$$

then from Theorem 3.1, we deduce that $\|u\|_{\mathbb{E}} \leq 0$, which implies that $u_1 = u_2$. □

Corollary 3.2 *The range $R(\bar{L})$ of \bar{L} is closed in \mathbb{F} and $\overline{R(\bar{L})} = R(\bar{L})$.*

Proof First, we prove that $R(\bar{L})$ is closed. Let $T \in \overline{R(\bar{L})}$, then there exists a sequence $U_n \in D(\bar{L})$ such that $\bar{L}U_n \xrightarrow[n \rightarrow \infty]{} T$, in \mathbb{F} . since $\|U\|_{\mathbb{E}} \leq c\|\bar{L}U\|_{\mathbb{F}}, \forall U \in D(\bar{L})$. Then $\|U_n\|_{\mathbb{E}} \leq c\|\bar{L}U_n\|_{\mathbb{F}}, \forall U_n \in D(\bar{L})$, we deduce that the convergence of $\bar{L}U_n$ in \mathbb{F} implies the convergence of U_n in \mathbb{E} , say $U_n \xrightarrow[n \rightarrow \infty]{} U$, in \mathbb{E} . Since \bar{L} is closed, (U_n) is a sequence in $D(\bar{L})$ and $U_n \xrightarrow[n \rightarrow \infty]{} U$, in \mathbb{E} , and $\bar{L}U_n \xrightarrow[n \rightarrow \infty]{} T$, in \mathbb{F} , we have $U \in D(\bar{L})$ and $\bar{L}U = T$, that is, $T \in R(\bar{L})$. Hence, $R(\bar{L})$ is closed in \mathbb{F} .

Now, to prove that $R(\bar{L}) = \overline{R(\bar{L})}$, we observe that \bar{L} is an extension of L ; therefore, $\Gamma(L) \subset \Gamma(\bar{L})$, where $\Gamma(L)$ is the graph of L , hence $R(L) \subseteq R(\bar{L})$, which implies $\overline{R(\bar{L})} \subseteq \overline{R(L)} = R(\bar{L})$. On the other hand, let $T \in R(\bar{L})$, that is, $\bar{L}U = T$ for some $U \in D(\bar{L})$, which means that $(U, \bar{L}U) \in \Gamma(\bar{L}) = \overline{\Gamma(L)}$, therefore, there exists a sequence $(U_n, LU_n)_{n \in \mathbb{N}}$ in $\Gamma(L)$ such that $(U_n, LU_n) \xrightarrow[n \rightarrow \infty]{} (U, T)$ in $\mathbb{E} \times \mathbb{F}$, which implies that $LU_n \xrightarrow[n \rightarrow \infty]{} T$ but $U_n \in D(L), \forall n \in \mathbb{N}$, then we have $T \in \overline{R(L)}$ and hence $R(\bar{L}) \subset \overline{R(L)}$ □

Corollary 3.2 shows that, to prove that problem (1.1)–(1.4) has a strong solution for arbitrary F , it suffices to prove that the set $R(L)$ is dense in \mathbb{F} .

4 Solvability of the linear problem

In order to prove the solvability of problem (2.1)–(1.4) it is sufficient to show that $R(L)$ is dense in \mathbb{F} . The proof is based on the following lemma.

Lemma 4.1 *Suppose that the function a and its derivatives are bounded.*

Let $u \in D_0(L) = \{u \in D(L), u(x, 0) = 0\}$. If, for $u \in D_0(L)$ and some functions $w \in L^2(Q)$, we have

$$\int_Q \theta(x) f \bar{w} \, dx \, dt = 0, \tag{4.1}$$

where

$$\theta(x) = \begin{cases} \frac{x^2}{\alpha^2}, & x \in (0, \alpha), \\ w, & x \in (\alpha, \beta), \\ \frac{(1-x)^2}{(1-\beta)^2}, & x \in (\beta, 1), \end{cases}$$

then w vanishes almost everywhere in Ω .

Proof Equality (4.1), can be written as follows:

$$\int_Q \frac{\partial u}{\partial t} \bar{\rho} \, dx \, dt = \int_Q A(t) u \bar{\rho} \, dx \, dt, \tag{4.2}$$

where

$$\rho = \theta(x)w$$

and

$$A(t)u = \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u}{\partial x} \right).$$

We introduce the smoothing operators $J_\epsilon^{-1} = (I - \epsilon \frac{\partial}{\partial t})^{-1}$ and $(J_\epsilon^{-1})^* = (I + \epsilon \frac{\partial}{\partial t})^{-1}$ from $L^2(0, T)$ into the space $H^1(0, T)$ with respect to t , then these operators provide the solution of the problems:

$$\begin{cases} u_\epsilon(t) - \epsilon \frac{\partial u_\epsilon}{\partial t} = u(t), & u_\epsilon(0) = 0, \\ v_\epsilon^*(t) + \epsilon \frac{\partial v_\epsilon^*}{\partial t} = v(t), & v_\epsilon^*(T) = 0. \end{cases}$$

We also have the following properties: If $g \in D(L)$, then $J_\epsilon^{-1}g \in D(L)$ and we have

$$\begin{cases} \lim \|J_\epsilon^{-1}g - g\|_{L^2(0,T)} = 0, & \text{for } \epsilon \rightarrow 0, \\ \lim \|(J_\epsilon^{-1})^*g - g\|_{L^2(0,T)} = 0, & \text{for } \epsilon \rightarrow 0. \end{cases}$$

Substituting the function u in (4.2) by the smoothing function u_ϵ and using the relation

$$A(t)u_\epsilon = J_\epsilon^{-1}A(t)u - \epsilon J_\epsilon^{-1}B_\epsilon(t)u_\epsilon,$$

where

$$B_\epsilon(t)u_\epsilon = \frac{\partial A(t)}{\partial t}u_\epsilon = \frac{\partial}{\partial x} \left(\frac{\partial a}{\partial t} \frac{\partial u_\epsilon}{\partial x} \right),$$

we obtain

$$- \int_\Omega u \frac{\partial \rho_\epsilon^*}{\partial t} dx dt = \int_\Omega (A(t)u - \epsilon B_\epsilon(t)u_\epsilon) \overline{\rho_\epsilon^*} dx dt. \tag{4.3}$$

Since the operator $A(t)$ has a continuous inverse in $L^2(0, 1)$ defined by

$$A^{-1}(t)g = \int_0^x \frac{d\zeta}{a} \int_0^\zeta g(\eta) d\eta + C_1(t) \int_0^x \frac{d\zeta}{a},$$

where the functions $C_1(t)$ satisfy the following expression:

$$C_1(t) = \frac{\int_0^1 \frac{K(x)}{a} \int_0^x g(\eta) d\eta dx}{\int_0^1 \frac{K(x)}{a} dx},$$

the function $K(x)$ is given by

$$K(x) = \begin{cases} x - \alpha, & (0, \alpha), \\ 0, & (\alpha, \beta), \\ x - 1, & (\beta, 1). \end{cases}$$

Then we have $\int_0^\alpha A^{-1}(t)u dx + \int_\beta^1 A^{-1}(t)u dx = 0$, hence, the function $J_\epsilon^{-1}u = u_\epsilon$ can be represented in the form

$$u_\epsilon = J_\epsilon^{-1}A^{-1}(t)A(t)u,$$

then

$$B_\epsilon(t)g = \frac{\partial^2 a}{\partial t \partial x} J_\epsilon^{-1} \frac{C_1(t) + \int_0^\zeta g(\eta) d\eta}{a} + \frac{\partial a}{\partial t} J_\epsilon^{-1} \frac{g}{a} - \frac{\partial a}{\partial t} J_\epsilon^{-1} \frac{\frac{\partial a}{\partial x} C_1(t) + \int_0^\zeta g(\eta) d\eta}{a}.$$

Consequently, equality (4.3) becomes

$$- \int_\Omega u \frac{\partial \rho_\epsilon^*}{\partial t} dx dt = \int_\Omega A(t)u \overline{h_\epsilon} dx dt, \tag{4.4}$$

where

$$h_\epsilon = \rho_\epsilon^* - \epsilon B_\epsilon^*(t)\rho_\epsilon^*$$

and

$$\begin{aligned}
 B_\epsilon^*(t)\rho_\epsilon^* &= \frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial a}{\partial t} \rho_\epsilon^* - \int_0^x \left(\frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial^2 a}{\partial t \partial \zeta} \rho_\epsilon^* - \frac{1}{a^2} \frac{\partial a}{\partial \zeta} (J_\epsilon^{-1})^* \frac{\partial a}{\partial t} \rho_\epsilon^* \right) d\zeta \\
 &\quad + \frac{\int_0^1 \frac{K}{a} dx + \int_x^1 \frac{K}{a} d\zeta}{\int_0^1 \frac{K}{a} dx} \int_0^1 \left(\frac{1}{a}(J_\epsilon^{-1})^* \frac{\partial^2 a}{\partial t \partial \zeta} \rho_\epsilon^* - \frac{1}{a^2} \frac{\partial a}{\partial \zeta} (J_\epsilon^{-1})^* \frac{\partial a}{\partial t} \rho_\epsilon^* \right) dx.
 \end{aligned}$$

The left-hand side of (4.4) is a continuous linear functional of u , hence the function h_ϵ has the derivatives $\frac{\partial h_\epsilon}{\partial x}, \frac{\partial^2 h_\epsilon}{\partial x^2} \in L^2(Q)$ and the following conditions are satisfied:

$$h_\epsilon(0, t) = h_\epsilon(\alpha, t) = h_\epsilon(\beta, t) = h_\epsilon(1, t) = 0, \quad \frac{\partial h_\epsilon}{\partial x}(1, t) = 0.$$

For a sufficiently small ϵ and the operator $\frac{(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}}{a}$ is bounded in $L^2(Q)$, we have $\|\epsilon \frac{(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}}{a}\|_{L^2(\Omega)} < 1$, hence, the operator $I - \epsilon \frac{(J_\epsilon^{-1})^* \frac{\partial a}{\partial t}}{a}$ has a bounded inverse in $L^2(Q)$, we deduce that $\frac{\partial \rho_\epsilon^*}{\partial x}, \frac{\partial^2 \rho_\epsilon^*}{\partial x^2} \in L^2(Q)$ and the following conditions are satisfied:

$$\begin{cases} \rho_\epsilon^*(0, t) = \rho_\epsilon^*(\alpha, t) = \rho_\epsilon^*(\beta, t) = \rho_\epsilon^*(1, t) = 0, \\ \frac{\partial \rho_\epsilon^*}{\partial x}(\alpha, t) = \frac{\partial \rho_\epsilon^*}{\partial x}(\beta, t) = \frac{\partial \rho_\epsilon^*}{\partial x}(1, t) = 0. \end{cases} \tag{4.5}$$

We introduce the function v such that

$$v = \begin{cases} \frac{x}{\alpha} w + \frac{1}{\alpha} \int_\alpha^x w d\zeta, & x \in (0, \alpha), \\ w, & x \in (\alpha, \beta), \\ v = \frac{1-x}{1-\beta} w - \frac{1}{1-\beta} \int_\beta^x w d\zeta, & x \in (\beta, 1), \end{cases}$$

then the function $\rho(x)$ can be expressed as follows:

$$\rho(x) = \begin{cases} \frac{x^2}{\alpha^2} w = \frac{x}{\alpha} v - \frac{1}{\alpha} \int_\alpha^x v, & x \in (0, \alpha), \\ w = v, & x \in (\alpha, \beta), \\ \frac{(1-x)^2}{(1-\beta)^2} w = \frac{1-x}{1-\beta} v + \frac{1}{1-\beta} \int_\beta^x v, & x \in (\beta, 1). \end{cases}$$

Then we deduce that

$$\begin{cases} v(0, t) = v(\alpha, t) = v(\beta, t) = v(1, t) = 0, \quad \int_0^\alpha v dx + \int_\beta^1 v dx = 0, \\ \frac{\partial v}{\partial x}(\alpha, t) = \frac{\partial v}{\partial x}(\beta, t) = \frac{\partial v}{\partial x}(1, t) = 0, \end{cases}$$

and

$$\frac{\partial \rho}{\partial x} = H(x) \frac{\partial v}{\partial x}, \quad \text{where } H(x) = \begin{cases} \frac{x}{\alpha}, & x \in (0, \alpha), \\ 1, & x \in (\alpha, \beta) \\ \frac{1-x}{1-\beta}, & x \in (\beta, 1). \end{cases}$$

Putting

$$u = \int_0^t \exp(c\tau) v dt,$$

in (4.2) and integrating with respect to x and t , using (4.5) we obtain

$$\begin{aligned} & \operatorname{Re} \int_Q A(t)u\bar{\rho} \, dx \, dt \\ &= - \int_{\Omega} \frac{H(x)}{2} \left(ca - \frac{\partial a}{\partial t} \right) e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt - \int_0^1 \frac{H(x)}{2} a e^{-ct} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \Big|_{t=T} \end{aligned}$$

and

$$\operatorname{Re} \int_0^s \int_0^\alpha \frac{\partial u}{\partial t} \bar{\rho} \, dx \, dt = \int_0^s \int_0^\alpha e^{ct} H(x) |v|^2 \, dx \, dt,$$

as we choose

$$c > \frac{a_3}{a_0}$$

then we get

$$\int_Q \exp(ct) H(x) |v|^2 \, dx \, dt = 0,$$

so $v = 0$ a.e., which implies $\omega = 0$. □

Theorem 4.1 *The range $R(L)$ of the operator L is dense in \mathbb{F} .*

Proof Since \mathbb{F} is a Hilbert space, we have $\overline{R(L)} = \mathbb{F}$ if and only if the relation

$$\int_Q (1-x)^2 f \bar{g} \, dx \, dt + \int_0^1 (1-x)^2 \frac{d\varphi}{dx} \frac{d\bar{\psi}}{dx} \, dx + \int_0^1 \varphi \bar{\psi} \, dx = 0, \tag{4.6}$$

for an arbitrary $u \in D(L)$ and $(g, \psi) \in \mathbb{F}$, implies that $g = 0$ and $\psi = 0$.

Putting $u \in D_0(L)$ in (4.6), we conclude from Lemma 4.1 that $(1-x)^2 g = \theta(x)\omega = 0$, a.e. then $g = 0$.

Taking $u \in D(L)$ in (4.6) yields

$$\int_0^1 (1-x)^2 \frac{d\varphi}{dx} \frac{d\bar{\psi}}{dx} \, dx + \int_0^1 \varphi \bar{\psi} \, dx = 0, \tag{4.7}$$

Since the two terms in the previous equality vanish independently and since the range of the trace operator is everywhere dense in Hilbert space with the norm

$$\int_0^1 (1-x)^2 \left| \frac{d\varphi}{dx} \right|^2 \, dx + \int_0^1 |\varphi|^2 \, dx,$$

hence, $\psi = 0$. Thus $\overline{R(A)} = \mathbb{F}$. □

5 Study of the nonlinear problem

This section is devoted to the proof of the existence, uniqueness of the solution of the problem (1.1)–(1.4).

If the solution of problem (1.1)–(1.4) exists, it can be expressed in the form $\theta = w + U$, where U is a solution of the homogeneous problem

$$\epsilon U = \frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial U}{\partial x} \right) = 0, \tag{5.1}$$

$$U_0 = U(x, 0) = \varphi(x), \tag{5.2}$$

$$U(0, t) = 0, \tag{5.3}$$

$$\int_0^\alpha U(x, t) dx + \int_\beta^1 U(x, t) dx = 0, \tag{5.4}$$

and w is a solution of the problem

$$\epsilon w = \frac{\partial w}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial w}{\partial x} \right) = F \left(x, t, w, \frac{\partial w}{\partial x} \right), \tag{5.5}$$

$$w(x, 0) = 0, \tag{5.6}$$

$$w(0, t) = 0, \tag{5.7}$$

$$\int_0^\alpha w(x, t) dx + \int_\beta^1 w(x, t) dx = 0, \tag{5.8}$$

where $F(x, t, w, \frac{\partial w}{\partial x}) = f(x, t, w + U, \frac{\partial(w+U)}{\partial x})$ satisfying the condition

$$|F(x, t, u_1, v_1) - F(x, t, u_2, v_2)| \leq d(|u_1 - u_2| + |v_1 - v_2|) \quad \text{for all } x, t \in Q. \tag{5.9}$$

According to Theorem 3.1 and Lemma 4.1, the problem (5.1)–(5.4) has a unique solution that depends continuously on $U_0 \in V^{1,0}(0, 1)$ where $V^{1,0}(0, 1)$ is a Hilbert space with the scalar product

$$(u, v)_{V^{1,0}(0,1)} = \int_0^1 (1-x)^2 \frac{\partial u}{\partial x} \frac{\partial \bar{v}}{\partial x} dx + \int_0^1 u \bar{v} dx$$

and with the associated norm

$$\|u\|_{V^{1,0}(0,1)}^2 = \int_0^1 (1-x)^2 \left| \frac{\partial u}{\partial x} \right|^2 dx + \int_0^1 |u|^2 dx.$$

We shall prove that the problem (5.5)–(5.8) has a weak solution by using an approximation process and passing to the limit.

Assume that v and $w \in C^1(Q)$, and the following conditions are satisfied:

$$\begin{cases} v(x, T) = 0, \int_0^\alpha v(x, t) dx + \int_\beta^1 v(x, t) dx = 0, \\ w(x, 0) = 0, \quad w(0, t) = 0. \end{cases} \tag{5.10}$$

Taking the scalar product in $L^2(Q)$ of Eq. (5.5) and the integrodifferential operator

$$Nv = \lambda(1-x) \int_0^x v d\zeta - x \int_x^1 g(\zeta, t) d\zeta,$$

where

$$g(\zeta, t) = k \int_0^\zeta v d\mu - (k - \lambda) \int_\alpha^\zeta v d\mu + (k - \lambda) \int_\beta^\zeta v d\mu d\zeta,$$

by taking the real part, we obtain

$$\begin{aligned} H(w, v) &= \operatorname{Re} \int_Q F\left(x, t, w, \frac{\partial w}{\partial x}\right) \overline{Nv} dx dt \\ &= \operatorname{Re} \int_Q \frac{\partial w}{\partial t} \overline{Nv} dx dt - \operatorname{Re} \int_Q \frac{\partial}{\partial x} \left(a \frac{\partial w}{\partial x} \right) \overline{Nv} dx dt. \end{aligned} \tag{5.11}$$

Substituting the expression of Nv in the first integral of the right-hand side of (5.11), integrating by parts with respect to t , using the condition (5.10), we get

$$\operatorname{Re} \int_Q \frac{\partial w}{\partial t} \overline{Nv} = -\operatorname{Re} \int_Q w \left(\lambda(1-x) \int_0^x \frac{\partial v}{\partial t} d\zeta + x \int_x^1 \frac{\partial g}{\partial t} d\zeta \right) dx dt. \tag{5.12}$$

Substituting the expression of Nv in the second integral of the right hind-side of (5.11), integrating by parts with respect to x , using the condition (5.10), we get

$$\begin{aligned} & -\operatorname{Re} \int_Q \frac{\partial}{\partial x} \left(a \frac{\partial w}{\partial x} \right) \overline{Nv} dx dt \\ &= \operatorname{Re} \int_Q \left[(2\lambda - kx)a - \lambda(1-x) \frac{\partial a}{\partial x} \right] wv - \lambda \operatorname{Re} \int_Q (1-x)aw \frac{\partial v}{\partial x} dx dt \\ & \quad - \operatorname{Re} \int_Q \left[2a + x \frac{\partial a}{\partial x} \right] wg dx dt + \lambda \operatorname{Re} \int_Q \frac{\partial a}{\partial x} w \int_0^\zeta v d\mu dx dt \\ & \quad - \operatorname{Re} \int_Q \frac{\partial a}{\partial x} w \int_x^1 \overline{g} d\zeta dx dt. \end{aligned} \tag{5.13}$$

Insertion of (5.12), (5.13) into (5.11) yields

$$\begin{aligned} H(w, v) &= -\operatorname{Re} \int_Q w \left(\lambda(1-x) \int_0^x \frac{\partial v}{\partial t} d\zeta - x \int_x^1 \frac{\partial g}{\partial t} d\zeta \right) dx dt \\ & \quad - \lambda \operatorname{Re} \int_Q (1-x)aw \frac{\partial v}{\partial x} dx dt + \operatorname{Re} \int_Q \left[(2\lambda - kx)a - \lambda(1-x) \frac{\partial a}{\partial x} \right] wv \\ & \quad - \operatorname{Re} \int_Q \left[2a + x \frac{\partial a}{\partial x} \right] wg dx dt + \lambda \operatorname{Re} \int_Q \frac{\partial a}{\partial x} w \int_0^\zeta v d\mu dx dt \\ & \quad - \operatorname{Re} \int_Q \frac{\partial a}{\partial x} w \int_x^1 \overline{g} d\zeta dx dt, \end{aligned}$$

where

$$\begin{aligned} H(w, v) &= \lambda \operatorname{Re} \int_Q \overline{v} \int_x^1 (1-\zeta)F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta \\ & \quad - \operatorname{Re} \int_Q g \int_0^x \zeta F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta dx dt, \end{aligned} \tag{5.14}$$

obtained by integrating by parts the right-hand side of (5.11) with respect to x .

Definition 5.1 By a weak solution of problem (5.5)–(5.8) we mean a function $w \in L^2(0, T : V^{1,0}(0, 1))$ satisfying the identity (5.14) and the integral condition (5.8).

We will construct an iteration sequence in the following way.

Starting with $w_0 = 0$, the sequence $(w_n)_{n \in \mathbb{N}}$ is defined as follows: given w_{n-1} , then, for $n \geq 1$, we solve the problem

$$\mathcal{L}w_n = \frac{\partial w_n}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial w_n}{\partial x} \right) = F \left(x, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial x} \right), \tag{5.15}$$

$$w_n(x, 0) = 0, \tag{5.16}$$

$$w_n(0, t) = 0, \tag{5.17}$$

$$\int_0^\alpha w_n(x, t) dx + \int_\beta^1 w_n(x, t) = 0. \tag{5.18}$$

From Theorem 3.1 and Lemma 4.1, we deduce that, for fixed n , each problem (5.15)–(5.18) has a unique solution $w_n(x, t)$. If we set $V_n(x, t) = w_{n+1}(x, t) - w_n(x, t)$, we obtain the new problem

$$\mathcal{L}V_n = \frac{\partial V_n}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial V_n}{\partial x} \right) = \sigma_{n-1}, \tag{5.19}$$

$$V_n(x, 0) = 0, \tag{5.20}$$

$$V_n(0, t) = 0, \tag{5.21}$$

$$\int_0^\alpha V_n(x, t) dx + \int_\beta^1 V_n(x, t) = 0, \tag{5.22}$$

where

$$\sigma_{n-1} = F \left(x, t, w_n, \frac{\partial w_n}{\partial x} \right) - F \left(x, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial x} \right). \tag{5.23}$$

Theorem 5.1 Assume that the condition (5.9) holds, for the linearized problem (5.19)–(5.22), there exists a positive constant k , such that

$$\|V_n\|_{L^2(0, T; V^{1,0}(0, 1))} \leq k \|V_{n-1}\|_{L^2(0, T; V^{1,0}(0, 1))}. \tag{5.24}$$

Proof We denote

$$M \frac{\partial V_n}{\partial t} = \frac{(1-x)^2}{2} \frac{\partial V_n}{\partial t} + \lambda(1-x) \int_0^x \frac{\partial V_n}{\partial t} d\zeta - x e^{\delta(x-1)} \int_x^1 g(\zeta, t) d\zeta,$$

where

$$g(\zeta, t) = k \int_0^\zeta \frac{\partial V_n}{\partial t} d\mu - (k - \lambda) \int_\alpha^\zeta \frac{\partial V_n}{\partial t} d\mu + (k - \lambda) \int_\beta^\zeta \frac{\partial V_n}{\partial t} d\mu d\zeta,$$

and λ, k and δ are scalars parameters such that

$$3 < \frac{\lambda}{k} < 4e^{-\delta} \quad \text{with } \delta > \ln\left(\frac{4}{3}\right).$$

We consider the quadratic form obtained by multiplying Eq. (5.19) by $e^{-ct} \overline{M \frac{\partial V_n}{\partial t}}$, with the constant c satisfying (3.11), integrating over $Q_s = [0, 1] \times [0, s]$, with $0 \leq s \leq T$, taking the real part, we obtain

$$\begin{aligned} \Phi(V_n, V_n) &= \operatorname{Re} \int_{Q_s} e^{-ct} \sigma_{n-1} M \overline{\frac{\partial V_n}{\partial t}} dx dt \\ &= \operatorname{Re} \int_{Q_s} e^{-ct} x^2 (1-x)^2 \frac{\partial V_n}{\partial t} M \overline{\frac{\partial V_n}{\partial t}} dx dt \\ &\quad - \operatorname{Re} \int_{Q_s} e^{-ct} \frac{\partial}{\partial x} \left(a \frac{\partial V_n}{\partial x} \right) M \overline{\frac{\partial V_n}{\partial t}} dx dt. \end{aligned} \tag{5.25}$$

Integrating with respect to x and t , using the conditions (5.20), (5.21) and (5.22) we get

$$\begin{aligned} &\frac{\lambda}{2} \int_{Q^s} \frac{(1-x)^2}{2} \exp(-ct) \left| \frac{\partial V_n}{\partial t} \right|^2 dx dt + \frac{\lambda}{2} \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) \frac{(1-x)^2}{2} e^{-ct} \left| \frac{\partial V_n}{\partial x} \right|^2 dx dt \\ &\quad + \frac{1}{2k} \int_{Q^s} \left(\frac{\lambda}{k} - 2xe^{\delta(x-1)} \right) e^{-ct} |g(x, t)|^2 dx dt + \frac{\lambda}{2k^2} \int_{Q^s} e^{-ct} |g(0, t)|^2 dx dt \\ &\quad + \frac{1}{2k} e^{-\delta} \int_0^s e^{-ct} \left| \int_0^1 g(x, t) \right|^2 dx dt \\ &\quad + \frac{1}{2} \int_{Q^s} \left(ca - \frac{\partial a}{\partial t} \right) (\lambda - kxe^{\delta(x-1)}) e^{-ct} |V_n|^2 dx dt \\ &\quad + \frac{1}{2k} \int_{Q^s} (2\delta + \delta^2 x) e^{\delta(x-1)} e^{-ct} \left| \int_x^1 g(\zeta, t) d\zeta \right|^2 dx dt \\ &\quad + \frac{\lambda}{2} \int_0^1 \frac{(1-x)^2}{2} e^{-cs} a(x, s) \left| \frac{\partial V_n}{\partial x} \right|^2 \Big|_{t=s} dx \\ &\quad + \frac{1}{2} \int_0^1 (\lambda - kxe^{\delta(x-1)}) e^{-cs} a(x, s) |V_n|^2 \Big|_{t=s} dx \\ &\quad + \lambda \operatorname{Re} \int_{Q^s} \frac{\partial a(x, t)}{\partial x} e^{-ct} V_n \int_0^x \frac{\partial u}{\partial t} d\zeta dx dt \\ &\quad - \operatorname{Re} \int_{Q^s} \left(x \frac{\partial a}{\partial x} + 2a(1 + \delta x) \right) e^{\delta(x-1)} e^{-ct} V_n \overline{g(x, t)} dx dt \\ &\quad - \frac{\lambda}{2k^2} \operatorname{Re} \int_0^s e^{-ct} g(0, t) \int_0^1 \overline{g(x, t)} dx dt \\ &\quad - \operatorname{Re} \int_{Q^s} \left((1 + \delta x) \frac{\partial a}{\partial x} + (2\delta + \delta^2 x) a \right) e^{\delta(x-1)} e^{-ct} V_n \int_x^1 g(\zeta, t) d\zeta dx dt \\ &= \operatorname{Re} \int_{Q^s} e^{-ct} \sigma_{n-1} M \overline{\frac{\partial V_n}{\partial t}} dx dt. \end{aligned}$$

Following the same procedure as performed in establishing the proof of Theorem 3.1, we get

$$\int_0^T \int_0^1 \left(\frac{(1-x)^2}{2} \left| \frac{\partial V_n}{\partial x} \right|^2 + |V_n|^2 \right) dx \leq K \int_Q \frac{(1-x)^2}{2} |\sigma_{n-1}|^2 dx dt,$$

where

$$K = \frac{\max\left(\frac{2k^2}{\lambda^2-3k} + \frac{k^2}{128\lambda} + 66\lambda\right)}{\min\left(M, \frac{\lambda-k}{2}e^{-\delta}(ca_0 - c_2)\right)} e^{cT},$$

using (5.9), the above inequality becomes

$$\|V_n\|_{L^2(0,T;V^{1,0}(0,1))}^2 \leq k^2 \|V_{n-1}\|_{L^2(0,T;V^{1,0}(0,1))}^2, \tag{5.26}$$

where

$$k^2 = 2d^2 \frac{\frac{2k^2}{\lambda^2-3k} + \frac{k^2}{128\lambda} + 66\lambda}{\min\left(M, \frac{\lambda-k}{2}e^{-\delta}(ca_0 - c_2)\right)} e^{cT}.$$

From the criterion of convergence of the series, we see that the series $\sum_{n \geq 1} V_n(x, t)$ converges if

$$d^2 < \frac{1}{2} \frac{\min\left(M, \frac{\lambda-k}{2}e^{-\delta}(ca_0 - c_2)\right)}{\frac{2k^2}{\lambda^2-3k} + \frac{k^2}{128\lambda} + 66\lambda} e^{-cT}.$$

Since $V_n(x, t) = w_{n+1}(x, t) - w_n(x, t)$, it follows that the sequence $w_n(x, t)$ defined by

$$w_n(x, t) = \sum_{k=1}^{k=n-1} V_k + w_0(x, t)$$

converges to an element $w \in L^2(0, T : V^{1,0}(0, 1))$. Now to prove that this limit function w is a solution of the problem under consideration (5.19)–(5.22), we should show that w satisfies (5.8) and (5.14).

For problem (5.15)–(5.18), we have

$$\begin{aligned} &H(w_n - w, v) + H(w, v) \\ &= \lambda \operatorname{Re} \int_Q \bar{v} \int_x^1 (1 - \eta) \left(F\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\eta, t, w, \frac{\partial w}{\partial \eta}\right) \right) d\eta \, dx \, dt \\ &\quad - \operatorname{Re} \int_Q g \int_0^x \eta \left(F\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\eta, t, w, \frac{\partial w}{\partial \eta}\right) \right) d\eta \, dx \, dt \\ &\quad + \lambda \operatorname{Re} \int_Q \bar{v} \int_x^1 (1 - \zeta) F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta \\ &\quad - \operatorname{Re} \int_Q g \int_0^x \zeta F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta \, dx \, dt. \end{aligned} \tag{5.27}$$

From Eq. (5.15), we have

$$\begin{aligned} H(w_n - w, v) &= \operatorname{Re} \int_Q \frac{\partial(w_n - w)}{\partial t} \left(\lambda(1 - x) \int_0^x \bar{v} d\zeta - x \int_x^1 \bar{g}(\zeta, t) d\zeta \right) dx \, dt \\ &\quad - \operatorname{Re} \int_Q \frac{\partial}{\partial x} \left(a \frac{\partial(w_n - w)}{\partial x} \right) \left(\lambda(1 - x) \int_0^x \bar{v} d\zeta - x \int_x^1 \bar{g}(\zeta, t) d\zeta \right) dx \, dt. \end{aligned}$$

Integrating by parts each term in the previous equality with respect to t and x using the condition (5.10), we obtain

$$\begin{aligned}
 &H(w_n - w, v) \\
 &= -\operatorname{Re} \int_Q (w_n - w) \left(\lambda(1-x) \int_0^x \frac{\partial v}{\partial t} d\zeta - x \int_x^1 \frac{\partial g}{\partial t}(\zeta, t) d\zeta \right) dx dt \\
 &\quad - \lambda \operatorname{Re} \int_Q (1-x)a(w_n - w) \frac{\partial v}{\partial x} dx dt \\
 &\quad + \operatorname{Re} \int_Q \left[(2\lambda - kx)a - \lambda(1-x) \frac{\partial a}{\partial x} \right] (w_n - w)v dx dt \\
 &\quad - \operatorname{Re} \int_Q \left[2a + x \frac{\partial a}{\partial x} \right] (w_n - w)g dx dt + \lambda \operatorname{Re} \int_Q \frac{\partial a}{\partial x} (w_n - w) \int_0^\zeta v d\mu dx dt \\
 &\quad - \operatorname{Re} \int_Q \frac{\partial a}{\partial x} (w_n - w) \int_x^1 \bar{g} d\zeta dx dt, \tag{5.28}
 \end{aligned}$$

where each term of the left-hand side of (5.28) is controlled by

$$\begin{aligned}
 &-\operatorname{Re} \int_Q (w_n - w) \left(\lambda(1-x) \int_0^x \frac{\partial v}{\partial t} d\zeta - x \int_x^1 \frac{\partial g}{\partial t}(\zeta, t) d\zeta \right) dx dt \\
 &\leq 2\sqrt{2} \max(\lambda, 1) \left(\int_Q |w_n - w|^2 dx dt \right)^{\frac{1}{2}} \left(\int_Q (1-x)^2 \left| \frac{\partial v}{\partial t} \right|^2 dx dt + \int_Q \left| \frac{\partial g}{\partial t} \right|^2 dx dt \right)^{\frac{1}{2}}, \\
 &\operatorname{Re} \int_Q \left[(2\lambda - kx)a - \lambda(1-x) \frac{\partial a}{\partial x} \right] (w_n - w)v dx dt \\
 &\leq ((2\lambda - k)a_1 + \lambda b) \left(\int_Q |w_n - w|^2 dx dt \right)^{\frac{1}{2}} \left(\int_Q |v|^2 dx dt \right)^{\frac{1}{2}} \\
 &\quad - \lambda \operatorname{Re} \int_Q (1-x)a(w_n - w) \frac{\partial v}{\partial x} dx dt \\
 &\leq \lambda a_1 \left(\int_Q |w_n - w|^2 dx dt \right)^{\frac{1}{2}} \left(\int_Q (1-x)^2 \left| \frac{\partial v}{\partial x} \right|^2 dx dt \right)^{\frac{1}{2}}, \\
 &-\operatorname{Re} \int_Q \left[2a + x \frac{\partial a}{\partial x} \right] (w_n - w)g dx dt \\
 &\leq (2a_1 + b) \int_Q |w_n - w|^2 dx dt^{\frac{1}{2}} \left(\int_Q |g|^2 dx dt \right)^{\frac{1}{2}}, \\
 &\lambda \operatorname{Re} \int_Q \frac{\partial a}{\partial x} (w_n - w) \int_0^\zeta v d\mu dx dt \\
 &\leq \lambda b \int_Q |w_n - w|^2 dx dt^{\frac{1}{2}} \left(\int_Q |v|^2 dx dt \right)^{\frac{1}{2}} + \operatorname{Re} \int_Q \frac{\partial a}{\partial x} (w_n - w) \int_x^1 \bar{g} d\zeta dx dt \\
 &\leq b \left(\int_Q |w_n - w|^2 dx dt \right)^{\frac{1}{2}} \left(\int_Q \left| \frac{\partial g}{\partial t} \right|^2 dx dt \right)^{\frac{1}{2}}.
 \end{aligned}$$

From the previous inequalities, we deduce that

$$|H(w_n - w, v)| \leq C \|w_n - w\|_{L^2(0,T;V^{1,0}(0,1))} \left[\int_Q \left((1-x)^2 \left(\left| \frac{\partial v}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial t} \right|^2 \right) + |v|^2 \right) dx dt + \int_Q \left(\left| \frac{\partial g}{\partial t} \right|^2 + |g|^2 \right) dx dt \right]^{\frac{1}{2}}, \tag{5.29}$$

where

$$C = \max \{ 2\sqrt{2} \max(\lambda, 1), (2\lambda - k)a_1 + \lambda b, \lambda a_1, (2a_1 + b) \}.$$

Using the condition (5.9) and the Cauchy–Schwarz inequality in the first two terms in the left-hand side in (5.27), we get

$$\begin{aligned} & \lambda \operatorname{Re} \int_Q \bar{v} \int_x^1 (1-\zeta) \left[F\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) \right] d\zeta \\ & - \operatorname{Re} \int_Q g \int_0^x \zeta \left[F\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) \right] d\zeta dx dt \\ & \leq (\lambda + 1) d \|w_n - w\|_{L^2(0,T;V_0^1(0,1))} \left(\int_Q |v|^2 dx dt + \int_Q |g|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned} \tag{5.30}$$

From (5.29), (5.30) and passing to the limit in (5.27) as $n \rightarrow +\infty$, we deduce that

$$\begin{aligned} H(w, v) &= \lambda \operatorname{Re} \int_Q \bar{v} \int_x^1 (1-\zeta) F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta \\ & - \operatorname{Re} \int_Q g \int_0^x \zeta F\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta dx dt. \end{aligned}$$

Now we show that (5.8) holds. Since $\lim_{n \rightarrow +\infty} \|w_n - w\|_{L^2(0,T;V^{1,0}(0,1))} = 0$,

$$\lim_{n \rightarrow +\infty} \left| \int_0^\alpha (w_n - w) dx + \int_\beta^1 (w_n - w) dx \right|^2 \leq \lim_{n \rightarrow +\infty} \int_0^1 |w_n - w|^2 dx. \tag{5.31}$$

So

$$\int_0^\alpha w dx + \int_\beta^1 w dx = 0. \tag{5.32}$$

Let us now prove the uniqueness of the solution.

Theorem 5.2 *If condition (5.9) is satisfied, then the solution of problem (5.5)–(5.8) is unique.*

Proof Suppose that $w_1, w_2 \in L^2(0, T : V^{1,0}(0, 1))$ are two solutions of (5.5)–(5.8), the function $v = w_1 - w_2$ is in $L^2(0, T : V^{1,0}(0, 1))$ and satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial v}{\partial x} \right) = G(x, t), \tag{5.32}$$

$$v(x, 0) = 0, \quad (5.33)$$

$$v(0, t) = 0, \quad (5.34)$$

$$\int_0^\alpha v \, dx + \int_\beta^1 v \, dx = 0, \quad (5.35)$$

where $G(x, t) = F(x, t, w_1, \frac{\partial w_1}{\partial x}) - F(x, t, w_2, \frac{\partial w_2}{\partial x})$.

Taking the inner product in $L^2(Q)$ of Eq. (5.32) and the integro-differential operator

$$M \frac{\partial v}{\partial t} = \frac{(1-x)^2}{2} \frac{\partial v}{\partial t} + \lambda(1-x) \int_0^x \frac{\partial v}{\partial t} \, d\zeta - x \int_x^1 g(\zeta, t) \, d\zeta,$$

where λ satisfied (3.2) and following the same procedure as done in establishing the proof of Theorem 3.1, we get

$$\|v\|_{L^2(0, T; V^{1,0}(0,1))}^2 \leq k^2 \|v\|_{L^2(0, T; V^{1,0}(0,1))}^2,$$

where

$$k^2 = \frac{2(\frac{2k^2}{\lambda^2 - 3k} + \frac{k^2}{128\lambda} + 66\lambda)}{\min(M, \frac{\lambda-k}{2}e^{-\delta}(ca_0 - c_2))} d^2 e^{cT}.$$

Since $k^2 < 1$, we have $v = 0$, which implies that $w_1 = w_2 \in L^2(0, T; V^{1,0}(0, 1))$. \square

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