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A weak solution for a $(p(x), q(x))$-Laplacian elliptic problem with a singular term

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Abstract

Here, we consider the following elliptic problem with variable components:

$$-a(x)\Delta_{p(x)}u - b(x)\Delta_{q(x)}u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u),$$

with Dirichlet boundary condition in a bounded domain in $\mathbb{R}^N$ with a smooth boundary. By applying the variational method, we prove the existence of at least one nontrivial weak solution to the problem.

MSC: 35J75; 35J25

Keywords: $(p(x), q(x))$-Laplacian problem; Singular term; Variational method

1 Introduction

The quasilinear operator (p, q)-Laplacian has been used to model steady-state solutions of reaction–diffusion problems arising in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction–diffusion system:

$$u_t = -\operatorname{div}\bigl[D(u)\nabla u\bigr] + h(x,u),$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$ is the diffusion coefficient, function u describes the concentration, and the reaction term $h(x,u)$ has a polynomial form with respect to the concentration u. The differential operator $\Delta_p + \Delta_q$ is known as the (p,q)-Laplacian operator, if $p \neq q$, where Δ_j, $j > 1$ denotes the j-Laplacian defined by $\Delta_j u := \operatorname{div}(|\nabla u|^{j-2}\nabla u)$. It is not homogeneous, thus some technical difficulties arise in applying the usual methods of the theory of elliptic equations (for further details, see [1, 2, 5, 7, 8, 10, 12–16, 19–23] and references therein).

Our main interest in this work is to prove the existence of a weak solution of the weighted $(p(x), q(x))$-Laplacian problem

$$\begin{cases} -a(x)\Delta_{p(x)}u - b(x)\Delta_{q(x)}u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary, $a, b \in L^\infty(\Omega)$ are positive functions with $a(x) \geq 1$ a.e. on Ω, $\lambda > 0$ is a real parameter, $\Delta_{r(x)} u = \operatorname{div}(|\nabla u|^{r(x)-2}\nabla u)$ denotes $r(x)$-Laplacian operator, for $r \in \{p, q\}$, where $p, q \in C_+(\bar{\Omega})$, $1 < s < q(x) < p(x) < \infty$ a.e. on Ω and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following growth condition:

(f_1) $\quad \bigl|f(x,t)\bigr| \leq \alpha + \beta |t|^{h(x)-1}$

for all $(x,t) \in \Omega \times \mathbb{R}$ where a_1, a_2 are two nonnegative constants, $h \in C_+(\bar{\Omega})$ with $h(x) < p^*(x)$ a.e. in Ω and

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases}$$

In [11] the existence and multiplicity of solutions for the following problem have been established

$$\begin{cases} -\Delta_{p(x)} u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < s < p(x) < \infty$. In [4] the authors proved the existence of two weak solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u + \frac{|u|^{p-2}u}{|x|^p} + \frac{|u|^{q-2}u}{|x|^q} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2 \leq q < p < N$. Motivated by their works, we want to verify the existence of at least one solution for the weighted problem (1.1). To this end, we introduce our notations and also bring some definitions and results.

Throughout this note, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. We set

$$p^- = \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ = \sup_{x\in\Omega} p(x),$$

for $p \in C_+(\bar{\Omega}) = \{g \in C(\overline{\Omega}) : g^- > 1\}$. For $p \in C_+(\bar{\Omega})$, the Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as follows:

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega \bigl|u(x)\bigr|^{p(x)}\,dx < \infty \right\},$$

endowed with the following norm:

$$\|u\|_p := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

For any $u \in L^{p(\cdot)}(B)$ and $v \in L^{p'(\cdot)}(B)$, where $L^{p'(\cdot)}(B)$ is the conjugate space of $L^{p(\cdot)}(B)$, the Hölder type inequality

$$\left|\int_B uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_p\|v\|_{p'}$$

holds true.

The Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

and the norm in $W^{1,p(\cdot)}(\Omega)$ is taken to be

$$\|u\|_{W^{1,p(\cdot)}} := \|u\|_p + \||\nabla u|\|_p,$$

where $\nabla u = (\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_N}(x))$ is the gradient of u at $x = (x_1, \dots, x_N)$ and, as usual, $|\nabla u| = (\sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2)^{\frac{1}{2}}$. Also, we set

$$W_0^{1,p(\cdot)}(\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) : u|_{\partial\Omega} = 0\},$$

with the norm $\||\nabla u|\|_p$.

Proposition 1.1 ([9]) *Let $p, q \in C_+(\bar{\Omega})$.*

(i) *If $q(x) \le p(x)$ a.e. on Ω, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$,*

(ii) *If $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then the embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact;

(iii) *If $q(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then there is a constant $k_q > 0$ such that*

$$\|u\|_q \le k_q \||\nabla u|\|_p,$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \ge N. \end{cases}$$

Remark 1.1 Notice that if $q(x) \le p(x)$ a.e. on Ω, then, by Proposition 1.1, one has

$$W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega).$$

Here, we recall the classical Hardy inequality (see [3, 17]).

Lemma 1.1 *Let $1 < s < N$. Then*

$$\int_\Omega \frac{|u(x)|^s}{|x|^s}\,dx \le \frac{1}{H}\int_\Omega |\nabla u(x)|^s\,dx, \tag{1.2}$$

for $u \in W_0^{1,s}(\Omega)$, where $H := (\frac{N-s}{s})^s$.

Remark 1.2 Let $p \in C_+(\Omega)$ and $s < p(x)$ a.e. on Ω. From Remark 1.1 and Lemma 1.1, we gain

$$\int_\Omega \frac{|u(x)|^s}{|x|^s}\,dx \le \frac{1}{H}\int_\Omega \left|\nabla u(x)\right|^s dx,$$

for $u \in W_0^{1,p(.)}(\Omega)$, where H is as in Lemma 1.1.

Definition 1.1 The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, if

- $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$.
- $t \to f(x,t)$ is continuous for almost every where $x \in \Omega$.

Definition 1.2 ([6]) Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$I = \Phi - \Psi$$

and fix some $r \in [-\infty, +\infty]$. We say that I satisfies the Palais–Smale condition upper cut off at r (in short, the $(PS)^{[r]}$ condition), if any sequence u_n in X such that $I(u_n)$ is bounded, $I'(u_n) \to 0$, and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$ admits a convergent subsequence.

Finally, we recall the following theorem [6, Theorem 2.4] which is the main tool in this paper.

Theorem 1.1 *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functional such that*

$$\inf_{x\in\Omega} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\overline{x} \in X$, with $0 < \Phi(\overline{x}) < r$, such that

(i) $\frac{\sup_{\Phi(x)<r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) *for each $\lambda \in \Lambda :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)<r}\Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda$, there is $x_0 \in \Phi^{-1}(]0,r[)$ such that $I'_\lambda(x_0) = 0$ and $I_\lambda(x_0) \le I_\lambda(x)$, for all $x \in \Phi^{-1}(]0,r[)$.

In the sequel we set $X := W_0^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\| = \left\| |\nabla u| \right\|_p.$$

2 Existence of a solution

In this section we prove the existence of at least one nontrivial weak solution of the problem (1.1).

Let $\Phi : X \to \mathbb{R}$ be a functional defined by

$$\Phi(u) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{b(x)}{q(x)}|\nabla u|^{q(x)}\,dx + \int_\Omega \frac{|u(x)|^s}{s|x|^s}\,dx,$$

where $1 < s < q^- < q(x) < p(x) < p^+ < \infty$.

Remark 2.1 Under the above assumptions, we get

$$\frac{1}{p^+}\big[\|u\|\big]_p \le \Phi(u) \le \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\big(\|u\|^{p^+} + \|u\|^s\big).$$

Proof First, let $\|u\| > 1$. So, we have

$$\begin{aligned}\frac{1}{p^+}\|u\|^{p^-} &\le \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx \le \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}\,dx \\ &\le \Phi(u) \\ &\le \frac{\|a\|_\infty}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{\|b\|_\infty}{q^-}\int_\Omega |\nabla u|^{q(x)}\,dx + \frac{1}{Hs}\int_\Omega |\nabla u|^s\,dx \\ &\le \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\|u\|^{p^+}.\end{aligned}$$

Now, let $\|u\| \le 1$. Then we have

$$\begin{aligned}\frac{1}{p^+}\|u\|^{p^+} &\le \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx \le \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}\,dx \\ &\le \Phi(u) \\ &\le \frac{\|a\|_\infty}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{\|b\|_\infty}{q^-}\int_\Omega |\nabla u|^{q(x)}\,dx + \frac{1}{Hs}\int_\Omega |\nabla u|^s\,dx \\ &\le \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\|u\|^{s}.\end{aligned}$$

Thus the proof is completed. □

It is known that Φ is a continuously Gâteaux differentiable functional; moreover,

$$\Phi'(u)(v) = \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u\nabla v + b(x)|\nabla u|^{q(x)-2}\nabla u\nabla v + \frac{|u(x)|^{s-2}uv}{|x|^s}\right)dx,$$

for $u, v \in X$, see [18]. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and define

$$F(x,t) := \int_0^t f(x,s)\,ds. \tag{2.1}$$

Then the functional $\Psi : X \to \mathbb{R}$ with

$$\Psi(u) := \int_\Omega F\big(x, u(x)\big)\,dx,$$

for every $u \in X$ is continuously Gâteaux differentiable, with the following compact derivative:

$$\Psi'(u)(v) := \int_\Omega f\big(x, u(x)\big)v(x)\,dx,$$

for every u, v in X, see [18]. Moreover, define

$$I := \Phi - \lambda\Psi.$$

If $I'(u) = 0$, we have

$$\int_\Omega \left(a(x)|\nabla u|^{p(x)-2}|\nabla u||\nabla v| + b(x)|\nabla u|^{q(x)-2}|\nabla u||\nabla v| + \frac{|u|^{s-2}uv}{|x|^s} \right) dx$$
$$= \lambda \int_\Omega f(x,u)v\,dx,$$

for every $u, v \in X$, and then the critical points of I are the weak solutions of problem (1.1).
Set

$$\delta(x) = \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\} \quad \text{and} \quad D := \sup_{x\in\Omega} \delta(x).$$

Obviously, there exists $x_0 \in \Omega$ such that

$$B(x_0, D) \subseteq \Omega.$$

For $\gamma > 0$ and $h \in C_+(\overline{\Omega})$ with $1 < h^-$, we define

$$[\gamma]^h := \max\{\gamma^{h^-}, \gamma^{h^+}\}, \qquad [\gamma]_h := \min\{\gamma^{h^-}, \gamma^{h^+}\}, \tag{2.2}$$

and similarly,

$$[\gamma]^{\frac{1}{h}} := \max\{\gamma^{\frac{1}{h^-}}, \gamma^{\frac{1}{h^+}}\}, \qquad [\gamma]_{\frac{1}{h}} := \min\{\gamma^{\frac{1}{h^-}}, \gamma^{\frac{1}{h^+}}\}.$$

Further,

$$m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

where Γ is the Euler function.

The following is the main result of this paper.

Theorem 2.1 *Assume that f is a Carathéodory function satisfying the growth condition (f_1) and F defined by (2.1) is such that*

$$\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} F(x,t)}{t^s} = +\infty. \tag{2.3}$$

Then, for every $\lambda \in]0, \lambda^*[$ with

$$\lambda^* := \frac{1}{\alpha k_1 (p^+)^{\frac{1}{p^-}} + \frac{\beta}{h^-}[k_h]^h (p^+)^{\frac{h+}{p^-}}},$$

the problem (1.1) has at least one nontrivial weak solution.

Proof We use Theorem 1.1 in the case $r = 1$.

Let X, Φ and Ψ be as above and fix $\lambda \in]0, \lambda^*[$. By (2.3), there exists

$$0 < \delta_\lambda < \min\left\{1, \left(\frac{s}{(\|a\|_\infty + \|b\|_\infty + \frac{1}{H})((\frac{2}{D})^{p^+} + (\frac{2}{D})^s)m(D^N - (\frac{D}{2})^N)}\right)^{\frac{1}{s}}\right\}$$

such that

$$\frac{s\inf_{x\in\Omega} F(x,\delta_\lambda)}{((\frac{2}{D})^{p^+} + (\frac{2}{D})^s)\delta_\lambda^s(2^N - 1)} > \frac{\|a\|_\infty + \|b\|_\infty + \frac{1}{H}}{\lambda}. \tag{2.4}$$

We define $\overline{u} \in X$ such that

$$\overline{u}(x) = \begin{cases} 0, & x \in \Omega \backslash B(x_0, D), \\ \delta_\lambda, & x \in B(x_0, \frac{D}{2}), \\ \frac{2\delta_\lambda}{D}(D - |x - x_0|), & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

where $|\cdot|$ is Euclidean norm on $\mathbb{R}^N$. By Remark 2.1, we have

$$\begin{aligned}
\Phi(\bar{u}) &= \int_\Omega \left(\frac{a(x)}{p(x)}|\nabla\bar{u}|^{p(x)} + \frac{b(x)}{q(x)}|\nabla\bar{u}|^{q(x)} + \frac{|\bar{u}|^s}{s|x|^s}\right) dx \\
&< \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\left(\|\bar{u}\|^{p^+} + \|\bar{u}\|^s\right) \\
&\leq \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\left(\left(\frac{2\delta_\lambda}{D}\right)^{p^+} + \left(\frac{2\delta_\lambda}{D}\right)^s\right)m\left(D^N - \left(\frac{D}{2}\right)^N\right) \\
&\leq \frac{1}{s}\left(\|a\|_\infty + \|b\|_\infty + \frac{1}{H}\right)\left(\left(\frac{2}{D}\right)^{p^+} + \left(\frac{2}{D}\right)^s\right)\delta_\lambda^s m\left(D^N - \left(\frac{D}{2}\right)^N\right).
\end{aligned}$$

Clearly, $0 < \Phi(\overline{u}) < 1$. Moreover, thanks to (2.4), one has

$$\Psi(\bar{u}) \geq \int_{B(x_0,\frac{D}{2})} F\big(x, \bar{u}(x)\big)\, dx \geq \inf_{x\in\Omega} F(x,\delta_\lambda) m\left(\frac{D}{2}\right)^N, \tag{2.5}$$

and so

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{s\inf_{x\in\Omega} F(x,\delta_\lambda)}{(\|a\|_\infty + \|b\|_\infty + \frac{1}{H})((\frac{2}{D})^{p^+} + (\frac{2}{D})^s)\delta_\lambda^s(2^N - 1)} > \frac{1}{\lambda}.$$

Using Remark 2.1 for each $u \in \Phi^{-1}((-\infty, 1[)$, we have

$$\|u\| \leq \left[p^+\Phi(u)\right]^{\frac{1}{p}} \leq \left(p^+\right)^{\frac{1}{p^-}}. \tag{2.6}$$

Hence, by the growth condition (f_1) and Proposition 1.1, we have

$$\begin{aligned}
\Psi(u) &= \int_\Omega F\big(x, u(x)\big)\, dx \\
&\leq \alpha \int_\Omega \big|u(x)\big|\, dx + \frac{\beta}{h^-}\int_\Omega \big|u(x)\big|^{h(x)}\, dx \\
&\leq \alpha k_1\|u\| + \frac{\beta}{h^-}[k_h]^h\big[\|u\|\big]^h.
\end{aligned} \tag{2.7}$$

Then, from (2.6) and (2.7), we deduce

$$\sup_{\Phi(u)<1} \Psi(u) \leq \alpha k_1 \left(p^+\right)^{\frac{1}{p^-}} + \frac{\beta}{h^-} [k_h]^h \left(p^+\right)^{\frac{h^+}{p^-}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}.$$

Thus

$$\sup_{\Phi(u)<1} \Psi(u) < \frac{1}{\lambda} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})},$$

and so Theorem 1.1 guarantees the existence of a local minimum point for I, completing the proof of Theorem 2.1. □

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Authors' contributions
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